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List colorings with distinct list sizes, the case of complete bipartite graphs

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Abstract

A graph G is *f-choosable* if for every collection of lists with list sizes specified by f there is a proper coloring using colors from the lists. The sum choice number, $\chi_{sc}(G)$, is the minimum of $\sum f(v)$, over all f such that G is f-choosable. In this paper we show that $\chi_{sc}(G)/|V(G)|$ can be bounded while the minimum degree $\delta_{\min}(G) \to \infty$. (This is not true for the list chromatic number, $\chi_\ell(G)$). Our main tool is to give tight estimates for the sum choice number for the complete bipartite graphs $K_{a,q}$.

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1 Average list sizes and planar graphs

Given a graph G and a list of colors $L(v)$ for each vertex $v \in V(G)$ we say that G is L-*choosable* if it is possible to choose $\ell(v) \in L(v)$ for all v so that ℓ is a proper coloring of G. The *choice number* (or *list chromatic number*) χ_ℓ is the minimum t such that every assignment L with $|L(v)| \geq t$ for all $v \in V$ the graph is L-choosable. It is well-known (Grötzsch's theorem) that

$$\chi_\ell(P) \leq 5 \tag{1}$$

for every planar graph P, and this is the best possible. Erdős, Rubin and Taylor (see, e.g., [1]) showed for the complete bipartite graph that

$$\chi_\ell(K_{q,q}) = \Theta(\log q). \tag{2}$$

However, if we allow distinct list sizes, then the average size can be smaller. For example, Thomassen's beautiful proof [6] for (1) gives that if P is an n-vertex planar graph, $v_1, \ldots, v_t$ are its outside vertices and the list sizes are

$$|L(v)| = \begin{cases} 1 & \text{for } v = v_1, \\ 2 & \text{for } v = v_2, \\ 3 & \text{for } v = v_3, \ldots, v_t, \\ 5 & \text{otherwise}, \end{cases} \tag{3}$$

then P is L-choosable.

Consider a function $f : V(G) \to \mathbb{N}$. An f-*assignment* is an assignment of lists $L(v)$ to the vertices $v \in V(G)$ such that $|L(v)| = f(v)$ for all v. The function f itself is *choosable* if G is L-choosable for all f-assignments L. We define the *sum choice number* of G, denoted $\chi_{sc}(G)$, to be the least k for which there exists a choosable f with $\sum_{v \in V(G)} f(v) = k$. Thomassen's theorem implies that $\chi_{sc}(P) \leq 5n - 9$.

In fact, more is true. It is easy to show (see, e.g., [5]) that for every graph

$$\chi_{sc}(G) \leq |V(G)| + |E(G)| \tag{4}$$

holds. Hence $\chi_{sc}(P) \leq 4n - 6$. Our first result is a slight improvement.

Theorem 1.1 *Let P be an n-vertex planar graph. There exists an $f : V(P) \to \mathbb{N}$ such that $\sum f(v) = 4n - 6$, $\max f(v) \leq 6$, and P is f-choosable.* □

2 Graphs with small average list sizes, $K_{a,q}$

Sum choice numbers were introduced by Isaak in [4] who proved that if G is the line-graph of $K_{2,q}$ then $\chi_{\rm sc}(G) = q^2 + \lceil 5q/3 \rceil$. Various classes of graphs were investigated by Isaak in [5], by Berliner, Bostelmann, Brualdi, and Deaett [2] and by Heinold [3]. Two results are of particular interest to us.

Theorem 2.1 (Berliner et al. [2]) $\chi_{sc}(K_{2,q}) = 2q + 1 + \lfloor \sqrt{4q+1} \rfloor$.

Theorem 2.2 (Heinold [3]) $\chi_{sc}(K_{3,q}) = 2q + 1 + \lfloor \sqrt{12q+4} \rfloor$.

Our main result extends Theorems 2.1 and 2.2 to arbitrary a.

Theorem 2.3 *There exist constants c_1 and c_2 such that for all $a \geq 4$ and $q \geq 50a^2 \log a$*

$$2q + c_1 a\sqrt{q \log a} \leq \chi_{sc}(K_{a,q}) \leq 2q + c_2 a\sqrt{q \log a}.$$

This implies that we can find a choosable f such that the average list size does not necessarily grow with the average degree. Indeed, with q tending to infinity, average degree of $K_{a,q}$ approaches $2a$. We obtain

$$\lim_{a\to\infty,\ q>>a^2 \log a} \frac{|E(K_{a,q})|}{a+q} = \infty, \qquad \lim_{a\to\infty,\ q>>a^2 \log a} \frac{\chi_{sc}(K_{a,q})}{a+q} = 2. \tag{5}$$

3 List chromatic number and average degree

Alon has shown in [1] that χ_l depends heavily on the average degree.

Theorem 3.1 (Alon, [1]) *For some constant c, every graph G with average degree d has $\chi_l(G) \geq c\frac{\log d}{\log \log d}$.*

One of the most interesting corollaries of our Theorem 2.3 is (5), that if different list sizes are allowed, the conclusion of Theorem 3.1 is no longer true. Sum-choice depends more on the structure of the graph than the list chromatic number.

* * *

Throughout this paper, log is the natural logarithm and the two partite sets of the complete bipartite graph $K_{a,q}$ is denoted by A and Q, with $|A| = a$ and $|Q| = q$.

4 Upper bound, there are choosable short lists

Theorem 4.1 *Suppose that $a, q \in \mathbb{N}$ with $q > a > 3$. Then*

$$\chi_{sc}(K_{a,q}) \leq 2q + a\lceil\sqrt{32q(1+\log a)}\rceil.$$

Proof. To prove the upper bound, we present a function f with $\sum_{v \in A \cup Q} f(v) \geq 2q + a\sqrt{32q(1+\log a)}$ such that every f-assignment is choosable.

Define f as

$$f(v) = \begin{cases} r \text{ for } v \in A; \\ 2 \text{ for } v \in Q \end{cases}$$

where r will be defined later in (6). Let L be an arbitrary f-assignment, i.e. $|L(v)| = f(v)$ for all v.

Consider $S := \cup_{v \in A \cup Q} L(v)$. The assignment L yields a multihypergraph and a multigraph on the vertex set S and with edge sets $\mathcal{L}_A := \{L(u) : u \in A\}$ and $\mathcal{L}_Q := \{L(v) : v \in Q\}$, respectively. Choosability of L means that one can find a set $T \subset S$ meeting all hyperedges of $\mathcal{L}_A$ such that $S \setminus T$ meets all edges of $\mathcal{L}_Q$, so T is an independent set in the graph $\mathcal{L}_Q$. Then the choice function ℓ can be defined as

$$\ell(u) \in L(u) \cap T, \text{ for } u \in A$$

and

$$\ell(v) \in L(v) \cap (S \setminus T), \text{ for } v \in Q.$$

We are going to construct such a T by a 2-step random process.

Let us pick, randomly and independently, each element of S with probability p. Let B be the random set of all elements picked. Define a random variable X_u for each $u \in A$ as $X_u = |L(u) \cap B|$, and the random variable Y by $Y := |\{v \in Q : L(v) \subseteq B\}|$, so Y is the number of edges of $\mathcal{L}_Q$ spanned by B. Remove an element $\ell(v) \in L(v)$ for each edge of $\mathcal{L}_Q$ spanned by B, the remaining set $T \subset B$ is certainly independent in $\mathcal{L}_Q$. If $Y < X_u$ for each $u \in A$, then T meets all $L(u) \in \mathcal{L}_A$ and we are done.

One needs a careful definition

$$p := \sqrt{\frac{2(1+\log a)}{q}} \quad \text{and} \quad r \geq 4pq = \sqrt{32(1+\log a)}q. \tag{6}$$

Standard probabilistic arguments (Chernoff inequality) complete the proof.□

5 Lower bound, much shorter lists are not choosable

To prove that $\chi_{sc}(G) \geq k$ for a particular k, we need to show that for every f with $\sum_{v \in G} f(v) = k$, there exists a non-choosable f-assignment. Here we only show how to construct a non-choosable assignment for a very special f.

Lemma 5.1 *Let $t \geq 2$ and $l \geq 1$. For $a = \binom{2t}{t}$ and $q = t\ell^2$, there exists a non-choosable assignment L with $L(v) = 2$ for $v \in Q$ and $L(v) = t\ell$ for $v \in A$.*

Note that with this choice of a and q, we have $|L(v)| \geq \sqrt{\frac{q \log_2 a}{2}}$ for $v \in A$.

Proof. Let us define the vertex set of a hypergraph $\mathcal{H}$ as $V(\mathcal{H}) = \cup_{i=1}^{2t} A_i$ where the A_i's are disjoint ℓ-sets. The edges of $\mathcal{H}$ are of the form $\cup_{i \in I} A_i$ for all subsets $I \subseteq \{1, \ldots, 2t\}$ of size t. Define a graph G on the vertex set $\cup_{i=1}^{2t} A_i$. Let $\{x, y\}$ be an edge if and only if $x \in A_{2i-1}$ and $y \in A_{2i}$.

Define the lists of the vertices $v \in A$ to be the sets in $E(\mathcal{H})$ and the two element sets in $E(G)$ to be the lists of the vertices $v \in Q$. □

This argument can be extended to every f if $\sum f(v)$ is sufficiently small.

Theorem 5.2 *If $a \geq 3$ and $q > 50a^2 \log a$, then*

$$\chi_{sc}(K_{a,q}) \geq 2q + 0.068a\sqrt{q \log a}. \qquad \square$$

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