A Proof and Generalizations of the Erdős–Ko–Rado Theorem Using the Method of Linearly Independent Polynomials

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Summary. Our aim is to exhibit a short algebraic proof for the Erdős–Ko–Rado theorem. First, we summarize the method of linearly independent polynomials showing that if $X_1, \ldots, X_m \subset [n]$ are sets such that X_i does not satisfy any of the set of s intersection conditions R_i but X_i satisfies at least one condition in R_j for all $j > i$ then $m \leq \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{0}$. The EKR theorem is follows by carefully choosing the intersection properties and adding extra polynomials. We also prove generalizations for non-uniform families with various intersection conditions.

1 Proofs of the EKR Theorem

In 1961, Erdős, Ko, and Rado [EKR61] proved that if $\mathcal{F}$ is a k-uniform family of subsets of a set of n elements with $k \leq \frac{1}{2}n$ and with every pair of members of $\mathcal{F}$ intersect, then $\mid \mathcal{F} \mid \leq \binom{n-1}{k-1}$. They also showed that for $k < \frac{1}{2}n$, equality holds if $\mathcal{F}$ consists of all k-sets containing a given element of the underlying set.

In addition to their remarkable proof (induction on k and, for a given k, left-shifting and induction on n), there are many interesting new proofs. For example, in 1972, Katona [Kat72] used a simple and elegant argument, the permutation method. Daykin [Day74] obtained Erdős–Ko–Rado from the Kruskal–Katona theorem. Hajnal and Rothschild [HR73] proved it for $n > n_0(k)$ by an early version of the kernel (or delta-system) method, developed and used very successfully by Frankl [Fra77] (the first full description of the method was published in Deza, Erdős, and Frankl [DEF78]). The most remarkable technique was due to Lovász, in his ground-breaking paper [Lov79], which used a geometric representation to prove that the Shannon capacity of the Kneser graph $K(n,k)$ is at most $\binom{n-1}{k-1}$ for all $k \leq \frac{1}{2}n$, thus yielding another proof (and generalization). Wilson [Wil84] gave an ingenious proof, using Delsarte's linear programming bound. (Actually, he proved much more concerning t-intersecting families.) Finding different ways to prove EKR has been the subject of a set of papers recently by Ehud Friedgut from (1) Graph Homomorphisms [DF06+] (a joint work with I. Dinur) and (2) Harmonic Analysis [Fri06+].

One of the most powerful methods for counting the number of objects in a certain combinatorial structure is to correspond polynomials to the objects and to show that these polynomials are, in fact, linearly independent in some space. See, for example, Delsarte, Goethals and Seidel [DGS77] as well as Larman, Rogers and Seidel [LRS77] for deep early results. This method has been used to prove intersection theorems by Blokhuis [Blo90], and then by Alon, Babai and Suzuki [ABS91], and most recently by Ramanan [Ram97], Snevily [Sne03] and others. See the monograph of Babai and Frankl [BF92] for more details. Interestingly, none of the new algebraic proofs can be directly applied as a new proof for the original Erdős–Ko–Rado.

The aim of this paper is to exhibit a short algebraic proof for the Erdős–Ko–Rado theorem, and then to give a number of Frankl–Wilson–Ray–Chaudhuri type generalizations. Before that, in section 2, we summarize the essence of the polynomial method in a powerful lemma, in a form that best fits our purposes. In section 3 the new proof for EKR is given, in section 4 we summarize old generalizations for non-uniform hypergraphs, and finally in section 5 some new generalizations are presented.

2 The Polynomial Method for Intersection Theorems

Suppose that $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$ is a family of finite sets where each A_i is a subset of $[n] := \{1, 2, \ldots, n\}$. We say that the set X satisfies the intersection property (P, α) (where P is a set and α is a non-negative integer) if $|X \cap P| = \alpha$. Suppose that for each $A_i \in \mathcal{A}$ a set of (at most) s intersection properties are given

$$R_i := \{(P_{i1}, \alpha_{i1}), \ldots, (P_{is}, \alpha_{is})\}.$$

(An intersection condition can be repeated, even for the same i).

Lemma 2.1. *Suppose that for each $A_i \in \mathcal{A}$ one can find a set $X_i \subset [n]$ such that*

1. *X_i does not satisfy any of the conditions in R_i, and*
2. *X_i satisfies at least one condition in R_j for all $j > i$.*

Then

$$m \leq \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{0}. \tag{1}$$

Proof. Define the (n-variable, real) polynomial $f_i(x)$ as follows.

$$f_i(x_1, x_2, \ldots, x_n) := \prod_{1 \leq u \leq s} \left(\left(\sum_{v \in P_{iu}} x_v \right) - \alpha_{iu} \right)$$

If we use the notation $\widehat{X}$ for the characteristic vector of $X \subset [n]$ (i.e., $\widehat{X}$ is a 0-1 vector from R^n with its tth coordinate is 1 if and only if $t \in X$), then the scalar product $\widehat{X}\widehat{Y}$ is equal to $|X \cap Y|$ for all $X, Y \subset [n]$. With this notation one can rewrite f_i as

$$f_i\left(\widehat{X}\right) = \prod_u \left(\widehat{X}\widehat{P_{iu}} - \alpha_{iu} \right) = \prod_u \left(|X \cap P_{iu}| - \alpha_{iu} \right).$$

In any case, it is obvious that our conditions imply that

$$f_j\left(\widehat{X_i}\right) \begin{cases} = 0 \text{ if } i < j \\ \neq 0 \text{ if } i = j. \end{cases} \tag{2}$$

Now define the (integer coefficient, real, n-variable) multilinear polynomial g_i by eliminating all the parentheses from f_i and repeatedly replacing a higher order factor x_v^2 by x_v (for all $1 \leq v \leq n$). Note that for a 0-1 vector x one has $f_i(x) = g_i(x)$, so (2) implies that

$$g_i\left(\widehat{X_i}\right) \neq 0 \quad \text{but} \quad g_j\left(\widehat{X_i}\right) = 0 \text{ for } i < j. \tag{3}$$

The multilinear, n variable, real polynomials of degree at most s form a vectorspace V (over R), of dimension

$$\dim V = \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{0}. \tag{4}$$

We claim that the polynomials $g_1, g_2, \ldots, g_m$ are linearly independent in this space. Then (4) gives the desired upper bound on m. Indeed, suppose on the contrary, that there exists a linear dependency, i.e.,

$$c_1 g_1(x) + c_2 g_2(x) + \cdots + c_m g_m(x) = 0 \tag{5}$$

holds for every $x \in R^n$, where the c_i's are reals, not all 0. Suppose that i is the smallest integer with $c_i \neq 0$, and substitute $\widehat{X_i}$ into (5). Since $c_j = 0$ for $j < i$ and $g_j\left(\widehat{X_i}\right) = 0$ for $j > i$ the equation becomes $c_i g_i\left(\widehat{X_i}\right) = 0$. However, this contradicts the fact that both c_i and $g_i\left(\widehat{X_i}\right)$ are non-zero.   □

3 The Polynomial Proof of EKR Theorem

Let $\mathcal{F}$ be an intersecting family of k-sets of $[n]$, $n \geq 2k$, $|\mathcal{F}| = m$. In the above Lemma 2.1 we will define $s = k - 1$. The dimension of the vector space of multilinear polynomials V is not a perfect binomial coefficient, so at first Lemma 2.1 does not seem to be applicable to prove $m \leq \binom{n-1}{k-1}$. Instead of narrowing the vectorspace, which does not look viable, one can add more polynomials to the g_i's defined by the members of $\mathcal{F}$, and show that the larger system is still linearly independent. This method appeared first in a paper of Blokhuis [Blo84]. In fact, we will join to $\mathcal{F}$ another

$$\binom{n}{k-1} + \binom{n}{k-2} + \cdots + \binom{n}{1} + \binom{n}{0} - \binom{n-1}{k-1} = 2 \times \left(\sum_{k-2 \geq u \geq 0} \binom{n-1}{u} \right) \quad (6)$$

sets, together with appropriate intersection conditions.

Select a member p of $[n]$ arbitrarily. Define the family $\mathcal{A}$ as the union of the following four hypergraphs, $\mathcal{F}_0$, $\mathcal{H}$, $\mathcal{F}_1$ and $\mathcal{G}$, where

- $\mathcal{F}_0 := \{F \in \mathcal{F} : p \notin F\}$,
- $\mathcal{H} := \{H \subset [n] : p \notin H, 0 \leq |H| \leq k - 2\}$,
- $\mathcal{F}_1 := \{F \in \mathcal{F} : p \in F\}$, and
- $\mathcal{G} := \{G \subset [n] : p \in G, 1 \leq |G| \leq k - 1\}$.

Order these sets linearly in the above order. First, we put the members of $\mathcal{F}_0$ (in arbitrary order), then the members of $\mathcal{H}$ increasing by size, (i.e., $H \in \mathcal{H}$ precedes $H' \in \mathcal{H}$ if $|H| \leq |H'|$) then $\mathcal{F}_1$ (again in arbitrary order) and, finally, the members of $\mathcal{G}$ again in increasing order. Next, for each set $A_i \in \mathcal{A}$ we associate another set $X_i \subset [n]$, and at most $k - 1$ intersection conditions (P_{iu}, α_{iu}).

- For $F \in \mathcal{F}_0$ we let $X := [n] \setminus \{p\} \setminus F$ with intersection conditions (F, α), $1 \leq \alpha \leq k - 1$.
- For $H \in \mathcal{H}$ we let $X := H$ with intersection conditions $(\{h\}, 0)$, (for each $h \in H$) and $([n], n - k - 1)$.
- For $F \in \mathcal{F}_1$ we let $X := F \setminus \{p\}$ with intersection conditions $(F \setminus \{p\}, \alpha)$, $0 \leq \alpha \leq k - 2$.
- For $G \in \mathcal{G}$ we let $X := G$ with intersection conditions $(\{g\}, 0)$ for each $g \in G$.

It is straightforward to check that the $\{A_i, X_i, (P_{iu}, \alpha_{iu})\}$ system defined above indeed satisfies the constrains of Lemma 2.1 with $s = k - 1$. This gives $|\mathcal{F}| + |\mathcal{G}| + |\mathcal{H}| \le \binom{n}{k-1} + \binom{n}{k-2} + \cdots + \binom{n}{0}$. since $|\mathcal{G}| = |\mathcal{H}| = \sum_{k-2 \ge u \ge 0} \binom{n-1}{u}$ (6) gives the desired upper bound for $|\mathcal{F}|$. $\square$

Summary

To make the construction more explicit and transparent we repeat the the definitions of the associated functions and their characteristic properties.

For $F \in \mathcal{F}_0$ we have $f_F(x) = \prod_{1 \le u \le k-1} \left(\left(\sum_{e \in F} x_e\right) - u\right)$ and $g_F\left(\widehat{X}\right) = 0$ if and only if $|X \cap F| \in \{1, \ldots, k-1\}$,

for $H \in \mathcal{H}$ we have $f_H(x) = \left(\left(\sum_{1 \le e \le n} x_e\right) - (n - k - 1)\right) \prod_{e \in H} x_e$ and $g_H\left(\widehat{X}\right) = 0$ if and only if $|X| = n - k - 1$ or $H \not\subseteq X$,

for $F \in \mathcal{F}_1$ we have $f_F(x) = \prod_{0 \le u \le k-2} \left(\left(\sum_{e \in F, e \ne p} x_e\right) - u\right)$ and $g_F\left(\widehat{X}\right) = 0$ if and only if $|X \cap (F \setminus \{p\})| \in \{0, 1, \ldots, k-2\}$,

for $G \in \mathcal{G}$ we have $f_G(x) = \prod_{e \in G} x_e$ and $g_G\left(\widehat{X}\right) = 0$ if and only if $G \not\subseteq X$.

Let us point out the part where the condition $n \ge 2k$ was used. We needed $g_H\left(\widehat{H}\right) \ne 0$ and this is only true if $|H| \ne n - k - 1$. Furthermore, as in most cases when one uses linear algebra, it does not seem immediate from the above proof that for $n > 2k$ equality can hold only for $\bigcap \mathcal{F} \ne \emptyset$.

4 Generalizations for Non-uniform Families

Here we briefly discuss some old generalizations. In this section $\mathcal{F} = \{F_1, F_2, \ldots F_m\}$ is a family of subsets of $[n]$, $K = \{k_1, \ldots, k_t\}$ and $L = \{l_1, \ldots, l_s\}$ are sets of non-negative integers. We call $\mathcal{F}$ an (n, K, L)-family if $|F| \in K$ for every $F \in \mathcal{F}$ and $|F_i \cap F_j| \in L$ for any distinct $F_i, F_j \in \mathcal{F}$.

A celebrated result of Frankl and Wilson [FW81] claims that

$$|\mathcal{F}| \le \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{0}, \tag{7}$$

independent of K. The determination of $\max |\mathcal{F}|$ is very much related to a basic coding problem when a binary code is given with given weights and distances. There are many improvements of (7).

For $L = \{1, 2, \ldots, s\}$ Frankl and Füredi [FF81] conjectured that

$$m \le \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{0}. \tag{8}$$

This can be achieved by the family $\{F : 1 \in F \subset [n], |F| \leq s+1\}$. They proved (8) for $n > 100s^2/\log(s+1)$, Pyber showed it for $n \geq 6(s+1)$ and finally Ramanan [Ram97] proved it for all n. Recently, Snevily [Sne03] showed that (8) holds for any L with $\min L \geq 1$.

Alon, Babai, and Suzuki [ABS91] proved that

$$m \leq \binom{n}{s} + \binom{n}{s-1} + \binom{n}{s-t+1} \tag{9}$$

holds for the case $\min K > s - t$.

Snevily [Sne95] proved that in the case $\min K > \max L$

$$|\mathcal{F}| \leq \binom{n-1}{s} + \binom{n-1}{s-1} + \cdots + \binom{n-1}{s-2t+1}.$$

Frankl [Fra85] showed that if $\mathcal{F}$ is any family with $|F_j \setminus F_i| \in L$ for all $1 \leq i < j \leq m$ then

$$|\mathcal{F}| \leq \sum_{0 \leq i \leq s} \binom{n}{i}. \tag{10}$$

5 New Generalizations

Here we prove other generalizations of the Erdős–Ko–Rado Theorem allowing different sizes of the subsets and introducing a new parameter r.

Again $\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$ is a family of subsets of $[n]$, $K = \{k_1, \ldots, k_t\}$ and $L = \{l_1, \ldots, l_s\}$ are sets of non-negative integers. We suppose that $|F| \in K$ for all $F \in \mathcal{F}$. Define

$$r := \min_{q \in [n]} \left| \{|F| : q \in F \in \mathcal{F}\} \right|,$$

the number of different sizes through q. Obviously, $r \leq t$. Choose a vertex $p \in [n]$ so that $\left| \{|F| : p \in F \in \mathcal{F}\} \right| = r$. Assume, without loss of generality, that $p \notin F_i$ for $1 \leq i \leq a$ and $p \in F_i$ for $a < i \leq |\mathcal{F}|$. Thus $\mathcal{F}$ has been split into two parts, $\mathcal{F}_0 = \{F : p \notin F \in \mathcal{F}\}$, and $\mathcal{F}_1 = \{F : p \in F \in \mathcal{F}\}$.

Theorem 5.1. *Suppose that the F_i's satisfy the following intersection properties:*

(i) $|F_j \setminus F_i| \in L$ for all $1 \leq i < j \leq a$,
(ii) $1 \leq |F_j \setminus F_i| \leq s$ for $1 \leq i \leq a < j \leq m$,
(iii) $|F_i \cap F_j| \in \{1, \ldots, s\}$ for $a < i < j \leq m$.

Suppose further that $\min L > 0$, $|F_i| > s$ for $a < i \leq m$, and $n - k_i - 1 > s - r$ for $1 \leq i \leq r$. Then

$$|\mathcal{F}| \leq \sum_{i=s-r+1}^{s} \binom{n-1}{i}.$$

In the proof of this theorem we define an n-variable multilinear polynomial f_F of degree at most s for each $F \in \mathcal{F}$. These polynomials are linearly independent (due to the intersection conditions) so we get an upper bound on $|\mathcal{F}|$. To decrease this bound we add more polynomials to this collection which form a space perpendicular to the space spanned by $\{f_F : F \in \mathcal{F}\}$, and then calculate the dimension of the spaces they span. The additional polynomials are defined using the families $\mathcal{H}$ and $\mathcal{G}$, the subsets $[n]$ of size at most $s - 1$.

Proof. We proceed exactly like in Section 3. We divide the proof into two cases: $r \leq s$ and $s < r$. First, we consider $r \leq s$. Define the families
$\mathcal{H} = \{H \subseteq [n] : p \notin H, 0 \leq |H| \leq s - r\}$ and
$\mathcal{G} = \{G \subseteq [n] : p \in G, 1 \leq |G| \leq s\}$.
Order both of them linearly in increasing order, (i.e., H precedes H' if $|H| \leq |H'|$). Put the four families $\mathcal{F}_0$, $\mathcal{H}$, $\mathcal{F}_1$ and $\mathcal{G}$ in this order (keeping their inner order). For each member A of these four families we associate an n variable polynomial $f_A(x_1, \ldots, x_n)$ and a special set X such that for the characteristic vector of X we have $f_A\left(\widehat{X}\right) \neq 0$ but $f_B\left(\widehat{X}\right) = 0$ for all B following A in the linear order. Reduce f_A, as it was done in Lemma 1, to a multilinear polynomial g_A by replacing every higher order term x_u^2 by x_u. We show that these polynomials are linearly independent, implying the upper bound for $|\mathcal{F}|$.

For each $F_i \in \mathcal{F}_0$, consider the polynomial

$$f_i(x) = \prod_{j=1}^{s} (v_i \cdot x - l_j),$$

where v_i is the characteristic vector of F_i. The special set is $[n] \setminus F_i \setminus \{p\}$.

For each $H \in \mathcal{H}$, we define the polynomial

$$f_H(x) = \prod_{b=1}^{r} \left(\sum_{l=1}^{n} x_l - (n - k_b - 1) \right) \prod_{j \in H} x_j.$$

The corresponding special set is H itself. Since $n - k_i - 1 > s - r$, $f_H\left(\widehat{H}\right) \neq 0$, and $f_{H'}\left(\widehat{H}\right) = 0$ for any $|H| \leq |H'|$. Thus $\{f_H(x) : H \in \mathcal{H}\}$ is a linearly independent family.

For each $F_i \in \mathcal{F}_1$ let

$$f_i(x) = \prod_{j=0}^{s-1} (v_i^* \cdot x - j),$$

where v_i^* is the characteristic vector of $F_i \setminus \{p\}$. The special set is $F_i \setminus \{p\}$.

For each $G \in \mathcal{G}$ let

$$f_G(x) = \prod_{j \in G} x_j,$$

with special vector $\widehat{G}$. Note that $f_G\left(\widehat{G}\right) \neq 0$ and $f_{G'}\left(\widehat{G}\right) = 0$ for any $|G| \leq |G'|$, and thus $\{f_G(x) : G \in \mathcal{G}\}$ is a linearly independent family of polynomials.

For the rest of the proof of this case (i.e., the case $r \leq s$) we continue as in the proof of the EKR theorem.

For the $r > s$ case, we only need the family $\mathcal{G}$ as above, and then we show that $\{f_i(x) : F_i \in \mathcal{F}\} \cup \{f_G(x) : G \in \mathcal{G}\}$ is linearly independent. We get the bound $|\mathcal{F}| \leq \sum_{i=0}^{s} \binom{n-1}{i}$. (Note that the summation index i starts at 0 because $s - r + 1 \leq 0$). $\qquad\square$

We give another generalization. Note that this result is almost identical to the previous one, but each is independent of the other (i.e., neither of them implies the other).

Theorem 5.2. *Suppose that with given* n, K, L *and* $p \in [n]$ *the family* $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1$ *satisfies the following intersection conditions*

(i) $|F_j \setminus F_i| \in L$ *for all* $1 \leq i < j \leq a$,
(ii) $1 \leq |F_j \setminus F_i| \leq s - r + 1$ *for* $1 \leq i \leq a < j \leq m$,
(iii) $|F_i \cap F_j| \in \{1, \ldots, s - r + 1\}$ *for* $a < i < j \leq m$.

Suppose further that $\min L > 0$, $|F_i| > s - r + 1$ *for* $a < i \leq m$, *and* $n - k_i - 1 > s - r$ *for* $1 \leq i \leq r$. *Then*

$$|\mathcal{F}| \leq \sum_{i=s-r+1}^{s} \binom{n-1}{i}.$$

Proof. The proof closely follows the proof of Theorem 2 above. We divide the proof into the same two cases: $r \leq s$ and $s < r$. For the first case, we construct the same two families ($\mathcal{G}$ and $\mathcal{H}$) and the same associated polynomials, and for the second case we only use the family $\mathcal{G}$. The only difference is that the polynomial f_i for the set $F_i \in \mathcal{F}_1$ is defined as follows.

$$f_i(x) = \prod_{j=0}^{s-r} (v_i^* \cdot x - j).$$

$\qquad\square$

Finally, the above method also gives the following bound for the general Erdős–Ko–Rado problem. For exact upper bound see Ahlswede and Khachatrian [AK97].

Theorem 5.3. *Let* $\mathcal{F} = \{F_1, \ldots, F_m\}$ *denote a family of* k-subsets of $[n]$. *Suppose that* $|F_i \cap F_j| \in \{d, d+1, \cdots, k-1\}$ *for all* $i < j$. *If* $n \geq 2k - d + 1$, *then* $|\mathcal{F}| \leq \binom{n-1}{k-d}$.

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