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Unavoidable subhypergraphs: a-clusters

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Abstract

One of the central problems of extremal hypergraph theory is the description of unavoidable subhypergraphs, in other words, the Turan problem. Let $\mathbf{a} = (a_1, \ldots, a_p)$ be a sequence of positive integers, $p \geq 2$, $k = a_1 + \ldots + a_p$. An **a**-*cluster* is a family of k-sets $\{F_0, \ldots, F_p\}$ such that the sets $F_i \setminus F_0$ are pairwise disjoint ($1 \leq i \leq p$), $|F_i \setminus F_0| = a_i$, and the sets $F_0 \setminus F_i$ are pairwise disjoint, too. Given **a** there is a unique **a**-cluster, and the sets $F_0 \setminus F_i$ form an **a**-partition of F_0. With an intensive use of the delta-system method we prove that for $k > p > 1$ and sufficiently large n, $(n > n_0(k))$, if $\mathcal{F}$ is an n-vertex k-uniform family with $|\mathcal{F}|$ exceeding the Erdős-Ko-Rado bound $\binom{n-1}{k-1}$, then $\mathcal{F}$ contains an **a**-cluster. The only extremal family consists of all the k-subsets containing a given element.

Keywords: Erdős-Ko-Rado, set systems, traces.

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1 Clusters

Suppose that $\mathcal{F}$ is a family of k subsets of the n-set $[n] = \{1, 2, \ldots, n\}$, $\mathcal{F} \subset \binom{[n]}{k}$, $n \geq k \geq 2$. Call a family of k-sets $\{F_1, \ldots, F_d\}$ a (k, d)-cluster if

(1) $$|F_1 \cup F_2 \cup \ldots \cup F_d| \leq 2k \quad \text{and} \quad F_1 \cap F_2 \ldots \cap F_d = \emptyset.$$

The Erdős–Ko–Rado (EKR) theorem states that if $\mathcal{F}$ has no $(k, 2)$-cluster for $n \geq 2k$, then $|\mathcal{F}| \leq \binom{n-1}{k-1}$. Katona proposed in 1980 the problem for $d = 3$. It was proved by Frankl and the first author [4] that then the same EKR-type upper bound holds for $|\mathcal{F}|$ (at least for $n > n_1(k)$). Several results for this problem can be found in [10,13]. Mubayi [11] showed that this bound also follows for $d = 4$ and $n > n_2(k)$. This led him to the following conjecture.

Conjecture 1.1 *Let $k \geq d \geq 2$, $n \geq dk/(d-1)$ and suppose that $\mathcal{F}$ is a k-uniform family on n elements containing no (k, d)-cluster. Then $|\mathcal{F}| \leq \binom{n-1}{k-1}$, with equality only if $\cap\mathcal{F} \neq \emptyset$.*

The case $d = k$ follows from a theorem of Chvatal [3] as it was observed by Chen, Liu, and Wang [2]. Keevash and Mubayi [9] proved Conjecture 1.1 when both k/n and $n/2-k$ bonded away from zero, and Mubayi and Ramadurai [12] for $n > n_0(k)$. The present authors also proved Conjecture 1.1 for $n > n_0(k)$ with a different approach (unpublished) and also settled the case $d = k + 1$.

Our main theorem here not only implies Conjecture 1.1 for sufficiently large n but also gives an explicit structure of the unavoidable subhypergraphs. Let $\mathbf{a} = (a_1, \ldots, a_p)$ be a sequence of positive integers, $k > p > 1$, $k = a_1 + \ldots + a_p$. An $\mathbf{a}$-*cluster* is a family of k-sets $\{F_0, \ldots, F_p\}$ such that the sets $F_i \setminus F_0$ are pairwise disjoint $(1 \leq i \leq p)$, $|F_i \setminus F_0| = a_i$, and the sets $F_0 \setminus F_i$ are pairwise disjoint as well.

Theorem 1.2 *Suppose that $k > p > 1$, $\mathcal{F} \subset \binom{[n]}{k}$ with $|\mathcal{F}| > \binom{n-1}{k-1}$ and n is sufficiently large, ($n > n_0(k)$). Then $\mathcal{F}$ contains an $\mathbf{a}$-cluster. Moreover, if $|\mathcal{F}| = \binom{n-1}{k-1}$, then $\mathcal{F}$ consists of all the k-subsets containing a given element.*

2 Trees and traces in hypergraphs

A system of k-sets $\mathbb{T} := \{E_1, E_2, \ldots, E_q\}$ is called a **tree** (k-tree) if for every $2 \leq i \leq q$ we have $|E_i \setminus \cup_{j<i} E_j| = 1$, and there exists an $\alpha = \alpha(i) < i$ such that

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$|E_\alpha \cap E_i| = k-1$. The case $k=2$ corresponds to the usual trees in graphs. Let $\mathbb{T}$ be a k-tree on v vertices, and let $\mathrm{ex}_k(n,\mathbb{T})$ denote the maximum size of k-family on n elements without $\mathbb{T}$. We have

$$\mathrm{ex}_k(n,\mathbb{T}) \geq (1+o(1))\frac{v-k}{k}\binom{n}{k-1}.$$

Indeed, consider a $P(n, v-1, k-1)$ packing $P_1, \ldots, P_m$ on the vertex set $[n]$. This means that $|P_i| = v-1$ and $|P_i \cap P_j| < k-1$ for $1 \leq i < j \leq m$. Rödl's [15] theorem gives a packing of the size $m = (1+o(1))\binom{n}{k-1}/\binom{v-1}{k-1}$, when $n \to \infty$. Put a complete k-hypergraph into each P_i, the obtained k-graph does not contain $\mathbb{T}$. Note that a $(1,1,\ldots,1)$-cluster is a k-tree with $v=2k$, so if a perfect packing $P(n, 2k-1, k-1)$ exists, then the above construction gives a cluster-free k-family of size $\binom{n}{k-1}$, slightly exceeding the EKR bound.

Conjecture 2.1 (Erdős and Sós for graphs, Kalai 1984 for all k, see in [6]) $\mathrm{ex}_k(n,\mathbb{T}) \leq \frac{v-k}{k}\binom{n}{k-1}$.

The k-graph case was proved for **star-shaped** trees by Frankl and the first author [6], i.e., whenever $\mathbb{T}$ contains an edge wich intersects all other edges in $k-1$ vertices. (For $k=2$ these are the the diameter 3 trees, i.e., 'brooms'.)

Theorem 1.2 is related to the **trace problem of uniform hypergraphs**. Given a hypergraph H, its trace on $S \subseteq V(H)$ is defined as the set $\{E \cap S : E \in \mathcal{E}(H)\}$. Let $\mathrm{Tr}(n,r,k)$ denote the maximum number of edges in an r-uniform hypergraph of order n not admitting the power set $2^{[k]}$ as a trace. For $k \leq r \leq n$, the bound $\mathrm{Tr}(n,r,k) \leq \binom{n}{k-1}$ was proved by Frankl and Pach [7]. Mubayi and Zhao [14] reduced this bound by $\log_p n - k!k^k$ in the case when $k-1$ is a power of a prime p and n is large. On the other hand, Ahlswede and Khachatrian [1] showed $\mathrm{Tr}(n,k,k) \geq \binom{n-1}{k-1} + \binom{n-4}{k-3}$ for $n \geq 2k \geq 6$.

3 The intersection structure of an a-cluster-free family

Definition 3.1 A family $\{D_1, D_2, \ldots, D_s\}$ of distinct sets forms a *delta-system* of size s and with center C if $D_i \cap D_j = C$ holds for all $1 \leq i < j \leq s$.

Definition 3.2 The *intersection structure* of $F \in \mathcal{F}$ with respect to the family $\mathcal{F}$ is defined as $\mathcal{I}(F,\mathcal{F}) := \{F \cap F' : F' \in \mathcal{F}, F \neq F'\}$. The *rank*, $r(\mathcal{I})$, of the intersection structure $\mathcal{I} = \mathcal{I}(F,\mathcal{F})$ is defined as $r(\mathcal{I}) = \min\{|A| : A \subset F, \nexists B \in \mathcal{I}, A \subset B\}$.

Definition 3.3 Let $\mathcal{F}$ be k-partite and $S \subset [n]$. The *projection* $\Pi(S)$ is defined as $\Pi(S) = \{i : S \cap X_i \neq \emptyset\}$, and $\Pi(\mathcal{I}(F,\mathcal{F})) = \{\Pi(S) : S \in \mathcal{I}(F,\mathcal{F})\}$.

The rank is k only if $\mathcal{I} = 2^F \setminus \{F\}$, otherwise it is at most $k-1$. A k-uniform family $\mathcal{F} \subset \binom{[n]}{k}$ is *k-partite* if one can find a k-partition $[n] = X_1 \cup \ldots \cup X_k$ with $|F \cap X_i| = 1$ for all $F \in \mathcal{F}$, $1 \leq i \leq k$.

Theorem 3.4 [8] *For any two positive integers k and s there exists a positive constant $c(k,s)$ such that every family $\mathcal{F} \subset \binom{[n]}{k}$ contains a subfamily $\mathcal{F}^* \subset \mathcal{F}$ satisfying*

(1) $|\mathcal{F}^*| \geq c(k,s)|\mathcal{F}|$,

(2) *$\mathcal{F}^*$ is k-partite,*

(3) *there is a family $\mathcal{J} \subset 2^{[k]} \setminus \{[k]\}$ such that $\Pi(\mathcal{I}(F,\mathcal{F}^*)) = \mathcal{J}$ holds for all $F \in \mathcal{F}^*$,*

(4) *$\mathcal{J}$ is closed under intersection, i.e. $A, B \in \mathcal{J}$ implies $A \cap B \in \mathcal{J}$,*

(5) *every member of $\mathcal{I}(F,\mathcal{F}^*)$ is the center of a delta-system of size s formed by members of $\mathcal{F}^*$.*

Lemma 3.5 *Suppose that $\mathcal{F} \subset \binom{[n]}{k}$ is an* **a***-cluster-free family (* $\mathbf{a} \neq \mathbf{1}$ *). Let $\mathcal{F}^* \subset \mathcal{F}$ be k-partite with $\Pi(\mathcal{I}(F,\mathcal{F}^*)) = \mathcal{J}$ for all $F \in \mathcal{F}^*$, where $\mathcal{J} \subset 2^{[k]} \setminus \{[k]\}$, and every member of $\mathcal{I}(F,\mathcal{F}^*)$ is the center of a delta-system of size $s \geq 2k$ formed by members of $\mathcal{F}^*$ Let $F = \{x_1, \ldots, x_k\} \in \mathcal{F}^*$. If $\mathcal{J}$ is closed under intersection and $r(\mathcal{J}) \geq k-1$, then*

1. *$r(\mathcal{J}) = k-1$, i.e.,, it is impossible that $(F \setminus \{x_i\}) \in \mathcal{I}(F,\mathcal{F}^*)$ for all $1 \leq i \leq k$.*

2. *If there are $k-1$ $(k-1)$-sets in $\mathcal{J}$, e.g., $F \setminus \{x_i\} \in \mathcal{I}(F,\mathcal{F}^*)$ for $2 \leq i \leq k$, then $F \setminus \{x_1\}$ is an own set of F in $\mathcal{F}$. (Note that it is own in $\mathcal{F}$, not only in $\mathcal{F}^*$.)*

3. *If there are $k-t$ $(k-1)$-sets in $\mathcal{J}$, say $F \setminus \{x_i\} \in \mathcal{I}(F,\mathcal{F}^*)$ for $t < i \leq k$ with $2 \leq t \leq k$, then either F has at least two own $(k-1)$-subsets in $\mathcal{F}$ among $\{F \setminus \{x_1\}, \ldots, F \setminus \{x_t\}\}$ or there is no collection $\{F(x_1), \ldots, F(x_t)\} \subset \mathcal{F}$ whose $t-1$ members have disjoint elements outside F.*

To prove this lemma, we mainly use the properties of an **a**-cluster-free family and the properties of the subfamily $\mathcal{F}^*$. The existence of the subfamily $\mathcal{F}^*$ is guaranteed by Theorem 3.4 for sufficiently large n.

Finally, in the proof of Theorem 1.2 we use the above tools and a complicated version of the stability method developed by Frankl and the first author in [6].

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