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Covering the n-space by convex bodies and its chromatic number

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Dedicated to Miklós Simonovits on his 60th birthday

Abstract

Rogers [A note on coverings, Matematika 4 (1957) 1–6] proved, for a given closed convex body C in n-dimensional Euclidean space $\mathbb{R}^n$, the existence of a covering for $\mathbb{R}^n$ by translates of C with density $cn \ln n$ for an absolute constant c. A few years later, Erdős and Rogers [Covering space with convex bodies, Acta Arith. 7 (1962) 281–285] obtained the existence of such a covering having not only low-density $cn \ln n$ but also low multiplicity $c'n \ln n$ for an absolute constant c'. In this paper, we give a simple proof of Erdős and Rogers' theorem using the Lovász Local Lemma. Furthermore, we apply the result to the chromatic number of the unit-distance graph under ℓ_p-norm.

Keywords: Convex body; Rogers; Lovász Local Lemma; Covering; Chromatic number of the unit-distance graph

1. Introduction

For a bounded domain $D \subset \mathbb{R}^n$ and for a collection $\mathscr{C} := \{C_1, C_2, \ldots\}$ of convex bodies C_i which covers D, i.e., $\bigcup_i C_i \supset D$, the *density* of the collection $\mathscr{C}$ with respect to D is defined as

$$d(\mathscr{C}, D) = \frac{\sum_i \mathrm{Vol}(C_i)}{\mathrm{Vol}(D)},$$

where $\mathrm{Vol}(\cdot)$ is the Euclidean volume of a body and the sum is taken over all i for which $C_i \cap D \neq \emptyset$. For the whole space, we define

$$\overline{d}(\mathscr{C}, \mathbb{R}^n) = \limsup_{r \to \infty} d(\mathscr{C}, B(r, o)),$$
$$\underline{d}(\mathscr{C}, \mathbb{R}^n) = \liminf_{r \to \infty} d(\mathscr{C}, B(r, o)),$$

where $B(r, x)$ is a *ball* with radius r in $\mathbb{R}^n$ with center x, and o is the origin in $\mathbb{R}^n$. If these two numbers are the same, then their common value is called the *density* of the collection $\mathscr{C}$ in $\mathbb{R}^n$, and is denoted by $d(\mathscr{C}, \mathbb{R}^n)$. As usual, *body* means a bounded set with positive volume.

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In 1957, Rogers [14] proved that, for a given closed convex body C in n-dimensional Euclidean space $\mathbb{R}^n$ and for $n \geqslant 3$, there is a covering for $\mathbb{R}^n$ by translates of C with density at most $cn \ln n$ for an absolute constant c. However, low density does not imply low *multiplicity*, the number of copies of $C \in \mathscr{C}$ containing each point, of the covering. Even though the global density of the covering is low, there can exist local clusters of high multiplicity. Even a partition of the space like the collection of unit cubes of $\mathbb{R}^n$ has the optimal density of 1 but the multiplicity can go up to 2^n at the vertices of the cubes. In 1962, Erdős and Rogers [4] showed that, for sufficiently large n, there is a covering for $\mathbb{R}^n$ by translates of C having not only density at most $cn \ln n$ but also multiplicity at most $c'n \ln n$ for an absolute constant c'. Their proof is clever but technical. In this paper, we give a combinatorial proof using Lovász Local Lemma.

Theorem 1. *For a given convex body C in the n-dimensional Euclidean space $\mathbb{R}^n$, there is a covering for $\mathbb{R}^n$ by translates of C such that each point $x \in \mathbb{R}^n$ is covered at most $10n \ln n$ times for sufficiently large n.*

Along with our main result, we have included in this article an upper bound on the chromatic number of the unit-distance graph under ℓ_p-norm as an application of Theorem 1.

2. Tools of proof

2.1. Large inscribed ball/ellipse

It was proved by Ball [3] (and see [2] for the symmetric case) that every convex body $C \subset \mathbb{R}^n$ has an affine image $\widehat{C} \subset \mathbb{R}^n$ satisfying the following two conditions (A1) and (A2):

(A1) $\mathrm{Vol}(\widehat{C}) = 1$,
(A2) $\widehat{C}$ has an inscribed ball B of radius r at least as large as the inscribed radius of the regular simplex of volume 1. Thus,

$$r \geqslant \left(\frac{n!}{n^{n/2}(n+1)^{(n+1/2)}}\right)^{1/n} > \frac{1}{e}.$$

Let us remark that instead of the deep theorem of Ball, one can start with the classical result of John [9] that there exists a ball B such that $B \subset \widehat{C} \subset nB$. Since $\mathrm{Vol}(nB) \geqslant 1$, this implies a lower bound $r > 1/\mathrm{O}(\sqrt{n})$, which would be sufficient for our arguments below.

2.2. Minkowski sum

As usual $C + D$ means the sum of the bodies C and D, $C + D := \{x + y : x \in C, y \in D\}$, and hC means $\{hx : x \in C\}$. The ε-neighborhood of C, $C^{+\varepsilon}$, is $C + B(\varepsilon, o)$. Here $\varepsilon \geqslant 0$. We define the *inner* ε-core, $C^{-\varepsilon}$, as $R^n \backslash (R^n \backslash C)^{+\varepsilon}$. We have

$$C^{+\varepsilon} := \cup\{B(\varepsilon, x) : x \in C\} \quad \text{and} \quad C^{-\varepsilon} := \{x : B(\varepsilon, x) \subset C\}.$$

Lemma 1. *Suppose that the convex body C contains the ball $B(r, o)$. Then the expansion $(1 + \varepsilon/r)C$ contains the epsilon neighborhood $C^{+\varepsilon}$. On the other hand, the contraction $(1 - \varepsilon/r)C$ is contained in $C^{-\varepsilon}$. See Fig. 1.*

Proof. We use the fact that $(a + b)C = aC + bC$ for any convex set and non-negative reals a and b. Then,

$$\left(1 + \frac{\varepsilon}{r}\right) C = C + \frac{\varepsilon}{r} C \supseteq C + \frac{\varepsilon}{r} B(r, o) = C^{+\varepsilon}.$$

Similarly,

$$\left(1 - \frac{\varepsilon}{r}\right) C + B(\varepsilon, o) \subseteq \left(1 - \frac{\varepsilon}{r}\right) C + \frac{\varepsilon}{r} C = C,$$

hence $(1 - \varepsilon/r)C \subseteq C^{-\varepsilon}$. $\square$

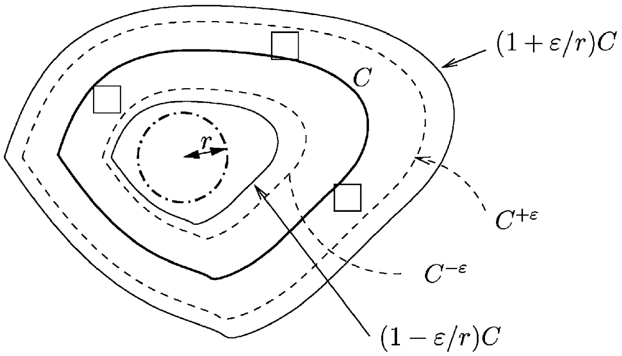

Fig. 1. $(1-\varepsilon/r)C \subset C^{-\varepsilon}$, $(1+\varepsilon/r)C \subset C^{+\varepsilon}$.

2.3. The Lovász Local Lemma

We follow the description from the monograph of Alon and Spencer [1].

Lemma 2. *Let $A_1, A_2, \ldots, A_N$ be events in an arbitrary probability space. A directed graph $D=(V,E)$ on the set of vertices $V=\{1,2,\ldots,N\}$ is called a dependency digraph for the events $A_1,\ldots,A_N$ if for each i, $1\leqslant i\leqslant N$, the event A_i is mutually independent of all the events $\{A_j : (i,j)\notin E\}$. Suppose that the maximum degree of D is at most d, and that $\mathrm{Prob}(A_i)\leqslant p$ for all $1\leqslant i\leqslant N$. If $ep(d+1)\leqslant 1$, then $\mathrm{Prob}(\bigcap_{i=1}^N \overline{A_i})>0$, i.e., with positive probability no event A_i holds.*

2.4. The volume of the difference body

For any convex set containing the center o the *difference* body $C-C$ is a centrally symmetric convex set containing it.

Lemma 3 (*Rogers and Shephard [15]*). *Let $C\subset\mathbb{R}^n$ be a closed convex body. Then $\mathrm{Vol}(C-C)\leqslant \binom{2n}{n}\mathrm{Vol}(C)$, with equality, if and only if C is a simplex.*

3. Proof

As a covering and an affine image of it have the same multiplicities, we can construct an appropriate covering by using any affine image of C. So, we may assume that C itself possesses the properties (A1) and (A2). For simplicity, we also assume that the ball B of (A2) is $B(r,o)$.

Let h be a small positive real number, we will use $h := 1/(4en\sqrt{n})$. Consider the lattice $h\mathbb{Z}^n := \{(hm_1,\ldots,hm_n) : m_1,\ldots,m_n$ are integers$\}$. We are going to construct a cover using only translates of C of the form $C+z$, $z\in h\mathbb{Z}^n$. Define Q_0 as the half closed, half open basis cube of this lattice:

$$Q_0 := \{(x_1,\ldots,x_n) : 0\leqslant x_i < h \text{ for all } i\}.$$

Then the translations of the form Q_0+z with $z\in h\mathbb{Z}^n$ define a partition Λ of $\mathbb{R}^n$. For $Q\in\Lambda$ with $Q=Q+z$, denote the translate $C+z$ by $C(Q)$.

We define a hypergraph $\mathscr{H}$ whose vertex set consists of all the cubes of Λ and whose edge set has two kinds of hyperedges induced by each $C(Q)$ as follows: $Q_1, Q_2, Q_3, \ldots \in \Lambda$ form a "small edge" of $C(Q)$, denoted by $e(C(Q))$ or $e(Q)$, if $Q_1, Q_2, Q_3, \ldots$ lie in $C(Q)$; $Q_1, Q_2, Q_3, \ldots \in \Lambda$ form a "big edge" of $C(Q)$, denoted by $E(C(Q))$ or $E(Q)$, if $Q_1, Q_2, Q_3, \ldots$ intersect $C(Q)$. Clearly, all the "small edges" have the same size, and so do all the "big edges"; their sizes are denoted by k and K, respectively.

Since $\mathrm{Vol}(C)/\mathrm{Vol}(Q)=1/h^n$, we have $K \geqslant 1/h^n \geqslant k$. The diameter of Q is $h\sqrt{n}$, so $C^{-h\sqrt{n}} \subset e(Q_0)$, and $E(Q_0) \subset C^{+h\sqrt{n}}$. Apply Lemma 1 to C with $\varepsilon := h\sqrt{n}$:

$$k \geqslant \frac{\mathrm{Vol}(C^{-h\sqrt{n}})}{\mathrm{Vol}(Q)} \geqslant \frac{\mathrm{Vol}((1-r^{-1}h\sqrt{n})C)}{\mathrm{Vol}(Q)} > \frac{(1-eh\sqrt{n})^n}{h^n} = \left(1-\frac{1}{4n}\right)^n \frac{1}{h^n} > .75\frac{1}{h^n},$$

$$K \leqslant \frac{\mathrm{Vol}(C^{+h\sqrt{n}})}{\mathrm{Vol}(Q)} \leqslant \frac{\mathrm{Vol}((1+r^{-1}h\sqrt{n})C)}{\mathrm{Vol}(Q)} < \frac{(1+eh\sqrt{n})^n}{h^n} = \left(1+\frac{1}{4n}\right)^n \frac{1}{h^n} < e^{1/4}\frac{1}{h^n}. \tag{1}$$

In particular, we have

$$k > \tfrac{1}{2}K. \tag{2}$$

Let ℓ be a positive integer and let $N := (2\ell)^n$ and consider the set $\Lambda_N := \{Q : Q = Q + hz \text{ with } z \in \mathbb{Z}^n, -\ell \leqslant z_i < \ell$ for all coordinates of $z\}$. Let $\mathscr{H}_N$ be the set of hyperedges of $\mathscr{H}$ containing any member of Λ_N, and let $\mathscr{C}_N$ be the translates of C (of the forms $C + hz$, $z \in \mathbb{Z}^n$) generating $\mathscr{H}_N$. Note that $|\mathscr{C}_N|$ (in general) is larger than N, but, obviously, it is finite. Any subcollection $\mathscr{C} \subset \mathscr{C}_N$ generates a subhypergraph of $\mathscr{H}_N$, denoted by $\mathscr{H}_{\mathscr{C}}$, in a natural way, namely the small and big edges of $\mathscr{H}_N$ generated by the members of $\mathscr{C}$.

To prove Theorem 1, we show that, for every N, there is a collection $\mathscr{C} \subset \mathscr{C}_N$ and hence a hypergraph $\mathscr{H}_{\mathscr{C}}$ such that each cube $Q \in \Lambda_N$ is covered by a "small edge" of $\mathscr{H}_{\mathscr{C}}$ but *not* covered by too many "big edges" of $\mathscr{H}_{\mathscr{C}}$, say not covered more than t times where $t = 10n \ln n$. Having such a cover of Λ_N for every $N = (2\ell)^n$, one can easily construct an appropriate infinite cover of $\mathbb{R}^n$ by letting $\ell \to \infty$ and using a standard compactness argument.

To construct such a cover of Λ_N, we consider a random subcollection $\mathscr{C}$ of $\mathscr{C}_N$ choosing its members randomly, independently with probability p. The value of p we use is $e^{-6/5}t/K$. To apply Lovász Local Lemma, for each cube $Q \in \Lambda_N$, let A_Q be the (first kind of bad) event that Q is not covered by any "small edge" of $\mathscr{H}_{\mathscr{C}}$, and let B_Q be the (second kind of bad) event that Q is covered by "big edges" more than t times. Since every $Q \in \Lambda_N$ is covered by exactly k small edges and K big edges of $\mathscr{H}_N$, it is immediate that

$$\mathrm{Prob}(A_Q) \leqslant (1-p)^k \leqslant e^{-pk}$$

and

$$\mathrm{Prob}(B_Q) \leqslant \binom{K}{T} p^T \leqslant \left(\frac{eKp}{t}\right)^t,$$

where $T := \lfloor t \rfloor + 1$. Furthermore, let d be the maximum degree in the dependency graph of the bad events. If we have

$$e\left(e^{-pk} + \left(\frac{eKp}{t}\right)^t\right)(d+1) < 1, \tag{3}$$

then, by the Local Lemma, there is a covering for Λ_N by members of of $\mathscr{C}_N$ having multiplicity less than t.

To bound d, for a given $Q \in \Lambda_N$, observe that the event $A_Q \cup B_Q$ is dependent on the other event $A_{Q'} \cup B_{Q'}$ only if there is a translate $C'' \in \mathscr{C}_N$ meeting both cubes Q and Q'. That is, there are $x, x' \in Q_0$ and $z, z', z'' \in h\mathbb{Z}^n$ such that $Q = Q_0 + z$, $Q' = Q_0 + z'$, $C'' = C + z''$, $z + x \in C + z''$ and $z' + x' \in C + z''$. Thus $(z + x - z'') - (z' + x' - z'') \in C - C$, $(z - z') \in (C - C) + (x' - x)$. Since $|x' - x|$ is at most $h\sqrt{n}$, the degree $d + 1$ is bounded by the number of lattice points $z - z'$ contained in $(C - C)^{+h\sqrt{n}}$. If we put a translation of Q_0 with these $z \in h\mathbb{Z}^n \cap (C - C)^{+h\sqrt{n}}$, then these cubes have disjoint interiors and are contained in the $2h\sqrt{n}$ neighborhood of $C - C$. See Fig. 2. (Actually, one can consider cubes with these *centers* and get a slightly better bound, but we do not need that.) Thus we get an upper bound

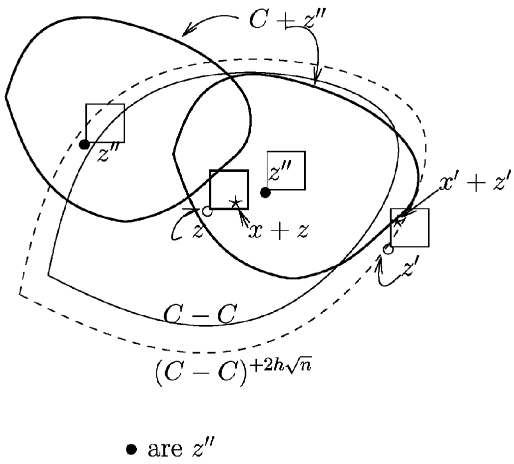

Fig. 2. Translates of C intersecting with $Q+z$.

for $d+1$ as the ratio of volumes. The difference body $C-C$ contains the ball $B(2r,o)$, so Lemma 1 gives that

$$d+1 \leqslant \frac{\mathrm{Vol}((C-C)^{+2h\sqrt{n}})}{\mathrm{Vol}(Q_0)} \leqslant (1+eh\sqrt{n})^n \,\mathrm{Vol}(C-C)\frac{1}{h^n}.$$

Lemma 3 gives that $\mathrm{Vol}(C-C) \leqslant \binom{2n}{n}$ which is at most $4^n/2$ (for every $n \geqslant 1$). We obtain

$$d+1 < \left(1+\frac{1}{4n}\right)^n \binom{2n}{n} \frac{1}{h^n} < e^{1/4}\frac{1}{2}4^n \frac{1}{h}^n < \left(\frac{4}{h}\right)^n. \tag{4}$$

Substituting the appropriate choices of h, t and p (i.e., $1/h = 4en^{3/2}$, $t = 10n\ln n$, $p = e^{-1.2}t/K$) and using (2), (1) and (4) one can obtain that the left-hand side of (3) is at most

$$\leqslant e\left(e^{-pK/2} + \left(\frac{eKp}{t}\right)^t\right)\left(\frac{4}{h}\right)^n$$
$$= e\left(\exp\left[-\frac{5}{e^{1.2}}n\ln n\right] + \exp[-2n\ln n]\right)\left((16e)^n \exp\left[\frac{3}{2}n\ln n\right]\right).$$

This is less than 1 for sufficiently large n. □

Remark. In the proof, we can also perform the computation with $t = (c + \mathrm{o}(1))n\ln n$, where c is the only root of the equation $(3/2)^{(c+1)/c} = c/e$ (with better choices of h and p) which will give a better bound for the multiplicity.

4. Unit-distance graph

A long-standing open problem in combinatorial geometry is the chromatic number of the unit-distance graph in $\mathbb{R}^n$; here points are adjacent if their distance in the ℓ_2-norm is 1. For $n=2$, we know the answer is between 4 and 7. Little is known about other dimensions.

More generally, for given integers n, p with $n \geqslant 2$ and $1 \leqslant p \leqslant \infty$, we can consider graphs on n-dimensional real space under the ℓ_p-norm. Specifically, we can define the graph $G(\mathbb{R}^n_p)$ with vertex set V and edge set E by

$$V = \mathbb{R}^n,$$

$$\vec{x}\vec{y} \in E \text{ if and only if } \|\vec{x}-\vec{y}\|_p = 1,$$

and consider $\chi(G(\mathbb{R}^n_p))$.

The present authors [10,6] examined the chromatic number of G, and proved a lower bound of $(1.067)^n$ and two upper bounds $\sqrt{p/(2\pi n)}(5(ep)^{1/p})^n$ and 9^n. We apply Theorem 1 above to obtain an improved upper bound in a more general form.

Theorem 2. *Let $\mathcal{N} = (\mathbb{R}^n, \|\cdot\|)$ be a normed vector space, $n \geqslant 2$. Let $G(\mathcal{N})$ denote the unit-distance graph in this normed space. Then for the chromatic number we have $\chi(G(\mathcal{N})) \leqslant c(n \ln n)5^n$ for large n.*

Proof. Let $\mathscr{C}$ be a covering for $\mathbb{R}^n$ by translates of $C := B_{\mathcal{N}}(\frac{1}{2} - \varepsilon)$ with multiplicity $c(n \ln n)$ where ε is a very small positive real number, and $B(r)$ is the ball with radius r centered at o in $\mathbb{R}^n$ with norm $\mathcal{N}$.

Define an auxiliary graph H such that

$$V(H) = \mathscr{C} \text{ and for } C + \vec{a},\ C + \vec{b} \in \mathscr{C},$$
$$(C + \vec{a}, C + \vec{b}) \in E(H) \text{ if and only if there are } \vec{x} \in C + \vec{a},\ \vec{y} \in C + \vec{b} \text{ such that } \|\vec{x} - \vec{y}\|_{\mathcal{N}} = 1. \tag{5}$$

It is easy to see that a proper coloring of H gives a proper coloring of $G(\mathcal{N})$; hence $\chi(G(\mathcal{N})) \leqslant \chi(H)$. We will bound $\chi(H)$ from above by its maximum degree.

Observe that $(C + \vec{a}, C + \vec{b}) \in E(H)$ implies that $\|\vec{a} - \vec{b}\|_{\mathcal{N}} \leqslant \|\vec{a} - \vec{x}\|_{\mathcal{N}} + \|\vec{x} - \vec{y}\|_{\mathcal{N}} + \|\vec{b} - \vec{y}\|_{\mathcal{N}} < 2$. So it is enough to count the number, say m, of the copies of $C \in \mathscr{C}$ with $B(\frac{5}{2} - \varepsilon) \cap C \neq \emptyset$. By Theorem 1, it is immediate that

$$\begin{aligned} m &\leqslant c(n \ln n) \frac{\mathrm{Vol}(B(5/2 - \varepsilon))}{\mathrm{Vol}(B(1/2 - \varepsilon))} \\ &\leqslant c(n \ln n)5^n. \qquad \square \end{aligned}$$

For more results on different kinds of proximity graphs of higher dimensions see Füredi and Loeb [7,5] or Guibas et al. [8] which are good sources of additional references.

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References

[1] N. Alon, J.H. Spencer, The Probabilistic Method, second ed., Wiley-Interscience, New York, 2000.
[2] K. Ball, Volumes of sections of cubes and related problems, in: Geometric Aspects of Functional Analysis, 1987–1988, pp. 251–260, Lecture Notes in Mathematics, vol. 1376, Springer, Berlin, 1989.
[3] K. Ball, Volume ratios and a reverse isoperimetric inequality, J. London Math. Soc. (2) 44 (1991) 351–359.
[4] P. Erdős, C. Rogers, Covering space with convex bodies, Acta Arith. 7 (1962) 281–285.
[5] Z. Füredi, The expected size of a random sphere-of-influence graph, in: I. Bárány et al. (Ed.), Intuitive Geometry Proceedings Colloquium, Budapest, Hungary, 1995, Bolyai Society Mathematical Studies, vol. 6, Budapest, Hungary, 1997, pp. 319–326.
[6] Z. Füredi, J.-H. Kang, Distance graph on $\mathbf{Z}^n$ with ℓ_1-norm, Theoret. Comput. Sci. (special issue on combinatorics of the discrete plane and tilings) (2004), to appear.
[7] Z. Füredi, P.A. Loeb, On the best constant for the Besicovitch covering theorem, Proc. Amer. Math. Soc. 121 (1994) 1063–1073.
[8] L. Guibas, J. Pach, M. Sharir, Sphere-of-influence graphs in higher dimensions, in: K. Böröczky, G. Fejes Tóth (Eds.), Intuitive Geometry, Proc. Colloquia Mathematica Societatis Janos Bolyai, vol. 63, North-Holland, Amsterdam, 1994, pp. 131–137.
[9] F. John, Extremum problems with inequalities as subsidiary conditions, in: Courant Anniversary Volume, New York, 1948, pp. 187–204.
[10] J.-H. Kang, Coloring of metric spaces and $L(2,1)$-labeling of graphs, Ph.D. Thesis, University of Illinois at Urbana-Champaign, 2004.
[14] C. Rogers, A note on coverings, Matematika 4 (1957) 1–6.
[15] C. Rogers, G. Shephard, The difference body of convex body, Arch. Math. 8 (1957) 220–233.