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Unavoidable subhypergraphs: **a**-clusters

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ABSTRACT

One of the central problems of extremal hypergraph theory is the description of unavoidable subhypergraphs, in other words, the Turán problem. Let $\mathbf{a} = (a_1, \dots, a_p)$ be a sequence of positive integers, $k = a_1 + \cdots + a_p$. An **a**-*partition* of a k-set F is a partition in the form $F = A_1 \cup \cdots \cup A_p$ with $|A_i| = a_i$ for $1 \leqslant i \leqslant p$. An **a**-*cluster* $\mathcal{A}$ with *host* F_0 is a family of k-sets $\{F_0, \dots, F_p\}$ such that for some **a**-partition of F_0, $F_0 \cap F_i = F_0 \setminus A_i$ for $1 \leqslant i \leqslant p$ and the sets $F_i \setminus F_0$ are pairwise disjoint. The family $\mathcal{A}$ has $2k$ vertices and it is unique up to isomorphisms. With an intensive use of the delta-system method we prove that for $k > p$ and sufficiently large n, if $\mathcal{F}$ is a k-uniform family on n vertices with $|\mathcal{F}|$ exceeding the Erdős–Ko–Rado bound $\binom{n-1}{k-1}$, then $\mathcal{F}$ contains an **a**-cluster. The only extremal family consists of all the k-subsets containing a given element.

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1. Introduction

1.1. History

Let $\mathcal{F}$ be a family of k subsets of the n-set $[n] = \{1, 2, \dots, n\}$, $\mathcal{F} \subset \binom{[n]}{k}$, $n \geqslant k \geqslant 2$. The Erdős–Ko–Rado (EKR) theorem [6] states that if any two sets intersect and $n \geqslant 2k$, then $|\mathcal{F}| \leqslant \binom{n-1}{k-1}$. Katona proposed in 1980 the following related problem: Suppose that every three members $F_1, F_2, F_3 \in \mathcal{F}$ meet ($F_1 \cap F_2 \cap F_3 \neq \emptyset$) whenever their union is small, $|F_1 \cup F_2 \cup F_3| \leqslant 2k$. It was proved by Frankl and the first author [8] that then the same EKR-type upper bound holds for $|\mathcal{F}|$ for $n > n_1(k)$.

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The case $3k/2 \leqslant n < 2k$ follows from a result of Frankl [7] (also see Mubayi and Verstraëte [19]), and finally Mubayi [16] gave a nice short proof that $|\mathcal{F}| \leqslant \binom{n-1}{k-1}$ holds for all $n \geqslant 2k$ (with equality only for $\bigcap \mathcal{F} \neq \emptyset$) so $n_1(k) = \lceil 3k/2 \rceil$. Mubayi [17] showed that the EKR bound also holds, if $|F_1 \cup F_2 \cup F_3 \cup F_4| \leqslant 2k$ implies $F_1 \cap F_2 \cap F_3 \cap F_4 \neq \emptyset$ (for $n > n_2(k)$). This led him to the following conjecture.

Conjecture 1. *Call a family of k-sets $\{F_1, \dots, F_d\}$ a (k, d)-cluster if*

$$|F_1 \cup F_2 \cup \dots \cup F_d| \leqslant 2k \quad \text{and} \quad F_1 \cap F_2 \cap \dots \cap F_d = \emptyset.$$

Let $k \geqslant d \geqslant 2$, $n \geqslant dk/(d-1)$ and suppose that $\mathcal{F}$ is a k-uniform family on n elements containing no (k, d)-cluster. Then $|\mathcal{F}| \leqslant \binom{n-1}{k-1}$, with equality only if $\bigcap \mathcal{F} \neq \emptyset$.

The case $d = k$ follows from a theorem of Chvátal [5] as it was observed by Chen, Liu, and Wang [4]. Keevash and Mubayi [14] proved Conjecture 1 when both k/n and $n/2 - k$ are bounded away from zero, and Mubayi and Ramadurai [18] for $n > n_3(k)$. The present authors also proved Conjecture 1 in 2007 for $n > n_4(k)$ with a different approach (unpublished). Recently, Jiang, Pikhurko, and Yilma [13] proved a more general result concerning the so-called strong simplices.

In Theorem 2, we give a stronger generalization which not only implies Conjecture 1 and all the above results for sufficiently large n but also gives an explicit structure of the unavoidable subhypergraphs.

In our notation, $A \subset B$ also includes the case that $A = B$. We write $A \subsetneq B$ for the case $A \subset B$ and $A \neq B$.

1.2. **a**-*Clusters*

Let $\mathbf{a} = (a_1, \dots, a_p)$ be a sequence of positive integers, $p \geqslant 2$, $k = a_1 + \dots + a_p$. An **a**-*partition* of a k-set F is a partition in the form $F = A_1 \cup \dots \cup A_p$ with $|A_i| = a_i$ for $1 \leqslant i \leqslant p$. An **a**-*cluster* $\mathcal{A}$ with *host* F_0 is a family of k-sets $\{F_0, \dots, F_p\}$ such that for some **a**-partition of F_0, $F_0 \cap F_i = F_0 \setminus A_i$ for $1 \leqslant i \leqslant p$ and the sets $F_i \setminus F_0$ are pairwise disjoint. The family $\mathcal{A}$ has $2k$ vertices and it is unique up to isomorphisms.

Theorem 2. *Suppose that $k > p > 1$, $\mathcal{F} \subset \binom{[n]}{k}$ with $|\mathcal{F}| > \binom{n-1}{k-1}$ and n is sufficiently large ($n > N(k)$). Then $\mathcal{F}$ contains any **a**-cluster, $\mathbf{a} \neq \mathbf{1}$. Moreover, if $|\mathcal{F}| = \binom{n-1}{k-1}$, **a**-cluster-free, then it consists of all the k-subsets containing a given element.*

Our $N(k)$ is very large, it is double exponential in k. In the proof of Theorem 2, we use the delta-system method and a complicated version of the stability method developed in [10] by Frankl and the first author of this paper. Note that the case $k = p$, i.e., $\mathbf{a} = (1, 1, \dots, 1)$, is different as described in Section 3.2.

1.3. *The delta-system method*

It is natural to investigate the intersection structure of $\mathcal{F}$. This is exactly where the delta-system method can be applied.

The *intersection structure* of $F \in \mathcal{F}$ with respect to the family $\mathcal{F}$ is defined as

$$\mathcal{I}(F, \mathcal{F}) = \{F \cap F' \colon F' \in \mathcal{F},\ F \neq F'\}.$$

If the set F is given, $A \subset F$ with $(F \setminus A) \in \mathcal{I}(F, \mathcal{F})$, then we use the notation $F(A)$ for a k-set in $\mathcal{F}$ such that $F(A) \cap F = F \setminus A$.

A k-uniform family $\mathcal{F} \subset \binom{[n]}{k}$ is *k-partite* if one can find a partition $[n] = X_1 \cup \dots \cup X_k$ with $|F \cap X_i| = 1$ for all $F \in \mathcal{F}$, $1 \leqslant i \leqslant k$. If $\mathcal{F}$ is k-partite, then for any set $S \subset [n]$, its *projection* $\Pi(S)$ is defined as

$$\Pi(S) = \{i \colon S \cap X_i \neq \emptyset\} \quad \text{and} \quad \Pi\big(\mathcal{I}(F, \mathcal{F})\big) = \big\{\Pi(S) \colon S \in \mathcal{I}(F, \mathcal{F})\big\}.$$

A family $\{D_1, D_2, \ldots, D_s\}$ is called a *delta-system* of size s and with center C if $D_i \cap D_j = C$ holds for all $1 \leqslant i < j \leqslant s$. The delta-system method is described in the following theorem due to the first author.

Theorem 3. *(See [12].) For any positive integers s and k with $s > k$, there exists a positive constant $c(k, s)$ such that every family $\mathcal{F} \subset \binom{[n]}{k}$ contains a subfamily $\mathcal{F}^* \subset \mathcal{F}$ satisfying*

(3.1) $|\mathcal{F}^*| \geqslant c(k, s)|\mathcal{F}|$,
(3.2) *$\mathcal{F}^*$ is k-partite,*
(3.3) *there is a family $\mathcal{J} \subset 2^{\{1,2,\ldots,k\}} \setminus \{[k]\}$ such that $\Pi(\mathcal{I}(F, \mathcal{F}^*)) = \mathcal{J}$ holds for all $F \in \mathcal{F}^*$,*
(3.4) *$\mathcal{J}$ is closed under intersection (i.e., $A, B \in \mathcal{J}$ imply $A \cap B \in \mathcal{J}$),*
(3.5) *every member of $\mathcal{I}(F, \mathcal{F}^*)$ is the center of a delta-system $\mathcal{D}$ of size s formed by members of $\mathcal{F}^*$ and containing F, $F \in \mathcal{D} \subset \mathcal{F}^*$.*

We call a family $\mathcal{F}^*$ *homogeneous* if $\mathcal{F}^*$ satisfies (3.2)–(3.5). In this paper, we fix $s = 2k$ in Theorem 3.

Lemma 4. *Suppose that $\mathcal{F}^* \subset \mathcal{F}$, where $\mathcal{F}^*$ is obtained by using Theorem 3 with $s = 2k$. If $G_1 \in \mathcal{F}^*$, $G_2 \in \mathcal{F}$, $M \in \mathcal{I}(G_1, \mathcal{F}^*)$, $M \subset G_2$ and $M \cap S = \emptyset$, where $|S| \leqslant k$, then there exists a $G_3 \in \mathcal{F}^*$ such that $G_2 \cap G_3 = M$ and $S \cap G_3 = \emptyset$.*

Proof. Let $\{F'_1, F'_2, \ldots, F'_{2k}\} \subset \mathcal{F}^*$ be a delta-system centered at M, where $F'_1 = G_1$. Since the sets $F'_1 \setminus M, \ldots, F'_{2k} \setminus M$ are pairwise disjoint, and $|G_2 \setminus M| < k$ and $|S| \leqslant k$ there is an F'_i avoiding both $(1 \leqslant i \leqslant 2k)$. Then $G_2 \cap F'_i = M$ and $S \cap F'_i = \emptyset$. □

2. Proof of the main theorem

2.1. Rank and shadow of **a**-cluster-free families

Throughout the proof of Theorem 2, we will be mostly interested in the *rank* of $\mathcal{J}$, which is defined as

$$r(\mathcal{J}) = \min\bigl\{|A|:\ A \subset [k],\ \nexists B \in \mathcal{J},\ A \subset B\bigr\}.$$

The rank of $\mathcal{J}$ is k only if $\mathcal{J} = 2^{[k]} \setminus \{[k]\}$; otherwise, it is at most $k - 1$.

From now on, $\mathcal{F} \subset \binom{[n]}{k}$ is an arbitrary k-family containing no **a**-cluster, where $\mathbf{a} = (a_1, \ldots, a_p)$ is a non-increasing sequence with $a_1 \geqslant 2$. We will show that $|\mathcal{F}| \geqslant \binom{n-1}{k-1}$ implies $\bigcap \mathcal{F} \neq \emptyset$ for sufficiently large n.

Frankl and the first author [9] developed a method while proving a conjecture of Erdős that is used in [10] to show that a family $\mathcal{F} \subset \binom{[n]}{k}$ has a common element ($\bigcap \mathcal{F} \neq \emptyset$) if certain intersection constraints are fulfilled. Here we revisit that result and modify that proof to obtain a version for **a**-cluster-free families.

For the rest of the paper, we let $\mathcal{F}^* \subset \mathcal{F}$ be a homogeneous subfamily of $\mathcal{F}$.

Corollary 5. *Let $F = \{x_1, \ldots, x_k\} \in \mathcal{F}^*$. If $r(\mathcal{J}) \geqslant k-1$, then $r(\mathcal{J}) = k-1$, i.e., it is impossible that $(F \setminus \{x_i\}) \in \mathcal{I}(F, \mathcal{F}^*)$ for all $1 \leqslant i \leqslant k$.*

Proof. Assume, on the contrary, that $r(\mathcal{J}) = k$. Because $\mathcal{J}$ is closed under intersection, we have $\mathcal{J} = 2^{[k]} \setminus \{[k]\}$. Therefore, $\mathcal{I}(F, \mathcal{F}^*)$ contains all proper subsets of F. Consider an **a**-partition of $F = (A_1, \ldots, A_p)$. Using Lemma 4 p times with $G_1 = F$, $M = F \setminus A_i$ and $S = \bigcup_{j<i}(F_j \setminus F)$ we obtain $F_1, \ldots, F_p \in \mathcal{F}^*$ such that, for $i \in [p]$, $F \cap F_i = F \setminus \{A_i\}$ and the sets $F_i \setminus F$ are disjoint. Therefore, $\{F_1, \ldots, F_p, F\}$ is an **a**-cluster with host F. □

We use the notation $\Delta_\ell(\mathcal{H})$ for the ℓ-*shadow* of the family $\mathcal{H}$, i.e.,

$$\Delta_\ell(\mathcal{H}) := \{L\colon |L| = \ell,\ \exists H \in \mathcal{H} \text{ with } L \subset H\}.$$

Lemma 6. *$\mathcal{F}$ is not too dense, i.e., $|\Delta_{k-1}(\mathcal{G})| \geqslant c_1(k)|\mathcal{G}|$ for all $\mathcal{G} \subset \mathcal{F}$, where $c_1(k) := c(k, 2k)$ from* (3.1).

Proof. Apply Theorem 3 to $\mathcal{G}$ to obtain a k-partite $\mathcal{G}^*$ with a homogeneous intersection structure $\mathcal{J} \subset 2^{[k]}$, i.e., $\Pi(\mathcal{I}(G, \mathcal{G}^*)) = \mathcal{J}$ for all $G \in \mathcal{G}^*$. Corollary 5 implies that the rank of $\mathcal{J}$ is at most $k - 1$ so each $G \in \mathcal{G}^*$ has a $(k - 1)$-subset that is not contained by another member of $\mathcal{G}^*$. We obtain $|\Delta_{k-1}(\mathcal{G}^*)| \geqslant |\mathcal{G}^*|$, and hence

$$\left|\Delta_{k-1}(\mathcal{G})\right| \geqslant \left|\Delta_{k-1}\left(\mathcal{G}^*\right)\right| \geqslant \left|\mathcal{G}^*\right| \geqslant c(k, 2k)|\mathcal{G}|. \qquad \square \tag{1}$$

2.2. The intersection structure of rank-$(k - 1)$ subfamilies

For a subset $S \subset F \in \mathcal{F}$, denote the *degree* of S in $\mathcal{F}$ by

$$\deg_{\mathcal{F}}(S) = \left|\{F\colon F \in \mathcal{F},\ S \subset F\}\right|.$$

A subset of $F \in \mathcal{F}$ is called an *own* subset of F, if its degree in $\mathcal{F}$ is one.

Lemma 7. *Let $F_0 \in \mathcal{F}^*$ and $\{A_1, \ldots, A_p\}$ an* **a***-partition of F_0. Assume that there exists an $H \in \mathcal{F}$ and $i \in [p]$ such that $F_0 \cap H = (F_0 \setminus A_i)$. Suppose $F_0 \setminus A_j \in \mathcal{I}(F_0, \mathcal{F}^*)$ for each $j \in [p]$ when $j \neq i$. Then there is an* **a***-cluster in $\mathcal{F}$ with host F_0.*

Proof. Call H to F_i. Use Lemma 4 $(p - 1)$ times to define F_j for $j \in [p] \setminus \{i\}$ with $G_1 = H$, $M = F_0 \setminus A_j \in \mathcal{I}(F_0, \mathcal{F}^*)$ and $S = (F_i \setminus F_0) \bigcup_{\ell<j}(F_\ell \setminus F_0)$. Note that $|S| < k$ at each step. $\square$

Lemma 7 can be generalized to allow more than one member with properties of H as used in the proof of Lemma 9.

Lemma 8. *Let $F = \{x_1, \ldots, x_k\} \in \mathcal{F}^*$. If $r(\mathcal{J}) = k - 1$, and there are $k - 1$ $(k - 1)$-sets in $\mathcal{J}$, say $F \setminus \{x_i\} \in \mathcal{I}(F, \mathcal{F}^*)$ for $2 \leqslant i \leqslant k$, then $F \setminus \{x_1\}$ is an own subset of F in $\mathcal{F}$. Moreover, in this case*

$$F_1 \in \mathcal{F}, \quad |F_1 \cap F| \geqslant k - 2 \quad \textit{imply}\ x_1 \in F_1. \tag{2}$$

Such an F (and $\mathcal{J}$ and $\mathcal{F}^*$) is called of type I. Note that we claim that $F \setminus \{x_1\}$ is an own subset of F in $\mathcal{F}$, not only in $\mathcal{F}^*$.

Proof. Suppose, on the contrary, that there exists an $F_1 \in \mathcal{F}$ such that $F_1 = \{y, x_2, \ldots, x_k\}$, $y \notin F_1$. This will enable us to find an **a**-cluster (with a host F_2 to be defined later), a contradiction.

Choose a subset M of F such that $x_1 \in M$ and $|M| = k - a_1 + 1$ $(< k)$. Note that (3.4) implies that

$$\{E\colon E \subsetneq F,\ x_1 \in E\} \subset \mathcal{I}\left(F, \mathcal{F}^*\right). \tag{3}$$

So $M \in \mathcal{I}(F, \mathcal{F}^*)$ and by Lemma 4 we can pick another member $F_2 \in \mathcal{F}^*$ such that $F \cap F_2 = M$ and $y \notin F_2$. We obtain

$$F_2 \cap F_1 = M \setminus \{x_1\} \quad \text{hence} \quad |F_2 \cap F_1| = k - a_1.$$

Consider an **a**-partition of F_2 such that $A_1 = F_2 \setminus F_1$, i.e. $F_1 = F_2(A_1)$. Since $F_2 \in \mathcal{F}^*$ and $\mathcal{F}^*$ is homogeneous, by (3) and (3.3) of Theorem 3, we have

$$\{E\colon E \subsetneq F_2,\ x_1 \in E\} \subset \mathcal{I}\left(F_2, \mathcal{F}^*\right).$$

Therefore, $F_2 \setminus A_i \in \mathcal{I}(F_2, \mathcal{F}^*)$ for $2 \leqslant i \leqslant p$ and we obtain an **a**-cluster by Lemma 7, a contradiction.

The proof of (2) when $|F_1 \cap F| = k-2$, assuming $x_1, x_2 \notin F_1$, is similar and we omit the details. To prove this case, one needs to follow the same steps assuming that $x_1, x_2 \in M$ and have to choose M and F_2 such that $|M| = k - a_1 + 2$ and $F_2 \cap F_1 = M \setminus \{x_1, x_2\}$, respectively, except in the case $a_1 = 2$ when we define $F_2 = F$. □

Lemma 9. *If $r(\mathcal{J}) = k-1$, and there are exactly $k-t$ $(k-1)$-sets in $\mathcal{J}$ with $2 \leqslant t \leqslant k$, say $F \setminus \{x_i\} \in \mathcal{I}(F, \mathcal{F}^*)$ for $t < i \leqslant k$ then*

$$\sum_{1 \leqslant i \leqslant t} \frac{1}{\deg_{\mathcal{F}}(F \setminus \{x_i\})} \geqslant 1 + \frac{1}{k-1}.$$

These $F \in \mathcal{F}^*$ (and $\mathcal{J}$ and $\mathcal{F}^*$) are called type II.

Proof. Define a bipartite graph G with partite sets $X = \{x_1, \ldots, x_t\}$ and $Y = [n] \setminus F$ and edges xy for $x \in X$ and $y \in Y$ if and only if $(F \setminus \{x\}) \cup \{y\} \in \mathcal{F}$. We claim that the maximum number of independent edges in this graph, $\nu(G)$, is at most $t-2$. This indeed implies Lemma 9 as follows. By the König–Hall theorem the size of a minimum vertex cover S of G is at most $t-2$. Let $|X \setminus S| = \ell$, we have $\ell \geqslant 2$ and $|S \cap Y| \leqslant \ell - 2$. Since each vertex $v \in X \setminus S$ has neighbors only in $S \cap Y$, we have

$$\deg_{\mathcal{F}}\big(F \setminus \{v\}\big) = \deg_G(v) + 1 \leqslant |S \cap Y| + 1 \leqslant \ell - 1.$$

This yields

$$\sum_{v \in X \setminus S} \frac{1}{\deg_{\mathcal{F}}(F \setminus \{v\})} \geqslant \frac{\ell}{\ell - 1} \geqslant \frac{k}{k-1}.$$

To prove $\nu(G) \leqslant t-2$ suppose, on the contrary, that there are $F_i := (F \setminus \{x_i\} \cup \{y_i\}) \in \mathcal{F}$ for $2 \leqslant i \leqslant t$, where y_i's are distinct elements outside F. We will see this leads to the existence of an **a**-cluster. First, we describe the intersection structure of F in $\mathcal{F}^*$ by using repeatedly the fact that $\mathcal{I}(F, \mathcal{F}^*)$ is closed under intersection.

Note that

$$\text{if } A \subseteq \{x_{t+1}, \ldots, x_k\} \quad \text{then} \quad F \setminus A \in \mathcal{I}\big(F, \mathcal{F}^*\big). \tag{4}$$

Also, if $A \subset F$, $|A| < k$ and

$$\big|A \cap \{x_1, \ldots, x_t\}\big| \geqslant 2 \quad \text{then} \quad (F \setminus A) \in \mathcal{I}\big(F, \mathcal{F}^*\big). \tag{5}$$

Indeed, the rank of $\mathcal{J}$ exceeds $k-2$, so we have that $F \setminus \{x_u\}, F \setminus \{x_v\} \notin \mathcal{I}(F, \mathcal{F}^*)$ $(1 \leqslant u < v \leqslant t)$, but $F \setminus \{x_u, x_v\} \in \mathcal{I}(F, \mathcal{F}^*)$. Also $F \setminus \{x_w\} \in \mathcal{I}(F, \mathcal{F}^*)$ for $t < w \leqslant k$. Since $\mathcal{J}$ is closed under intersection, we obtain that

$$F \setminus A = \Bigg(\bigcap_{x_u, x_v \in A,\, u < v \leqslant t} \big(F \setminus \{x_u, x_v\}\big) \Bigg) \cap \Bigg(\bigcap_{x_w \in A,\, w > t} \big(F \setminus \{x_w\}\big) \Bigg) \in \mathcal{I}\big(F, \mathcal{F}^*\big).$$

In the rest of the proof, we specify how one can build an **a**-cluster with host F using Lemma 7 if each A_i in an **a**-partition of F satisfies either one of (4) and (5) or $A_i = \{x_j\}$ with $1 < j \leqslant k$. There are several cases to consider.

Recall that $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_p$ and $a_1 \geqslant 2$. Define the positive integers i and ℓ as follows.

$$a_1 + \cdots + a_{i-1} < t \leqslant a_1 + \cdots + a_i,$$
$$\ell = t - (a_1 + \cdots + a_{i-1}).$$

Except the last case, the host of the **a**-cluster is F.

Case 1: $\ell \geqslant 2$. Then $a_1, \ldots, a_i \geqslant \ell \geqslant 2$.

Let $A_1, A_2, \ldots, A_{i-1} \subset X = \{x_1, \ldots, x_t\}$ and $|A_i \cap \{x_1, \ldots, x_t\}| = \ell$.

Case 2: $\ell = 1$ and $a_i = 1$.

By our assumption, there exist $F_i := (F \setminus \{x_i\} \cup \{y_i\}) \in \mathcal{F}$ for $2 \leqslant i \leqslant t$, where y_i's are distinct elements outside F. Let $A_1 \cup A_2 \cup \cdots \cup A_i = \{x_1, \ldots, x_t\}$, $x_1 \in A_1$.

From now on, $\ell = 1$ and $a_i \geqslant 2$ so $i \geqslant 2$.

Case 3: $\ell = 1$, $a_i \geqslant 2$ and $a_1 \geqslant 3$.

Let $A_1 \cup A_2 \cup \cdots \cup A_i \supseteq \{x_1, \ldots, x_t, x_{t+1}\}$, $x_{t+1} \in A_1$ and $A_2 \cup \cdots \cup A_{i-1} \subset \{x_1, \ldots, x_t\}$. We have that $|X \cap A_1|, |X \cap A_i| \geqslant 2$.

Case 4: $\ell = 1$, $a_i \geqslant 2$, $a_1 \leqslant 2$ and $a_p = 1$. Then $a_1 = \cdots = a_i = 2$.

Let $A_1 \cup A_2 \cup \cdots \cup A_{i-1} \cup A_p = \{x_1, \ldots, x_t\}$, $A_p := \{x_t\}$.

Case 5: $\ell = 1$, $a_1 = \cdots = a_p = 2$.

This implies that t is odd, $t \geqslant 3$, and $k = 2p$ is even so $t < k$. Pick a member F_0 from $\mathcal{F}^*$ such that $F_0 = F \setminus \{x_k\} \cup \{y\}$ for some $y \neq y_2$. Choose an $\mathbf{a}$-partition of F_0 such that $A_1 = \{y, x_2\}$, which means $F_2 = F_0(A_1)$. The other parts are $A_2 = \{x_1, x_3\}$ and $A_j = \{x_{2j-2}, x_{2j-1}\}$ for $3 \leqslant j \leqslant p$. By (3.3) of Theorem 3, the intersection structure $\mathcal{I}(F_0, \mathcal{F}^*)$ is isomorphic to $\mathcal{I}(F, \mathcal{F}^*)$ so (4) and (5) imply that $F \setminus A_j \in \mathcal{I}(F_0, \mathcal{F}^*)$ for $2 \leqslant j \leqslant p$. Then Lemma 7 implies that there is an $\mathbf{a}$-cluster with host F_0. □

2.3. Type I dominates, a partition of $\mathcal{F}$

Apply Theorem 3 to $\mathcal{F}$ to obtain $\mathcal{G}_1 := (\mathcal{F})^*$ with the intersection structure $\mathcal{J}_1 \subset 2^{[k]}$. Then we apply Theorem 3 again to $\mathcal{F} \setminus \mathcal{G}_1$ to obtain $\mathcal{G}_2 = (\mathcal{F} \setminus \mathcal{G}_1)^*$ and $\mathcal{J}_2$, then apply to $\mathcal{F} \setminus (\mathcal{G}_1 \cup \mathcal{G}_2)$ and so on, until either $\mathcal{F} \setminus (\mathcal{G}_1 \cup \cdots \cup \mathcal{G}_m) = \emptyset$ or $r(\mathcal{J}_{m+1}) \leqslant k-2$ for some m. Let $\mathcal{F}_1$ be the union of those $\mathcal{G}_i$'s, where $\mathcal{J}_i$ contains exactly $k-1$ $(k-1)$-sets (type I families) and let $\mathcal{F}_2$ be the union of the rest of these families (type II families)

$$\mathcal{F}_2 := \bigcup_j \bigl\{\mathcal{G}_j\colon\ r(\mathcal{J}_j) = k-1,\ \text{but } \mathcal{J}_j \text{ does not contain exactly } (k-1)\ (k-1)\text{-sets}\bigr\}.$$

Finally, let

$$\mathcal{F}_3 := \mathcal{F} \setminus (\mathcal{G}_1 \cup \cdots \cup \mathcal{G}_m) = \mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2).$$

Lemma 10. *If $\mathcal{F} \subset \binom{[n]}{k}$ is $\mathbf{a}$-cluster-free with $|\mathcal{F}| \geqslant \binom{n-1}{k-1}$, then*

$$|\mathcal{F}_2| + |\mathcal{F}_3| \leqslant \frac{k}{c_1(k)}\binom{n}{k-2} + (k-1)\binom{n-1}{k-2} < c_2(k)n^{k-2},$$

where $c_1(k) := c(k, 2k)$ from (3.1).

Proof. Since the rank of $\mathcal{J}_{m+1}$ is at most $k-2$, each member of $\mathcal{G}_{m+1}$ has its own $(k-2)$-subset in $\mathcal{G}_{m+1}$. We obtain as in (1) that

$$c(k, 2k)\bigl|\mathcal{F} \setminus (\mathcal{G}_1 \cup \cdots \cup \mathcal{G}_m)\bigr| \leqslant |\mathcal{G}_{m+1}| \leqslant \bigl|\Delta_{k-2}(\mathcal{G}_{m+1})\bigr| \leqslant \binom{n}{k-2},$$

therefore we can write

$$\frac{k}{k-1}|\mathcal{F}_3| \leqslant \frac{k}{(k-1)c_1(k)}\binom{n}{k-2}.$$

Lemma 8 implies that every $F \in \mathcal{F}_1$ contains an own $(k-1)$-set. This and Lemma 9 give

$$|\mathcal{F}_1| + \frac{k}{k-1}|\mathcal{F}_2| \leqslant \sum_{F \in \mathcal{F}} \left(\sum_{v \in F} \frac{1}{\deg_{\mathcal{F}}(F \setminus \{v\})} \right) = \bigl|\Delta_{k-1}(\mathcal{F})\bigr| \leqslant \binom{n}{k-1}.$$

Compare the sum of the above two inequalities to $\binom{n-1}{k-1} \leqslant |\mathcal{F}_1| + |\mathcal{F}_2| + |\mathcal{F}_3|$. A simple calculation completes the proof. □

2.4. Another partition, the stability of the extremum

For every $F \in \mathcal{F}_1$ there exists a type I family $\mathcal{G}_i \subset \mathcal{F}$, $F \in \mathcal{G}_i$. By the definition of type I family, there exists a (unique) $\ell := \ell(F)$ such that $\{E\colon\ \ell \in E \subset F\} \subset \mathcal{I}(F, \mathcal{G}_i)$. Classify the members $F \in \mathcal{F}_1$ according to $\ell(F)$, let $\mathcal{H}_i := \{F \in \mathcal{F}_1\colon\ \ell(F) = i\}$, $i \in [n]$. Let

$$\tilde{\mathcal{H}}_i := \bigl\{H \setminus \{i\}\colon\ H \in \mathcal{H}_i\bigr\}.$$

These families are pairwise disjoint, $\tilde{\mathcal{H}}_i \cap \tilde{\mathcal{H}}_j = \emptyset$. The shadows $\Delta_{k-2}(\tilde{\mathcal{H}}_i)$ are pairwise disjoint, too. Otherwise, for a set $H \in \Delta_{k-2}(\tilde{\mathcal{H}}_i) \cap \Delta_{k-2}(\tilde{\mathcal{H}}_j)$, $i \neq j$, (2) implies that $H' = H \cup \{i, j\} \in \mathcal{H}_i \cap \mathcal{H}_j$ contradicting with the uniqueness of $\ell(H')$.

Given a positive integer d and real x define $\binom{x}{d}$ as $x(x-1)\cdots(x-d+1)/d!$. We will need the following version of the Kruskal–Katona theorem due to Lovász.

Theorem 11. *(See [15].) Suppose that $\mathcal{H} \subset \binom{[n]}{d}$ and $|\mathcal{H}| = \binom{x}{d}$, $x \geqslant d$. Then $|\Delta_h(\mathcal{H})| \geqslant \binom{x}{h}$ holds for all $d > h \geqslant 0$.*

In case of $\mathcal{H}_i \neq \emptyset$ let x_i be a real number such that $x_i \geqslant k-1$ and $|\tilde{\mathcal{H}}_i| = \binom{x_i}{k-1}$. Without loss of generality, let x_1 be the maximal one, i.e. $n-1 \geqslant x_1 \geqslant x_i$. We obtain for all $i \in [n]$ that

$$|\mathcal{H}_i| = |\tilde{\mathcal{H}}_i| \leqslant \frac{\binom{x_i}{k-1}}{\binom{x_i}{k-2}} \bigl|\Delta_{k-2}(\tilde{\mathcal{H}}_i)\bigr| \leqslant \frac{x_1-k+2}{k-1} \bigl|\Delta_{k-2}(\tilde{\mathcal{H}}_i)\bigr| \leqslant \frac{n-k+1}{k-1} \bigl|\Delta_{k-2}(\tilde{\mathcal{H}}_i)\bigr|. \tag{6}$$

We assume that $|\mathcal{F}| \geqslant \binom{n-1}{k-1}$. Then Lemma 10 gives a lower bound for $|\mathcal{F}_1| = \sum |\mathcal{H}_i|$,

$$\binom{n-1}{k-1} - c_2 n^{k-2} \leqslant \sum_{i\in[n]} |\mathcal{H}_i| \leqslant \frac{x_1-k+2}{k-1}\biggl(\sum_{i\in[n]} \bigl|\Delta_{k-2}(\tilde{\mathcal{H}}_i)\bigr|\biggr) \leqslant \frac{x_1-k+2}{k-1}\binom{n}{k-2}.$$

This inequality implies that $x_1 > n - c_3$ for some constant $c_3 = c_3(k)$. Therefore there exists a constant $c_4 := c_4(k)$ such that

$$\sum_{2\leqslant i\leqslant k} |\mathcal{H}_i| = \sum_{2\leqslant i\leqslant k} |\tilde{\mathcal{H}}_i| \leqslant \binom{n}{k-1} - \binom{n-c_3}{k-1} < c_4 n^{k-2}.$$

This and Lemma 10 lead to

$$|\mathcal{F} \setminus \mathcal{H}_1| \leqslant (c_2 + c_4) n^{k-2}. \tag{7}$$

Note that (with minor modifications) the arguments in the above two sections lead to the following stability result.

Theorem 12. *For every $\varepsilon > 0$ there exists a $\delta > 0$ and $n_0 = n_0(k, \varepsilon)$ such that the following holds. If $\mathcal{F} \subset \binom{[n]}{k}$ contains no **a**-cluster and $|\mathcal{F}| > (1-\delta)\binom{n-1}{k-1}$, $n > n_0$, then there exists an element $v \in [n]$ such that all but at most $\varepsilon\binom{n-1}{k-1}$ members of $\mathcal{F}$ contains v.*

2.5. The extremal family is unique, the end of the proof

In this section we complete the proof of Theorem 2. We have given a family $\mathcal{F} \subset \binom{[n]}{k}$ containing no **a**-cluster and of size $|\mathcal{F}| \geqslant \binom{n-1}{k-1}$. In previous sections we have already defined $\mathcal{H}_1 \subset \mathcal{F}_1$, $\mathcal{F}_2$, and $\mathcal{F}_3$ and showed in (7) that $\mathcal{H}_1$ constitutes the bulk of $\mathcal{F}$. One can see (as we have seen in Lemma 8) that

$$F \in \mathcal{F},\ H \in \mathcal{H}_1, \quad |F \cap H| \geqslant k - a_1 \quad \text{imply } 1 \in F. \tag{8}$$

Let us split $\mathcal{F}$ into four subfamilies

$$\mathcal{B}=\{B\colon 1\notin B\in\mathcal{F}\},$$
$$\mathcal{C}=\bigl\{C\colon 1\in C\in\mathcal{F} \text{ and } |C\cap B|\geqslant k-a_1 \text{ for some } B\in\mathcal{B}\bigr\},$$
$$\mathcal{D}=\{D\colon 1\in D\in\mathcal{F}\setminus\mathcal{C} \text{ and every } S \text{ with } 1\in S\subsetneq D$$
$$\text{is a center of some delta-system of } \mathcal{F} \text{ of size } 2k\},$$
$$\mathcal{E}=\{E\colon 1\in E\in\mathcal{F}\}\setminus(\mathcal{C}\cup\mathcal{D}).$$

We have $\mathcal{H}_1\subset\mathcal{D}$. In (16), (17) and (20) we will prove that for sufficiently large n with respect to k, one has

$$|\mathcal{D}|+4|\mathcal{B}|\leqslant\binom{n-1}{k-1},\qquad |\mathcal{D}|+4|\mathcal{C}|\leqslant\binom{n-1}{k-1},\qquad |\mathcal{D}|+4|\mathcal{E}|\leqslant\binom{n-1}{k-1}. \tag{9}$$

By adding these three, we have

$$3|\mathcal{F}|+\bigl(|\mathcal{B}|+|\mathcal{C}|+|\mathcal{E}|\bigr)\leqslant 3\binom{n-1}{k-1}$$

implying $\mathcal{B}=\mathcal{C}=\mathcal{E}=\emptyset$. Thus $\mathcal{F}=\mathcal{D}$, $\bigcap\mathcal{F}\neq\emptyset$, and we are done.

Before starting the proof of (9), let us define the following subfamilies:

$$\tilde{\mathcal{C}}:=\bigl\{C\setminus\{1\}\colon C\in\mathcal{C}\bigr\},\qquad \tilde{\mathcal{D}}:=\bigl\{D\setminus\{1\}\colon D\in\mathcal{D}\bigr\},\qquad \tilde{\mathcal{E}}:=\bigl\{E\setminus\{1\}\colon E\in\mathcal{E}\bigr\}. \tag{10}$$

We also apply Theorem 3 with $c_1(k):=c(k,s)$ and $s=2k$ to $\tilde{\mathcal{C}}$ and $\tilde{\mathcal{E}}$ to obtain $(k-1)$-partite subfamilies $\mathcal{C}^*\subset\mathcal{C}$ and $\mathcal{E}^*\subset\mathcal{E}$. By (3.1), we have

$$\bigl|\mathcal{C}^*\bigr|\geqslant c_1(k)|\tilde{\mathcal{C}}|=c_1(k)|\mathcal{C}| \quad\text{and}\quad \bigl|\mathcal{E}^*\bigr|\geqslant c_1(k)|\tilde{\mathcal{E}}|=c_1(k)|\mathcal{E}|. \tag{11}$$

Since each member of $\tilde{\mathcal{D}}$ has $(k-1)$ subsets of size $k-2$ and every $(k-2)$-set is contained in at most $(n-k+1)$ members of $\tilde{\mathcal{D}}$ we have that $(n-k+1)|\Delta_{k-2}(\tilde{\mathcal{D}})|\geqslant(k-1)|\tilde{\mathcal{D}}|$. Rearranging and using $|\tilde{\mathcal{D}}|=|\mathcal{D}|$ we obtain

$$\frac{n-k+1}{k-1}\bigl|\Delta_{k-2}(\tilde{\mathcal{D}})\bigr|\geqslant|\mathcal{D}|. \tag{12}$$

Subfamily $\mathcal{B}$. By definition of $\mathcal{D}$ and Lemma 8, we have $|D\cap B|\neq k-2$ for all $D\in\tilde{\mathcal{D}}$ and $B\in\mathcal{B}$. In other words, $\Delta_{k-2}(\tilde{\mathcal{D}})\cap\Delta_{k-2}(\mathcal{B})=\emptyset$. Hence,

$$\binom{n-1}{k-2}\geqslant\bigl|\Delta_{k-2}(\tilde{\mathcal{D}})\bigr|+\bigl|\Delta_{k-2}(\mathcal{B})\bigr|.$$

Multiplying (14) with $(n-k+1)/(k-1)$ and using (12), we obtain

$$\binom{n-1}{k-1}\geqslant|\mathcal{D}|+\frac{n-k+1}{k-1}\bigl|\Delta_{k-2}(\mathcal{B})\bigr|. \tag{13}$$

Let $x\geqslant k-1$ be a real number such that $|\Delta_{k-1}(\mathcal{B})|=\binom{x}{k-1}$. By Theorem 11, we have

$$\bigl|\Delta_{k-2}(\mathcal{B})\bigr|\geqslant\frac{k-1}{x-k+2}\bigl|\Delta_{k-1}(\mathcal{B})\bigr|. \tag{14}$$

By Lemma 6,

$$\bigl|\Delta_{k-1}(\mathcal{B})\bigr|\geqslant c_1(k)|\mathcal{B}|. \tag{15}$$

Then (13)–(15) yield

$$\binom{n-1}{k-1} \geqslant |\mathcal{D}| + c_1(k)\frac{n-k+1}{x-k+2}|\mathcal{B}|. \tag{16}$$

Since $\mathcal{B}$ is contained in $\mathcal{F} \setminus \mathcal{H}_1$ inequality (7) gives

$$\binom{x}{k-1} = \left|\Delta_{k-1}(\mathcal{B})\right| \leqslant k|\mathcal{B}| < k(c_2 + c_4)n^{k-2}$$

implying that $x < c_5 n^{(k-2)/(k-1)}$ for some constant c_5. Therefore, the coefficient of $|\mathcal{B}|$ in (16) is at least 4 for sufficiently large n.

Subfamily $\mathcal{C}$. We denote the homogeneous intersection structure of $\mathcal{C}$ by $\mathcal{J}_C$.

Claim 13. *Each $C' \in \mathcal{C}^*$ has a $(k-2)$-set such that it is contained neither in $\Delta_{k-2}(\tilde{\mathcal{D}})$ nor in $\mathcal{I}(C', \mathcal{C}^*)$.*

Proof. Suppose, on the contrary, that for some $C' = \{x_1, \ldots, x_{k-1}\} \in \mathcal{C}^*$ with $C = C' \cup \{1\} \in \mathcal{C}$, we have

$$C' \setminus \{x_i\} \in \begin{cases} \mathcal{I}(C', \tilde{\mathcal{D}}), & i = 1, \ldots, r, \\ \mathcal{I}(C', \mathcal{C}^*), & i = r+1, \ldots, k-1. \end{cases}$$

All subsets of $C' \setminus \{x_i\}$ are contained in $\mathcal{I}(C', \tilde{\mathcal{D}})$, for $1 \leqslant i \leqslant r$, and all supersets of the set $\{x_1, \ldots, x_r\}$ in C', except C' itself, are contained in $\mathcal{I}(C', \mathcal{C}^*)$. So, for all $S \subset C'$, there is a delta-system of size $2k$ with center $S \cup \{1\}$.

We claim that $r \geqslant 1$. Otherwise $\mathcal{J}_C = 2^{[k-1]} \setminus \{[k-1]\}$ and there exists a member $C'' \in \mathcal{C}$ such that $C'' \setminus \{1\} \in \mathcal{C}^*$ and $|C'' \cap B| = k - a_1$ for some $B \in \mathcal{B}$. Then one can build an **a**-cluster with host C'' such that $C''(A_1) = B$.

Let $D_i \in \mathcal{D}$ such that $C \cap D_i = C \setminus \{x_i\}$, for $i = 1, \ldots, r$ and choose a $B \in \mathcal{B}$ with $|C \cap B| \geqslant k - a_1$. By definition of $\mathcal{D}$,

$$|D_i \cap B| \leqslant k - a_1 - 1.$$

We also have

$$|D_i \cap B| + 1 \geqslant \left|C' \cap B\right| = |C \cap B| \geqslant k - a_1.$$

Therefore, $x_i \in C \cap B$ for all $i = 1, \ldots, r$ and $|C \cap B| = k - a_1$ and one can build an **a**-cluster with host C and $C(A_1) = B$, a contradiction. □

By Claim 13, we have

$$\binom{n-1}{k-2} \geqslant \left|\Delta_{k-2}(\tilde{\mathcal{D}})\right| + \left|\mathcal{C}^*\right|.$$

Multiplying this by $\frac{n-k+1}{k-1}$ and applying (11) and (12) we obtain

$$\binom{n-1}{k-1} \geqslant |\mathcal{D}| + c_1(k)\frac{n-k+1}{k-1}|\mathcal{C}|. \tag{17}$$

Subfamily $\mathcal{E}$. First we show that each $E' \in \mathcal{E}^*$ has a $(k-2)$-subset that is neither in $\mathcal{I}(E', \mathcal{E}^*)$ nor in $\mathcal{I}(E', \tilde{\mathcal{D}})$. Suppose, on the contrary, that for some $E \in \mathcal{E}$, $E' := E \setminus \{1\} \in \mathcal{E}^*$, $E' = \{x_1, \ldots, x_{k-1}\}$ such that

$$E' \setminus \{x_i\} \in \begin{cases} \mathcal{I}(E', \tilde{\mathcal{D}}), & i = 1, \ldots, r, \\ \mathcal{I}(E', \mathcal{E}^*), & i = r+1, \ldots, k-1. \end{cases} \tag{18}$$

All subsets of $E' \setminus \{x_i\}$ are contained in $\mathcal{I}(E', \tilde{\mathcal{D}})$, for $1 \leqslant i \leqslant r$, and all supersets of the set $\{x_1, \dots, x_r\}$ in E', except E' itself, are contained in $\mathcal{I}(E', \mathcal{E}^*)$. So, for all $S \subset E'$, there is a delta-system of size $2k$ with center $S \cup \{1\}$. This contradicts to $E \notin \mathcal{D}$.

Since every $E' \in \mathcal{E}^*$ contains a $(k-2)$-set that is not contained in any member of $\tilde{\mathcal{D}}$ or another member of $\mathcal{E}^*$, we have

$$\binom{n-1}{k-2} \geqslant \left|\Delta_{k-2}(\tilde{\mathcal{D}})\right| + \left|\mathcal{E}^*\right|. \tag{19}$$

After multiplying (19) with $\frac{n-k+1}{k-1}$ and applying the inequalities (11) and (12), we obtain

$$\binom{n-1}{k-1} \geqslant |\mathcal{D}| + c_1(k)\frac{n-k+1}{k-1}|\mathcal{E}|. \tag{20}$$

3. Concluding remarks

3.1. Finding a $(k, k+1)$-cluster

Our first observation is, that in Conjecture 1 the constraint $d \leqslant k$ is not necessary. We prove the case $d = k+1$. It is not clear what is the possible maximum value of d. We need a classical result of Bollobás [3]. A *cross-intersecting set system*, $\{A_i, B_i\}$ for $i \in [m]$, is a collection of pairs of sets such that $A_i \cap B_i = \emptyset$ and $A_i \cap B_j \neq \emptyset$ for $i \neq j$. If $|A_i| \leqslant a$ and $|B_i| \leqslant b$ (for all $1 \leqslant i \leqslant m$) then

$$m \leqslant \binom{a+b}{a}.$$

Equality holds only if $\{A_1, \dots, A_m\} = \binom{[a+b]}{a}$ and $B_i = [a+b] \setminus A_i$.

Theorem 14. *If $\mathcal{F} \subset \binom{[n]}{k}$ contains no $(k, k+1)$-cluster and $n \geqslant k$, then $|\mathcal{F}| \leqslant \binom{n-1}{k-1}$. Here equality holds only if $\bigcap \mathcal{F} \neq \emptyset$.*

Proof. Every $F \in \mathcal{F}$ has a $(k-1)$-subset $B(F) \subset F$ that is not contained by any other member of $\mathcal{F}$, otherwise there are sets $F_1, \dots, F_k \in \mathcal{F}$ such that $F = \{x_1, \dots, x_k\}$ and $F \cap F_i = F \setminus \{x_i\}$, a contradiction. Therefore, the sets $\{B(F), [n] - F\}$ form an intersecting set pair system and the result of Bollobás yields $|\mathcal{F}| \leqslant \binom{(k-1)+(n-k)}{k-1} = \binom{n-1}{k-1}$. □

3.2. Trees in hypergraphs, Kalai's conjecture

A system of k-sets $\mathbb{T} := \{E_1, E_2, \dots, E_q\}$ is called a *tree* (k-tree) if for every $2 \leqslant i \leqslant q$ we have $|E_i \setminus \bigcup_{j<i} E_j| = 1$, and there exists an $\alpha = \alpha(i) < i$ such that $|E_\alpha \cap E_i| = k-1$. The case $k = 2$ corresponds to the usual trees in graphs. Let $\mathbb{T}$ be a k-tree on v vertices, and let $\mathrm{ex}_k(n, \mathbb{T})$ denote the maximum size of a k-family on n elements without $\mathbb{T}$. We have

$$\mathrm{ex}_k(n, \mathbb{T}) \geqslant \bigl(1 + o(1)\bigr)\frac{v-k}{k}\binom{n}{k-1}. \tag{21}$$

Indeed, consider a $P(n, v-1, k-1)$ packing $P_1, \dots, P_m$ on the vertex set $[n]$. This means that $|P_i| = v-1$ and $|P_i \cap P_j| < k-1$ for $1 \leqslant i < j \leqslant m$. Rödl's [21] theorem gives a packing of the size $m = (1+o(1))\binom{n}{k-1}/\binom{v-1}{k-1}$, when $n \to \infty$. Put a complete k-hypergraph into each P_i, the obtained k-graph does not contain $\mathbb{T}$.

Conjecture 15. *(Erdős and Sós for graphs, Kalai 1984 for all k, see in [10].)*

$$\mathrm{ex}_k(n, \mathbb{T}) \leqslant \frac{v-k}{k}\binom{n}{k-1}.$$

This was proved for *star-shaped* trees by Frankl and the first author [10], i.e., whenever $\mathbb{T}$ contains an edge which intersects all other edges in $k-1$ vertices. (For $k=2$ these are the diameter 3 trees, i.e., 'brooms'.)

Note that a $\mathbf{1}$-cluster is a k-tree with $v=2k$, here $\mathbf{1}:=(1,1,\dots,1)$. A Steiner system $S(n,k,t)$ is a *perfect* packing, a family of k-subsets of $[n]$ such that each t-subset of $[n]$ is contained in a unique member of that family. So if an $S(n,2k-1,k-1)$ exists then construction (21) gives a cluster-free k-family of size $\binom{n}{k-1}$, slightly exceeding the EKR bound. (Such designs exist, e.g., for $k=3$ and $n\equiv 1$ or 5 (mod 20), see [2].) On the other hand, the result of Frankl and the first author [10] (cited above) implies that if $\mathcal{F}\subset\binom{[n]}{k}$ is a family with more than $\binom{n}{k-1}$ members, then $\mathcal{F}$ contains every star-shaped tree with $k+1$ edges, especially it contains a $\mathbf{1}$-cluster.

3.3. Traces

Theorem 2 is related to the trace problem of uniform hypergraphs. Given a hypergraph H, its trace on $S\subseteq V(H)$ is defined as the set $\{E\cap S\colon\ E\in\mathcal{E}(H)\}$. Let $\mathrm{Tr}(n,r,k)$ denote the maximum number of edges in an r-uniform hypergraph of order n and not admitting the power set $2^{[k]}$ as a trace. For $k\leqslant r\leqslant n$, the bound $\mathrm{Tr}(n,r,k)\leqslant\binom{n}{k-1}$ was proved by Frankl and Pach [11]. Mubayi and Zhao [20] slightly reduced this upper bound by $\log_p n-k!k^k$ in the case when $k-1$ is a power of the prime p and n is large. On the other hand, Ahlswede and Khachatrian [1] showed $\mathrm{Tr}(n,k,k)\geqslant\binom{n-1}{k-1}+\binom{n-4}{k-3}$ for $n\geqslant 2k\geqslant 6$.

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