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2-Bases of Quadruples

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For Béla Bollobás on his 60th birthday

Let $\mathscr{B}(n, \leqslant 4)$ denote the subsets of $[n] := \{1, 2, \ldots, n\}$ of at most 4 elements. Suppose that $\mathscr{F}$ is a set system with the property that every member of $\mathscr{B}$ can be written as a union of (at most) two members of $\mathscr{F}$. (Such an $\mathscr{F}$ is called a 2-base of $\mathscr{B}$.) Here we answer a question of Erdős proving that

$$|\mathscr{F}| \geqslant 1 + n + \binom{n}{2} - \left\lfloor \frac{4}{3}n \right\rfloor,$$

and this bound is best possible for $n \geqslant 8$.

1. 2-bases

The n-element set $\{1, 2, \ldots, n\}$ is denoted by $[n]$. The family of all subsets of $[n]$ is called the Boolean lattice and is denoted by $\mathscr{B}(n)$. Its kth level is $\mathscr{B}(n, k) := \{B : B \subset [n] : |B| = k\}$, and $\mathscr{B}(n, \leqslant k) := \cup_{0 \leqslant i \leqslant k} \mathscr{B}(n, i)$. The set system $\mathscr{F}$ is called a 2-*base* of $\mathscr{A}$ if every member $A \in \mathscr{A}$ can be obtained as a union of two members of $\mathscr{F}$, in other words $A = F_1 \cup F_2$, $F_1, F_2 \in \mathscr{F}$. Note that we allow $F_1 = F_2$ and we do not insist that the 2-base is a subset of the set system.

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The interest is in how small a base one can find. Let $f(\mathscr{A}) := \min\{|\mathscr{F}| : \mathscr{F}$ is a 2-base of $\mathscr{A}\}$. This is known exactly in very few cases, even when the set system is a natural one. For example, it is not known even for the power-set itself (the discrete cube). In 1993 Erdős [2] proposed the problem of determining $f(\mathscr{B}(n))$ and also the problem of determining the minimum size of a 2-base of the small sets, $f(\mathscr{B}(n, \leqslant k))$. We also use $f_k(n)$ for $f(\mathscr{B}(n, \leqslant k))$. Erdős conjectured that

$$f(\mathscr{B}(n)) = 2^{\lfloor n/2 \rfloor} + 2^{\lceil n/2 \rceil} - 1,$$

and that the extremal family consists of all subsets of V_1 and V_2 where $V_1 \cup V_2 = [n]$ is a partition of $[n]$ into two almost equal parts. A lower bound $f(\mathscr{B}(n)) \geqslant (1 + o(1))2^{(n+1)/2}$ is obvious from the fact that

$$|\mathscr{A}| \leqslant \binom{|\mathscr{F}|}{2} + |\mathscr{F}|,$$

which holds for any 2-base $\mathscr{F}$ of $\mathscr{A}$.

The aim in this paper is to answer this question for the family $\mathscr{B}(n, \leqslant 4)$. The question of the smallest base for $\mathscr{B}(n, \leqslant k)$ is trivial for $k \leqslant 2$, and for $k = 3$ it turns out to be a question about graphs whose answer follows immediately from Turán's theorem. So the case $k = 4$ is the first nontrivial case. It boils down to an interesting question about 3-graphs (3-regular hypergraphs), and it might be somewhat surprising that it is possible to give an exact answer.

Let $f_4(n) := 1 + n + \binom{n}{2} - h(n)$. The main result of this paper can be summarized in the following table:

n	0	1	2	3	4	5	6	7	$n \geqslant 8$
$h(n)$	0	0	1	2	4	5	7	8	$\lfloor \frac{4}{3} n \rfloor$

Theorem 1.1. *For $n \geqslant 8$, $f_4(n) = 1 + n + \binom{n}{2} - \lfloor \frac{4}{3} n \rfloor$.*

Let $g_k(n) := f(\mathscr{B}(n, 4))$, the size of a minimum 2-base for the k-tuples. We will deduce from Theorem 1.1 that $g_4(n) + n + 1 = f_4(n)$ for $n \geqslant 5$.

Theorem 1.2. *We have $g_4(5) = 4$, $g_4(6) = 8$, $g_4(7) = 13$ and for $n \geqslant 8$, $g_4(n) = \binom{n}{2} - \lfloor \frac{4}{3} n \rfloor$.*

In the following section we discuss $f_k(n)$ in the (easy) case $k \leqslant 3$. Then give constructions for $f_4(n)$ separating the cases $n \leqslant 7$ and $n \geqslant 8$ and thus providing lower bounds for $h(n)$. In Section 2 the structure of minimal bases of $\mathscr{B}(n, \leqslant 4)$ is investigated, namely those with minimum deficiency with at least 2, and then (the upper bounds for) the values of $h(n)$ in the above table is proved in Section 3. In Section 4 the uniform case (the case of g_4) is considered, and in Section 5 we close with a few remarks on the case $k > 4$.

1.1. The case $\mathscr{B}(n, \leqslant 3)$

For $k \geqslant 1$ every 2-base of $\mathscr{B}(n, \leqslant k)$ must contain the $\emptyset$ and all singletons. This easily leads to

$$f_0(n) = 1, \quad f_1(n) = 1 + n, \quad f_2(n) = 1 + n.$$

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Suppose that $\mathscr{F}$ is a 2-base of $\mathscr{B}(n, \leqslant k)$, $1 < k \leqslant n$, such that $|\mathscr{F}| = f_k(n)$ and $\sum_{F \in \mathscr{F}} |F|$ is minimal. Such bases are called *minimal*. Then

(i) $\emptyset \in \mathscr{F}$, $\mathscr{B}(n,1) \subset \mathscr{F}$,

(ii) for every $F \in \mathscr{F}$ we have $|F| \leqslant k-1$.

Indeed, one need only observe that for $F \in \mathscr{F}$, $|F| = k$, $x \in F$ one can replace F by $F' := F \setminus \{x\}$, *i.e.*, $\mathscr{F} \setminus \{F\} \cup \{F'\}$ is also a 2-base.

Construction 1.3. *Consider a 2-partition $V_1 \cup V_2$ of $[n]$ with $\lfloor n/2 \rfloor \leqslant |V_1| \leqslant |V_2| \leqslant \lceil n/2 \rceil$ and let $\mathscr{F}$ be all the subsets of V_1 and V_2 of size at most 2. Every triple from $[n]$ meets a V_i in at least 2 elements so it also contains a 2-element member of $\mathscr{F}$. Hence $\mathscr{F}$ is a 2-base of $\mathscr{B}(n, \leqslant 3)$.*

Claim 1.4. $f_3(n) = 1 + n + \binom{\lfloor n/2 \rfloor}{2} + \binom{\lceil n/2 \rceil}{2}$.

Proof of Claim 1.4. Suppose that $\mathscr{F}$ is a minimal 2-base of $\mathscr{B}(n, \leqslant 3)$ satisfying (i) and (ii). Split it into subfamilies according to the sizes of its members, $\mathscr{F} = \mathscr{F}_0 \cup \mathscr{F}_1 \cup \mathscr{F}_2$ where $\mathscr{F}_i := \mathscr{F} \cap \mathscr{B}(n,i)$. Then $\mathscr{F}_2$ is a graph (*i.e.*, a 2-graph) with the property that every triple contains an edge, so its complement $\mathscr{H}_2$ is triangle-free ($\mathscr{H}_2 := \mathscr{B}(n,2) \setminus \mathscr{F}_2$). Then Turán's theorem [7] implies that $|\mathscr{H}_2| \leqslant \lfloor n^2/4 \rfloor$, hence

$$|\mathscr{F}| = |\mathscr{F}_0| + |\mathscr{F}_1| + |\mathscr{F}_2| \geqslant 1 + n + \binom{n}{2} - \left\lfloor \frac{n^2}{4} \right\rfloor. \qquad \square$$

1.2. Constructions for $\mathscr{B}(n, \leqslant 4)$ if $n \leqslant 7$

Let $\mathscr{F}$ be a minimal 2-base of $\mathscr{B}(n, \leqslant 4)$ satisfying (i) and (ii). Let $\mathscr{F}_i := \mathscr{F} \cap \mathscr{B}(n,i)$; then $\mathscr{F} = \mathscr{F}_0 \cup \mathscr{F}_1 \cup \mathscr{F}_2 \cup \mathscr{F}_3$ where $\mathscr{F}_0 = \{\emptyset\}$, $\mathscr{F}_1 = \mathscr{B}(n,1)$. Use the notation $\mathscr{H}_2 := \mathscr{B}(n,2) \setminus \mathscr{F}_2$. Then

$$|\mathscr{F}| = 1 + n + \binom{n}{2} - |\mathscr{H}_2| + |\mathscr{F}_3| := 1 + n + \binom{n}{2} - h(n).$$

Since $\mathscr{B}(n, \leqslant 2)$ is a 2-base of $\mathscr{B}(n, \leqslant 4)$ we have $h(n) \geqslant 0$.

Let us summarize the properties of $\mathscr{F}_2 \cup \mathscr{F}_3$:

for every triple $T \subset [n]$	either T contains a pair from $\mathscr{F}_2$	(1.1)
	or $T \in \mathscr{F}_3$,	(1.2)
for every quadruple $Q \subset [n]$	either Q contains a triple from $\mathscr{F}_3$	(1.3)
	or Q is a union of two edges from $\mathscr{F}_2$.	(1.4)

Construction 1.5. *For $n \geqslant 4$ let $\mathscr{H}_2$ be a Hamilton cycle, $|\mathscr{F}_3| = 0$.*

It is easy to show that this family $\mathscr{F}_2$ satisfies (1.1) and (1.4) so (together with $\mathscr{B}(n, \leqslant 1)$) it is a 2-base. This construction shows that $h(n) \geqslant n$ (for $\geqslant 4$), and one can see that this is the best possible for $n = 4$ and $n = 5$.

Claim 1.6. $h(0) = h(1) = 0$, $h(2) = 1$, $h(3) = 2$, $h(4) = 4$ *and* $h(5) = 5$.

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The proof of this (and the following two claims concerning $n = 6$ and 7) is a short, finite process. For completeness we sketch them in Section 3.

Construction 1.7. *For $n = 6$ let $\mathscr{F}_3$ be two disjoint triples F_1, F_2 and let $\mathscr{F}_2$ be the six pairs contained in either F_1 or F_2.*

Another construction of the same size can be obtained by considering a Hamilton cycle $\mathscr{F}_2 := \{12, 23, 34, 45, 56, 16\}$ with two triples $\mathscr{F}_3 := \{135, 246\}$.

Claim 1.8. $h(6) = 7$.

Construction 1.9. *For $n = 7$ label the seven elements by two coordinates, $V := \{v(1,1), v(1,2), v(1,3), v(2,1), v(2,2), v(3,1)\}$. Let $\mathscr{F}_2$ be the ten pairs $v(\alpha,\beta)v(\alpha',\beta')$ with $\alpha \neq \alpha'$ and $\beta \neq \beta'$, and let $\mathscr{F}_3$ be formed by the three triples having a constant coordinate, i.e., $\{v(1,1), v(1,2), v(1,3)\}$, $\{v(2,1), v(2,2), v(2,3)\}$ and $\{v(1,1), v(2,1), v(3,1)\}$. (This is a truncated version of Construction 1.13 for $n = 9$.)*

Claim 1.10. $h(7) = 8$.

Construction 1.11. *Let n_1, n_2 be nonnegative integers, $V^1 \cup V^2$ a partition of $[n]$ with $|V^i| = n_i$, $\mathscr{F}^i$ a minimal 2-base on V_i. Define $\mathscr{F}$ as $\mathscr{F}^1 \cup \mathscr{F}^2$ together with all pairs joining V^1 and V^2.*

It is easy to see that this construction satisfies (1.1)–(1.4): it is a 2-base. Indeed, it is sufficient to check a triple T and a quadruple Q meeting both V_1 and V_2. Then T contains a pair joining V^1 and V^2; thus it satisfies (1.1). If $|Q \cap V^1| = |Q \cap V^2| = 2$, then it is a union of two crossing pairs. Finally, if $Q = \{a,b,c,d\}$ and $Q \cap V^1 = \{a,b,c\}$, then since $\mathscr{F}^1$ is a 2-base, $Q \cap V^1$ satisfies either (1.1) or (1.2). In the first case $Q \cap V^1$ it contains a pair, say ab from $\mathscr{F}^1$; then $\{a,b\} \cup \{c,d\}$ is a partition of Q satisfying (1.4). In the second case $Q \cap V^1 \in \mathscr{F}^1$, so Q satisfies (1.3). We obtained the following.

Claim 1.12. *For n_1, n_2 nonnegative integers $h(n_1 + n_2) \geqslant h(n_1) + h(n_2)$.* □

1.3. Constructions for $n \geqslant 8$

Construction 1.13. *Suppose that $\mathscr{F}_3$ is a triple system on $[n]$ of* girth at least 4*, i.e., $|F' \cap F''| \leqslant 1$ for $F', F'' \in \mathscr{F}_3$, $F_1, F_2, F_3 \in \mathscr{F}_3$ and $F_1 \cap F_2 \neq \emptyset$, $F_1 \cap F_3 \neq \emptyset$, $F_2 \cap F_3 \neq \emptyset$ imply $F_1 \cap F_2 \cap F_3 \neq \emptyset$. Suppose further that every* degree *of $\mathscr{F}_3$ is at most two, i.e., every singleton is contained in at most two triples. Define $\mathscr{H}_2$ as the pairs covered by the members of $\mathscr{F}_3$.*

This construction (together with $\mathscr{B}(n, \leqslant 1)$) forms a 2-base. Indeed, if a triple $T \subset [n]$ contains no edge from $\mathscr{F}_2$, then it belongs to $\mathscr{F}_3$, so either (1.1) or (1.2) holds. Moreover, if $Q = \{a,b,c,d\} \subset [n]$ is a quadruple and contains no triple from $\mathscr{F}_3$, then the induced graph $\mathscr{H}_2|Q$ contains no triangle. So $\mathscr{F}_2|Q$ contains two disjoint edges (and thus fulfils (1.4)) unless $\mathscr{H}_2|Q$ has a vertex of degree 3, say, $ab, ac, ad \in \mathscr{H}_2$. Since the degree of $\mathscr{F}_3$ at the vertex a is at most two and the edges of $\mathscr{H}_2$ are obtained from the triples of $\mathscr{F}_3$ we get that there exists a triple $T \in \mathscr{F}_3$ with $a \in T \subset Q$. We have proved that Construction 1.13 indeed defines a 2-base.

For $n = 3k$, $k \geqslant 3$ we obtain $h(3k) \geqslant 4k$ as follows. Let $[n] = \{a_1, a_2, \ldots, a_k\} \cup \{b_1, b_2, \ldots, b_k\} \cup \{c_1, c_2, \ldots, c_k\}$. Define $\mathscr{F}_3$ as all triples of the form $a_i b_i c_i$ and $a_i b_{i+1} c_{i+2}$ (indices are taken modulo k). This satisfies the constraint of Construction 1.13. Since $|\mathscr{H}_2| = 3|\mathscr{F}_3|$, we get $h(n) \geqslant 2|\mathscr{F}_3| = 4k$.

If we leave out from the above construction the 2 triples of $\mathscr{F}_3$ and the 4 pairs of $\mathscr{H}_2$ containing the element $3k$ we obtain that $h(3k-1) \geqslant 4k-2$. Thus we already have the cases $n = 3k$ and $n = 3k-1$ in the following claim.

Claim 1.14. *$h(n) \geqslant \lfloor \frac{4}{3} n \rfloor$ for $n \geqslant 8$.*

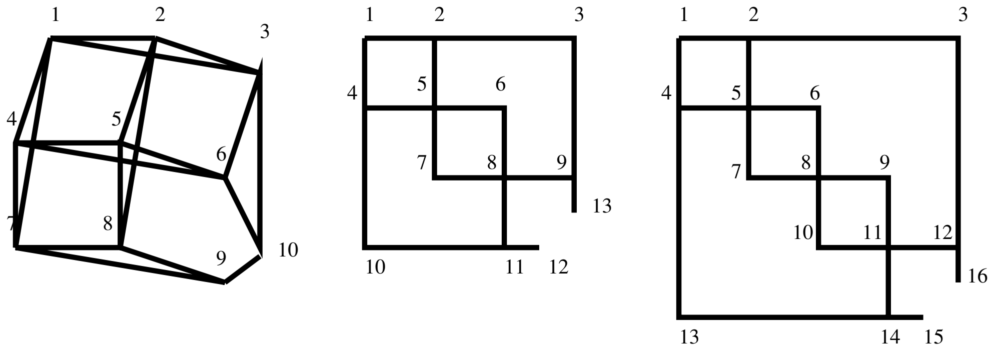

Figure 1.

Proof. We only need a construction for $n = 3k+1$, $k \geqslant 3$ to show $h(3k+1) \geqslant 4k+1$. It is enough to show $h(10) \geqslant 13$, $h(13) \geqslant 17$ and $h(16) \geqslant 21$; then the general case follows from $h(9) \geqslant 12$ using Claim 1.12.

Define the six triples of $\mathscr{F}_3$ as $\{1,2,3\}$, $\{4,5,6\}$, $\{7,8,9\}$, $\{1,4,7\}$, $\{2,5,8\}$ and $\{3,6,10\}$ and $\mathscr{H}_2$ as the 18 pairs covered by these triples and $\{9,10\}$. The graph $\mathscr{H}_2$ has only these 6 triangles, so (1.1)–(1.2) hold, and it is not difficult to check the four-tuples, too.

The other cases are similar: for $n = 13$ we can define $\mathscr{F}_3 := \{1,2,3\}$, $\{4,5,6\}$, $\{7,8,9\}$, $\{10,11,12\}$ and $\{1,4,10\}$, $\{2,5,7\}$, $\{6,8,11\}$, $\{3,9,13\}$ and $\mathscr{H}_2$ consists of these triangles and the pair $\{12,13\}$.

Finally, for $n = 16$ we define $\mathscr{F}_3$ as $\{1,2,3\}$, $\{4,5,6\}$, $\{7,8,9\}$, $\{10,11,12\}$, $\{13,14,15\}$ and $\{1,4,13\}$, $\{2,5,7\}$, $\{6,8,10\}$, $\{9,11,14\}$, and $\{3,12,16\}$. Again $\mathscr{H}_2$ consists of the triangles obtained from $\mathscr{F}_3$ and the edge $\{15,16\}$. □

2. Bases with deficiency at least 2

The aim of this paper is to prove Theorem 1.1, so suppose that $\mathscr{F}$ is a minimal 2-base of $\mathscr{B}(n, \leqslant 4)$ and that $\mathscr{F}_2 \cup \mathscr{F}_3$ satisfies (1.1)–(1.4).

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Lemma 2.1. *If* $abc \in \mathscr{F}_3$, *then either* $\{ab, bc, ca\} \subset \mathscr{F}_2$ *or* $\{ab, bc, ca\} \subset \mathscr{H}_2$.

Proof. Suppose, on the contrary, that $ab \in \mathscr{F}_2$, $ac \notin \mathscr{F}_2$. Replace abc by ac in $\mathscr{F}$. Since $\sum_{F\in\mathscr{F}} |F|$ is minimal the family $\mathscr{F}' := \mathscr{F} \setminus \{abc\} \cup \{ac\}$ is not a 2-base. What can go wrong? Since we added a new pair, conditions (1.1) and (1.2) still hold. The only condition we can violate is (1.3)–(1.4). We removed abc, so there exists an $Q = abcd$ not a union of two members of $\mathscr{F}'$. So $abcd$ does not contain any triple from $\mathscr{F}'$ and also bd, $cd \notin \mathscr{F}'$. Consider bcd. We have $bcd \notin \mathscr{F}$ so (1.1) implies that $bc \in \mathscr{F}_2$. Consider acd. Since ac, cd, and $acd \notin \mathscr{F}$ again (1.1) implies that $ad \in \mathscr{F}_2$. However, then $Q = ad \cup bc$, a contradiction. □

Use the notation $\deg_2^-(x)$ for the degree of the vertex x in the graph $\mathscr{H}_2$ and $\deg_3(x)$ for the degree of x in $\mathscr{F}_3$. The difference $\deg_2^-(x) - \deg_3(x)$ is called the *deficiency* of the vertex $x \in V$. From now on in this section we suppose that

$$\deg_2^-(x) - \deg_3(x) \geqslant 2 \text{ for every } x \in [n]. \tag{2.1}$$

Let $N(x)$ denote the neighbourhood of x in $\mathscr{H}_2$, $N(x) := \{y : xy \in \mathscr{H}_2\}$, $\deg_2^-(x) = |N(x)|$. Let $\mathscr{T}(x)$ denote the set of triples T from $\mathscr{F}_3$ with $x \in T \subset N(x) \cup \{x\}$, and let $t(x) := |\mathscr{T}(x)|$. Suppose that $D = \max_{x\in[n]} \deg_2^-(x)$, and a has maximum degree in $\mathscr{H}_2$. Consider $A = \{a\} \cup N(a)$, $|A| = D + 1$: let $t := t(a)$. Then (2.1) implies $t, t(x) \leqslant D - 2$.

2.1. Eliminating the case $D \geqslant 5$

Claim 2.2. (2.1) *implies that* $D \leqslant 4$.

Proof. Consider the $\binom{D}{3}$ four-tuples of A containing x: let $\mathscr{B} := \{Q : a \in Q \subset A, |Q| = 4\}$. Note that none of these can satisfy (1.4), so each of them contains a member of $\mathscr{F}_3$. Classify them into two groups as follows:

$\mathscr{B}_1 := \{abcd : b, c, d \in A$ and there exists a $T \in \mathscr{F}_3$ with $a \in T \subset \{a, b, c, d\}\}$,
$\mathscr{B}_2 := \{abcd : abcd \subset A,\ abc, abd, acd \notin \mathscr{F}_3\}$.

Each $Q \in \mathscr{B}_2$ contains a member of $\mathscr{F}_3|N(a)$, hence

$$|\mathscr{B}_2| \leqslant |\mathscr{F}_3|N(a)|.$$

Each member of $\mathscr{T}(a)$ is contained in $D - 2$ four-tuples from $\mathscr{B}_1$, hence

$$|\mathscr{B}_1| \leqslant t(D-2). \tag{2.2}$$

Here the sum of the left-hand sides is $\binom{D}{3}$. The sum of the right-hand sides can be estimated by the degrees of $\mathscr{F}_3$ on A. Using $\deg_3(x) \leqslant D - 2$ we obtain

$$\begin{aligned}\binom{D}{3} &= |\mathscr{B}_1| + |\mathscr{B}_2| \leqslant t(D-2) + |\mathscr{F}_3|N(a)| = t(D-3) + |\mathscr{F}_3|A| \\ &\leqslant t(D-3) + \frac{1}{3}\sum_{x\in A} \deg_3(x) \leqslant t(D-3) + \frac{1}{3}(t + D(D-2)).\end{aligned} \tag{2.3}$$

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Hence

$$\frac{1}{6}D(D-2)(D-3) \leqslant t\frac{3D-8}{3}. \tag{2.4}$$

Since $t \leqslant D-2$ we get $D \leqslant 6$. In the case of $t \leqslant D-3$ (2.4) implies $D \leqslant 4$. So two cases are left in the proof of the claim, namely $(D,t) = (6,4)$ and $(5,3)$.

In the case of $D = 6$, $t = 4$ the right-hand side of (2.2) can be improved by 2, since there are at least 2 coincidences when estimating the cardinality of $\mathscr{B}_1$. So $|\mathscr{B}_1| \leqslant 14$, and we can decrease the right-hand sides of (2.3) and (2.4) by 2, and that leads to the contradiction $12 \leqslant 4 \times \frac{10}{3} - 2$.

In the case of $D = 5$, $t = 3$ we use two things. The first one is implied by Lemma 2.1 and (1.1).

(C1) If $abc \in \mathscr{T}(a)$ then $bc \in \mathscr{H}_2$; if $abc \notin \mathscr{T}(a)$ and $b, c \in N(a)$ then $bc \in \mathscr{F}_2$. Thus $\mathscr{F}_2|N(a)$ has exactly $\binom{D}{2} - t$ edges.

(C2) If $\deg_3(x) \geqslant 3$, then $t(x) = 3$. Indeed, (2.1) implies $\deg_2^-(x) \geqslant \deg_3(x) + 2 \geqslant 5$.

Consequently $\deg_2^-(x) = 5 = D$, x has maximum degree, D, and then the previous considerations for a are valid for x, too, *i.e.*, (2.4) implies that $t(x) = 3$ is the only possibility.

Now we are ready to show that, in fact, $(D,t) = (5,3)$ is impossible. Suppose, on the contrary, that there is such a construction and let $N(a) = \{b,c,d,e,f\}$. Consider the 3-edge graph $G := \{xy : axy \in \mathscr{F}_3\}$. There are 4 non-isomorphic possibilities for G:

(α) G is a triangle, $\{bc, cd, bd\}$,
(β) G is a path of length 3, $\{bc, cd, de\}$,
(γ) G is a star, $\{bc, bd, be\}$,
(δ) G has 2 components, $\{bc, cd, ef\}$.

In each case we will find one or more $x \in N(a)$ with $t(x) = 3$. Then the triples containing x cover no pair from $\mathscr{F}_2$ and this will lead to a contradiction.

For (α), by (1.3) we have $bef, cef, def \in \mathscr{F}_3$. Hence $\deg_3(f) \geqslant 3$. Then (C2) implies that $t(f) = 3$ and then Lemma 2.1 gives that $\{b,c,d,e\} \subset N(f)$, $ef \notin \mathscr{F}_2$. However, $ef \in \mathscr{F}_2$ by (C1), a contradiction.

The other cases can be handled in the same way. For (β) we have $bdf, bef, cef \in \mathscr{F}_3$, hence $\deg_3(f) \geqslant 3$. Then $t(f) = 3$ and $\{b,c,d,e\} \subset N(f)$, $ef \notin \mathscr{F}_2$. For (γ) we have $cdf, cef, def \in \mathscr{F}_3$, hence $\deg_3(f) \geqslant 3$. Then $t(f) = 3$ and $\{c,d,e\} \subset N(f)$, $ef \notin \mathscr{F}_2$. For (δ) we have $bde, bdf \in \mathscr{F}_3$, hence $\deg_3(b) \geqslant 3$. Then $t(b) = 3$ and $\{c,d,e,f\} \subset N(b)$, $bf \notin \mathscr{F}_2$. This final contradiction completes the proof of the case $(D,t) = (5,3)$ and Claim 2.2. □

2.2. The case $D \leqslant 4$

From now on in this section we suppose that $D \leqslant 4$.

Claim 2.3. (2.1) *and* $\deg_2^-(a) = 4$ *imply that* $t(a) = \deg_3(a) = 2$ *and the two triples containing the element a meet only in a, e.g.,* $N(a) = bcde$ *and* $\mathscr{T}(a) = \{abc, ade\}$.

Proof. Suppose first that $t(a) = 0$. Then all the four triples of the form xyz, $x,y,z \in N(a)$ belong to $\mathscr{F}_3$. Hence $\deg_3(b) \geqslant 3$, contradicting $D \geqslant \deg_2(x) \geqslant 2 + \deg_3(x)$. If $t(a) = 1$, say $abc \in \mathscr{T}(a)$, then $bde, cde \in \mathscr{F}_3$ is implied by (1.3). Hence $\deg_3(e) \geqslant 2$, so $\deg_2^-(e) = 4$. Since

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(1.1) implies that $be, ce, de \in \mathscr{F}_2$ we get that $N(e) \cap \{b,c,d\} = \emptyset$, so $t(e) = 0$. However, we have seen that $\deg_2^-(e) = D = 4$ implies $t(e) > 0$.

So we get $t(a) \geqslant 2$, *i.e.*, by $t(x) \leqslant D - 2$ we have $t(a) = 2$. The only case left to exclude is when the triples in $\mathscr{T}(a)$ meet in two elements, say $\mathscr{T}(a) = \{abc, acd\}$. Then $bde \in \mathscr{F}_3$, so $\deg_3(b) \geqslant 2$. Hence we get $\deg_2^-(b) = 4$, this implies $t(b) = 2$ and $\{c,d,e\} \subset N(b)$. We get $ab, ae, be \in \mathscr{H}_2$, $abe \notin \mathscr{F}_3$, contradicting (1.1). □

Claim 2.4. (2.1) *and* $\deg_2^-(x) = 3$ *imply that* $\deg_3(x) = 1$.

Proof. Suppose, on the contrary, that $\deg_3(x) = 0$. Consider $N(x) = abc$, we have $ab, bc, ca \in \mathscr{F}_2$ by (1.1) and $abc \in \mathscr{F}_3$ by (1.3). Then $ab \in \mathscr{F}_2$ implies that $abc \notin \mathscr{T}(a)$ Therefore $t(a)$ cannot be $D - 2 = 2$. So Claim 2.3 gives that $\deg_2^-(a) \neq 4$. Since $\deg_3(a) \geqslant 1$ we get that $\deg_2^-(a) = 3$. Consider $N(a) = xyz$. Note that $y, z \notin \{x, a, b, c\}$. Then $xyz \in \mathscr{F}_3$ by (1.3). This contradicts $\deg_3(x) = 0$, so we have $\deg_3(x) \geqslant 1$. On the other hand, (2.1) implies $\deg_3(x) \leqslant 1$. □

Claim 2.5. (2.1) *implies that* $h(\mathscr{F}) \leqslant \frac{4}{3}n$.

Proof. For $x \in [n]$ define $\varphi(x) := \frac{1}{2}\deg_2^-(x) - \frac{1}{3}\deg_3(x)$. We are going to prove that $\varphi(x) \leqslant 4/3$ for every x. This implies the claim as follows:

$$h(\mathscr{F}) = |\mathscr{H}_2| - |\mathscr{F}_3| = \sum_{x\in[n]} \varphi(x) \leqslant \frac{4}{3}n. \tag{2.5}$$

Using the previous three claims one can split $[n]$ into three parts, $[n] = P \cup Q \cup R$, where $P := \{x : \deg_2^-(x) = 4,\ \deg_3(x) = 2\}$, $Q := \{x : \deg_2^-(x) = 3,\ \deg_3(x) = 1\}$, and $R := \{x : \deg_2^-(x) = 2,\ \deg_3(x) = 0\}$. For each case we have $\varphi \leqslant 4/3$. □

Note that $h(\mathscr{F}) = \frac{4}{3}n$ in Claim 2.5 is only possible for Construction 1.13, especially

$$P = [n] \text{ and } Q = R = \emptyset. \tag{2.6}$$

3. Proof of the main result

Let $\mathscr{F}$ be a minimal 2-base for $\mathscr{B}(n, \leqslant 4)$. Then

$$\begin{aligned} 1 + n + \binom{n}{2} - h(n) = |\mathscr{F}| &= |\mathscr{F}|([n] \setminus \{x\})| + 1 + (n - 1 - \deg_2^-(x)) + \deg_3(x) \\ &\geqslant 1 + n + \binom{n}{2} - h(n-1) - (\deg_2^-(x) - \deg_3(x)) \end{aligned} \tag{3.1}$$

gives that the deficiency of every vertex is at least $h(n) - h(n-1)$.

Proof of Theorem 1.1. We use induction on n to show that $h(n) \leqslant \frac{4}{3}n$. This is certainly true for $n \leqslant 2$. Suppose that $h(n-1) \leqslant \frac{4}{3}(n-1)$ and consider $h(n)$. If $h(n) \leqslant h(n-1) + 1$, then we are done. If $h(n) \geqslant h(n-1) + 2$, then, as we have seen in (3.1), there exists a minimal 2-base $\mathscr{F}$ on $[n]$ with deficiency at least 2. Then Claim 2.5 gives $h(n) = h(\mathscr{F}) \leqslant \frac{4}{3}n$. □

Proofs of Claims 1.6, 1.8 and 1.10. The case $n \leqslant 4$ is trivial. Suppose that $5 \leqslant n \leqslant 7$ and let $\mathscr{F}$ be a minimal 2-base on n vertices.

The case $n = 5$ is easy. $h(\mathscr{F}) \geqslant 6$ implies $|\mathscr{F}_2| + |\mathscr{F}_3| \leqslant 4$. If $|\mathscr{F}_2| = 4$, then there is a unique way to satisfy (1.1) (namely, $\mathscr{F}_2$ is a union of an edge and a triangle) and then (1.4) is violated. If $|\mathscr{F}_2| = 3$, then there are at least 2 triples not containing any member of $\mathscr{F}_2$, so (1.2) gives $|\mathscr{F}_3| \geqslant 2$. If $|\mathscr{F}_2| \leqslant 2$, then they satisfy (1.1) with at most $3|\mathscr{F}_2|$ triples. Hence, (1.2) gives $|\mathscr{F}_3| \geqslant 10 - 3|\mathscr{F}_2|$. Then $|\mathscr{F}_2| + |\mathscr{F}_3|$ exceeds 4, a final contradiction.

If the minimum deficiency of $\mathscr{F}$ is (at most) 1 then (3.1) gives $h(n) \leqslant h(n-1) + 1$, and we are done. From now on suppose that the deficiency of $\mathscr{F}$ is at least 2, *i.e.*, (2.1) holds.

For $n = 6$ Claim 2.5 gives that $h(\mathscr{F}) \leqslant \frac{4}{3} \times 6 = 8$. By (2.6) $h(\mathscr{F}) = 8$ is only possible if $P = [n]$, *i.e.*, $\mathscr{H}_2$ is a 4-regular graph, and $\mathscr{F}_3$ consists of four triples. Then $\mathscr{F}_2$ is a matching, say, $\mathscr{F}_2 = \{a_1a_2, b_1b_2, c_1c_2\}$. Then (1.2) implies that all the eight triples of the form $a_ib_jc_k$ should belong to $\mathscr{F}_3$, a contradiction. We have obtained $h(\mathscr{F}) = h(6) \leqslant 7$.

For $n = 7$ Theorem 1.1 implies $h(\mathscr{F}) \leqslant \lfloor 7 \times \frac{4}{3} \rfloor = 9$. We claim that $h(7) = 8$. Suppose, on the contrary, that $h(\mathscr{F}) = 9$. Consider the partition of $[n] = P \cup Q \cup R$ defined in the proof of Claim 2.5. For $R \neq \emptyset$ (2.5) gives $|R| = 1$, $|P| = 6$, $Q = \emptyset$. Then $\mathscr{H}_2|P$ is a 4-regular graph, not joined to R, so $\deg_2^-(R) = 2$ is impossible. Finally, if $R = \emptyset$, $|Q| = 2$ and $|P| = 5$ then we get $|\mathscr{F}_3| = 4$. The four members of $\mathscr{F}_3$ can pairwise meet in at most 1 vertex (by Claims 2.3 and 2.4) and have girth 4. But such an $\mathscr{F}_3$ does not exist on 7 vertices.

So we have obtained the exact value of $h(n)$ for every n. □

4. 2-bases for quadruples

Here we prove Theorem 1.2. Suppose that $\mathscr{F}$ is an extremal 2-base for $\mathscr{B}(n,4)$, *i.e.*, $|\mathscr{F}| = g_4(n)$, such that $|\mathscr{F}_1| + |\mathscr{F}_4|$ is minimal. The case $n = 5$ is a short finite process, the unique 2-base with 4 members {12, 34, 135, 245}.

In the case $n = 6$ the 6 pairs of a hexagon and the 2 disjoint triples of the second example in Construction 1.7 shows $g_4(6) \leqslant 8$. Consider a minimal 2-base $\mathscr{F}$. If $\deg_{\mathscr{F}}(x) \geqslant 3$, then

$$|\mathscr{F}| = \deg_{\mathscr{F}}(x) + |\mathscr{F}|([n] \setminus \{x\})| \geqslant \deg_{\mathscr{F}}(x) + g_4(n-1) \tag{4.1}$$

implies $|\mathscr{F}| \geqslant 3 + 4$. The impossibility of this case with $|\mathscr{F}| = 7$ follows easily from the uniqueness of the 2-base on 5 elements. Moreover, it is easy to check that a hypergraph of 7 edges on 6 elements with maximum degree 2 cannot be a 2-base, so $g_4(6) \geqslant 8$. From now on we may suppose that $n \geqslant 7$.

The upper bounds for $g_4(n)$ follows by leaving out the singletons and the empty set from Constructions 1.9 and 1.13 in Section 1. To prove a lower bound we proceed as in Section 2. The main idea of the proof is that we first investigate the minimal 2-bases with a maximum degree condition

$$\deg_{\mathscr{F}}(x) \leqslant n - 3 \tag{4.2}$$

for all $x \in [n]$.

We claim that (4.2) implies that $\mathscr{F}_4 = \emptyset$. Indeed, suppose, on the contrary, that $Q \in \mathscr{F}_4$. If Q contains any proper subset $F \in \mathscr{F}$, $x \in F \subset Q$, $Q \neq F$, then one can replace Q by $Q \setminus \{x\}$ to obtain another 2-base with smaller $|\mathscr{F}_1| + |\mathscr{F}_4|$. So we may suppose that such a proper subset does not exist. Consider $Q \setminus \{x\} \cup \{y\}$ for some $x \in Q$, $y \in [n] \setminus Q$. This is a union of (at most) two sets $A, B \in \mathscr{F}$. Both of them contain y. We obtain that the sets $\{F : y \in F \subset Q \cup \{y\},\ |F| > 1\}$ cover Q, and some vertex of Q is covered at least twice. Hence there exists an $x \in Q$ covered by these sets more than $n - 4$ times while y runs through $[n] \setminus Q$. Taking Q itself, we get that $\deg_{\mathscr{F}}(x) > n - 3$, contradicting (4.2).

Use the notation of the previous section, *e.g.*, $D := \max \deg_2^-(x)$ and $\deg_2^-(a) = D$. We claim that (4.2) implies that

$$D \leqslant 4.$$

In the proof of this one cannot use Lemma 2.1, either (1.1) or (1.2); however, (2.2)–(2.4) still hold, implying $D \leqslant 6$. Furthermore, $ab, ac, ad \notin \mathscr{F}_2$, and $abc, abd, acd \notin \mathscr{F}_3$ imply not only $bcd \in \mathscr{F}_3$ but $a \in \mathscr{F}_1$. Thus, in the case $\mathscr{B}_2 \neq \emptyset$ (*e.g.*, for $D > 4$), one gets $a \in \mathscr{F}_1$. Then (4.2) gives $t(a) \leqslant \deg_2^-(a) - 3 = D - 3$. So (2.4) gives $D \leqslant 4$.

Using the same idea one can see that Claim 2.3 remains true. The following analogue of Claim 2.4 is obviously true: $\deg_2^-(x) = 3$ implies $\deg_1(x) + \deg_3(x) = 1$.

As in Claim 2.5 we show that (4.2) implies

$$|\mathscr{F}| \geqslant \binom{n}{2} - \frac{4}{3}n. \tag{4.3}$$

Indeed, for $x \in [n]$ define $\varphi(x) := \frac{1}{2}\deg_2^-(x) - \frac{1}{3}\deg_3(x) - \deg_1(x)$. As before we have that (4.2) implies that $\varphi(x) \leqslant 4/3$ for every x, completing the proof of (4.3) for this case.

Finally, for hypergraphs with maximum degree at least $n - 2$ one can use induction on n. Inequality (4.1) implies that (4.3) always holds.

The case $n = 7$ can be finished as in the proof of Claim 2.5, by considering a partition of $[n]$ into three parts, $[n] = P \cup Q \cup R$, where now $Q := \{x : \deg_2^-(x) = 3,\ \deg_1(x) + \deg_3(x) = 1\}$. The details are omitted. □

5. More hypergraphs

Let $T(n, k, r)$ denote the minimum size of a hypergraph $\mathscr{F} \subseteq \mathscr{B}(n, r)$ such that every k-subset of $[n]$ contains a member of $\mathscr{F}$. The determination of $T(n, k, r)$ is proposed by Turán [8], who solved the case $r = 2$ (the case of graphs – see [7]) and has a longstanding conjecture $T(n, 4, 3) = \left(\frac{4}{9} + o(1)\right)\binom{n}{3}$. For a survey on this see Sidorenko [6].

One can prove for every odd integer k that our $f_k(n)$ equals $(1 + o(1))T(n, k, (k + 1)/2)$, but the even case is more involved and apparently leads to a new Turán-type problem. The authors intend to return to this topic in a future work.

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