Combinatorics, Probability and Computing (2009) 18, 725–729. © 2009 Cambridge University Press
doi:10.1017/S0963548309009985 Printed in the United Kingdom

On 14-Cycle-Free Subgraphs of the Hypercube

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Received 8 September 2008; revised 13 January 2009; first published online 22 May 2009

It is shown that the size of a subgraph of Q_n without a cycle of length 14 is of order $o(|E(Q_n)|)$.

1. Subgraphs of the hypercube with no C_4 or C_6

For given two graphs, Q and P, let $\mathrm{ex}(Q, P)$ denote the *generalized Turán number*, *i.e.*, the maximum number of edges in a P-free subgraph of Q. The *n-dimensional hypercube*, Q_n, is the graph with vertex set $\{0, 1\}^n$ and edges assigned between pairs differing in exactly one coordinate. Let $e(G) = |E(G)|$ be the size of the graph G. We use $N(G, P)$ for the number of subgraphs of G that are isomorphic to P.

Erdős [9] conjectured that $\mathrm{ex}(Q_n, C_4) = (\frac{1}{2} + o(1))e(Q_n)$. The best upper bound, $(0.6226 + o(1))e(Q_n)$, is due to Thomason and Wagner [17], while Brass, Harborth and Nienborg [6] showed $\frac{1}{2}(n + \sqrt{n})2^{n-1} \leqslant \mathrm{ex}(Q_n, C_4)$, when n is a positive integer power of 4, and $\frac{1}{2}(n + 0.9\sqrt{n})2^{n-1} \leqslant \mathrm{ex}(Q_n, C_4)$ for all $n \geqslant 9$.

Monotonocity implies that the limit $c_\ell := \lim_{n\to\infty} \mathrm{ex}(Q_n, C_\ell)/e(Q_n)$ exists. It is known that $1/3 \leqslant c_6 < 0.3941$ (Conder [8] and Lu [14], respectively), $c_{4k} = 0$ for any integer $k \geqslant 2$ (Chung [7]), and $c_{4k+2} \leqslant 1/\sqrt{2}$ for $k \geqslant 1$ (Axenovich and Martin [3]).

Theorem 1.1. *If G is a subgraph of Q_n containing no cycle of length* 14, *then*

$$e(G) = O(n^{6/7}2^n).$$

Hence $e(G) = o(e(Q_n))$, i.e., $c_{14} = 0$.

† Research supported in part by the Hungarian National Science Foundation under grants OTKA 062321, 060427 and by the National Science Foundation under grant NSF DMS 06-00303.

In fact, our proof gives $\mathrm{ex}(Q_n, \Theta_{14}) = O(n^{6/7}2^n)$, where Θ_{14} is the 14-cycle with a longest diagonal. Further related hypercube results can be found, *e.g.*, in Alon et al. [1, 2], Bialostocki [4], Kostochka [13], Johnson and Entringer [12], Harborth and Nienborg [11], Offner [15] and Schelp and Thomason [16].

2. The density of a C_{14}-free subgraph of Q_n is 0

2.1. Subgraphs with large girth

Lemma 2.1. *Let G be a subgraph of Q_n. Then there is a subgraph $G_8 \subset G$ with girth at least 8 such that $e(G_8) \geqslant (1/3)e(G)$.*

Proof. By a theorem of Conder [8], there is a C_4, C_6-free subgraph H of Q_n with at least $(1/3)e(Q_n)$ edges. Then there is a permutation $\pi \in \mathrm{Aut}(Q_n)$ such that

$$|E(\pi(H)) \cap E(G)| \geqslant \frac{1}{|\mathrm{Aut}(Q_n)|} \sum_{\rho \in \mathrm{Aut}(Q_n)} |E(\rho(H)) \cap E(G)| = \frac{e(H)}{e(Q_n)} e(G) \geqslant \frac{1}{3} e(G). \qquad \square$$

2.2. The intersection structure of C_8s

Lemma 2.2. *Let G be a subgraph of the hypercube with no C_4, C_6 or C_{14}. Let C' and C'' be two 8-cycles of G with a common edge. Then $E(C') \cap E(C'')$ forms a path of length 2, 3, or 4.*

Proof. There are two vertices u and v dividing C' into two paths of lengths a and b and a path $P \subset C''$ of length c such that $V(C') \cap V(P) = \{u, v\}$, $a, b, c \geqslant 1$, $a + b = 8$, $a \geqslant 4 \geqslant b$. The condition on the girth of G implies $c + b \geqslant 8$, hence $c \geqslant a \geqslant 4$. Thus C'' can possess only one such path P, we have $C'' \subset C \cup P$, and $E(C') \cap E(C'')$ is a path of length b. If $b = 1$, then the symmetric difference of C' and C'' is a cycle of length 14, a contradiction. $\square$

Let $\mathcal{C}_8(G)$ or just $\mathcal{C}$ denote the set of 8-cycles in the graph G. $\mathcal{C}[e]$ and $\mathcal{C}[e, f]$ denote the set of 8-cycles containing the edge e, or containing the edges e and f, respectively. We have the following obvious corollary of Lemma 2.2.

Lemma 2.3. *Let G be a subgraph of the hypercube with no C_4, C_6 or C_{14}. Let C be an 8-cycle of G with three consecutive edges e, f and g. Then $\mathcal{C}[f] = \mathcal{C}[e, f] \cup \mathcal{C}[f, g]$.*

2.3. An upper bound on $N(G, C_8)$

There is a partition of $E(Q_n)$ into n matchings M_i, $i \in [n]$, which we call *directions*, where M_i is formed of the edges with endpoints differing in the ith coordinate. In every 8-cycle C in Q_n each direction must occur an even number of times, so C has at most 4 directions, and C is contained in a (unique) 4- or 3-dimensional subcube. Since $N(Q_3, C_8) = 6$ and the number of 4-dimensional 8-cycles in Q_4 is 648, we obtain that

$$N(Q_n, C_8) = 648 \binom{n}{4} 2^{n-4} + 6 \binom{n}{3} 2^{n-3}.$$

This easily implies that, for any two edges e and f of Q_n sharing a vertex,

$$|\mathcal{C}_8(Q_n)[e,f]| = (27/8)(n-2)(n-3) + (1/4)(n-2) = O(n^2). \tag{2.1}$$

Lemma 2.4. *Let G be a subgraph of Q_n with no C_4, C_6 or C_{14}. Then the number of C_8s in G is at most $O(n^2) \times e(G)$.*

Proof. It is sufficient to prove that $|\mathcal{C}[f]| = O(n^2)$ for each edge $f \in E(G)$. Let C be an 8-cycle of G containing f and let e, f and g be the three consecutive edges of C. Then Lemma 2.3 and (2.1) complete the proof. □

2.4. A lower bound on the number of C_4s

Lemma 2.5. *Let H be a graph with e edges and n vertices. Then*

$$N(H, C_4) \geqslant 2\frac{e^3(e-n)}{n^4} - \frac{e^2}{2n} \geqslant 2\frac{e^4}{n^4} - \frac{3}{4}en. \tag{2.2}$$

Proof. This result goes back to Erdős (1962) and was published, *e.g.*, in Erdős and Simonovits [10] in an asymptotic form. As we use it for arbitrary n and e, we revisit the proof. Denote the average degree of H by $\overline{d} = 2e/n$ and the number of x, y-paths of length two by $d(x, y)$ and let $\overline{\overline{d}}$ be its average. We have

$$\overline{\overline{d}} = \binom{n}{2}^{-1} \sum_{x,y \in V(H)} d(x,y) = \binom{n}{2}^{-1} \sum_{x \in V(H)} \binom{\deg(x)}{2} \geqslant \binom{n}{2}^{-1} n \binom{\overline{d}}{2}. \tag{2.3}$$

Therefore, $\overline{\overline{d}} \geqslant \frac{2e(2e-n)}{n^2(n-1)}$. Moreover,

$$N(H, C_4) = \frac{1}{2} \sum_{x,y \in V(H)} \binom{d(x,y)}{2} \geqslant \frac{1}{2}\binom{n}{2}\binom{\overline{\overline{d}}}{2}. \tag{2.4}$$

We may suppose that the middle term in (2.2) is positive, which implies that

$$\frac{2e(2e-n)}{n^2(n-1)} \geqslant 1/2.$$

The paraboloid $\binom{x}{2}$ is increasing for $x \geqslant 1/2$. So we may substitute the lower bound of $\overline{\overline{d}}$ from (2.3) into (2.4) and a little algebra gives (2.2). □

2.5. A lower bound on the number of C_8s

For a graph $G \subset Q_n$, we define a graph $H_x = H_x(G)$ for each vertex $x \in Q_n$ as it was used by Chung in [7]. The vertex set of H_x consists of the n neighbours of x in Q_n. Consider two vertices y and z in H_x; there is a unique 4-cycle C containing x, y and z in Q_n, say $C = yxzw$, $w = w(y, z)$. (As vectors, $w = y + z - x$.) If wz and $wy \in E(G)$ then we put an edge yz in H_x. Every ywz path in G generates an edge in H_x, so we have

$$\sum_{x \in V(Q_n)} e(H_x) = \sum_{w \in V(Q_n)} \binom{\deg_G(w)}{2}.$$

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This implies

$$\bar{h} \geqslant \binom{\bar{d}}{2}, \tag{2.5}$$

where $\bar{h} := \sum_x e(H_x)/2^n$, and $\bar{d} := 2e(G)/2^n$.

A cycle C_ℓ, $V(C_\ell) = \{y_1, y_2, \dots, y_\ell\}$, $\ell \geqslant 3$, in H_x corresponds to a cycle y_1, $w(y_1, y_2)$, y_2, $w(y_2, y_3), \dots, w(y_\ell, y_1)$ of length 2ℓ in G. We have

$$N(G, C_8) \geqslant \sum_{x \in V(Q_n)} N(H_x, C_4).$$

By applying Lemma 2.5 and convexity, we get

$$N(G, C_8) \geqslant \sum_{x \in V(Q_n)} \left(2\frac{e(H_x)^4}{n^4} - \frac{3}{4} e(H_x) n \right) \geqslant 2^{n+1} \frac{1}{n^4} \bar{h}^4 - O(n\bar{h}2^n). \tag{2.6}$$

The inequality (2.5) and monotonicity in (2.6) give

$$N(G, C_8) \geqslant 2^{n+1} \frac{1}{n^4} \binom{\bar{d}}{2}^4 - O(n\bar{d}^2 2^n). \tag{2.7}$$

2.6. The end of the proof of Theorem 1.1

Let G be a C_{14}-free subgraph of Q_n of girth at least 8 and let $\bar{d}$ be its average degree. Compare (2.7) to the upper bound from Lemma 2.4, $O(n^2\bar{d}2^n) \geqslant N(G, C_8)$. Therefore, $\bar{d}(G) = O(n^{6/7})$ and $e(G) = o(e(Q_n))$. By Lemma 2.1, we get three times this upper bound for $\bar{d}(G)$ for an arbitrary C_{14}-free subgraph of Q_n, completing the proof of the theorem. □

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