Minimum Vertex-Diameter-2-Critical Graphs

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Abstract: We prove that the minimum number of edges in a vertex-diameter-2-critical graph on $n \geq 23$ vertices is $(5n-17)/2$ if n is odd, and is $(5n/2)-7$ if n is even. © 2005 Wiley Periodicals, Inc. J Graph Theory 50: 293–315, 2005.

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1. INTRODUCTION

For a graph G, let $V(G), E(G), n(G)$, and $e(G)$ denote its vertex set, edge set, number of vertices, and number of edges, respectively. We define the open neighborhood $N(x)$ of a vertex $x \in V(G)$ to be the set of the vertices adjacent to x, and the closed neighborhood to be $N[x] := N(x) \cup \{x\}$. We define the degree

$\deg(x)$ of x to be the number of its neighbors, $\deg(x) :=| N(x) |$. We denote the minimum degree of G by $\delta(G) = \min\{\deg(x): x \in V(G)\}$.

If $A, B \subset V(G)$, we define $G[A,B]$ to be the subgraph with vertex set $A \cup B$ and edge set $E(G[A,B])$ consisting of the edges of G between A and B, that is, $E(G[A,B]) = \{ab \in E(G) : a \in A, b \in B\}$. If $A = B$, then we abbreviate the subgraph $G[A,A]$ as $G[A]$ and its edge set $E(G[A,A])$ as $E(G[A])$. Also, $|E(G[A,B])| = e(G[A,B])$ and $|E(G[A])| = e(G[A])$ or sometimes just $e(A)$. The distance $d(a,b) := d_G(a,b)$ between two vertices a and b of the graph G is defined to be the length of a shortest (a,b)-path; if there is no connecting path then we define $d(a,b)$ to be infinite. We define the *farthest distance* between A, B to be $d(A,B) := \max\{d(a,b): a \in A, b \in B\}$. The diameter $\mathrm{diam}(G)$ of a graph G is the maximum distance over all pairs of vertices, $\mathrm{diam}(G) = d(V(G), V(G))$.

A graph is *vertex-diameter-k-critical* if for each vertex x, $\mathrm{diam}(G - x) > \mathrm{diam}(G) := k$. The study of diameter-critical graphs is one of the oldest subjects of extremal graph theory, initiated by Erdős and Rényi [17] with T. Sós [18], Murty and Vijayan [27], Murty [24–26], and Ore [28] in the 1960s. Most of the research dealt with minimal graphs with given diameter and maximum degree, e.g., Erdős and Rényi [17], Bollobás [4,5] with Eldridge [6] and with Erdős [7], and edge-critical graphs. For example, Ore [28], Plesník [30], Murty and Simon (see [12]) conjectured that the maximum number of edges in an edge-diameter-2-critical graph on n vertices is $\lfloor n^2/4 \rfloor$. Plesník [30], Caccetta and Häggkvist [12], and Fan [19] obtained upper bounds, and the conjecture was proved for $n > n_0$ in [20]. Extremal problems concerning diameter and connectivity were studied in a series of papers of Caccetta [9–11] with Huang [13]. Stability and vulnerability questions, especially their connections with communication networks were studied by Chung [14] and Chung and Garey [15]. Erdős and Howorka [16] asked for the maximum number of edges in a distance-critical graph on n vertices. Vertex-critical graphs have been studied by Gliviak [21] with Plesník [22] and Boals et al. [2].

The aim of this paper is to determine the minimum number of edges in vertex-diameter-2-critical graphs on n vertices for $n \geq 23$. Huang and Yeo [23] proved that this minimum is between $(5n - 12)/2$ and $(5n - 29)/2$. Ando and Egawa [1] proved the lower bound $(5n - 17)/2$ for $n \geq 23$. When n is odd, Huang and Yeo [23] and Ando and Egawa [1] both found vertex-diameter-2-critical graphs with $(5n - 17)/2$ edges shown later in Figure 3 (with $m = 0$). We prove the following.

Theorem 1.1. *Suppose G is a vertex-diameter-2-critical graph with $n = n(G) \geq 23$. Then*

$$e(G) \geq \begin{cases} (5n - 17)/2 & \text{if } n \text{ is odd,} \\ (5n/2) - 7 & \text{if } n \text{ is even.} \end{cases} \tag{1.1}$$

In next section, we present several graphs achieving these bounds, they are called *extremal graphs*. To prove the lower bound it is sufficient to show that if a vertex-diameter-2-critical graph G on $n = n(G) \geq 23$ vertices has at most $(5n/2) - 8$ edges, then it has at least $(5n - 17)/2$ edges, and n is odd. For completeness, we include the odd case and reprove Ando and Egawa's result [1]. This makes this paper only a few lines longer.

2. EXTREMAL GRAPHS

In this section, we present some extremal vertex-diameter-2-critical graphs achieving the bounds of Theorem 1.1. We believe that these examples give all extremal graphs, at least for $n \geq 23$. For smaller n's, as it was pointed out in [1] and [23], there are graphs with fewer edges. For example for $n = 10$, the Petersen graph gives $e(G) = (5n - 20)/2$.

Example 2.1. Suppose that $k \geq 0$ and let $n = 14 + 2k$. We shall define the vertex-diameter-2-critical graph G^n on n vertices. Arrange the vertices of G^n into six subsets, $C = \{v, u, w\}$, $S = \{s_0, s_1, s_2\}$, $A = \{a_0, a_1, a_2\}$, $B = \{b_0, b_1, b_2\}$, $X = \{x_0, x_1, \ldots, x_k\}$, and $Y = \{y_0, y_1, \ldots, y_k\}$.

For the edge set of G^n, we join u to every vertex in S and X; w to every vertex in S and in Y; v to all the vertices but S and C; we join a_i and b_i to s_i for $0 \leq i \leq 2$; we join a_1 and a_2 to a_0; we join b_1 and b_2 to b_0; and we join s_1 to s_2. Finally, we make a matching between X and Y by joining x_i to y_i for $0 \leq i \leq k$.

G^n is shown in Figure 1. We see that $e(G^n) = 28 + 5k = (5n/2) - 7$. To show that G^n is vertex-diameter-2-critical, it is sufficient to check that u, v, a_0, a_1, s_0, s_1, x_0 are critical. Deleting these vertices results in each of the distances $d(s_0, x_0)$, $d(a_0, x_0)$, $d(s_0, a_1)$, $d(s_1, a_0)$, $d(u, a_0)$, $d(a_1, s_2)$, $d(u, y_0)$ being greater than 2.

Example 2.2. Suppose that $k \geq 0$ and $m \geq 0$ and let $n = 9 + 2k + 2m$. We shall define the vertex-diameter-2-critical graph G^n_k on n vertices. Arrange

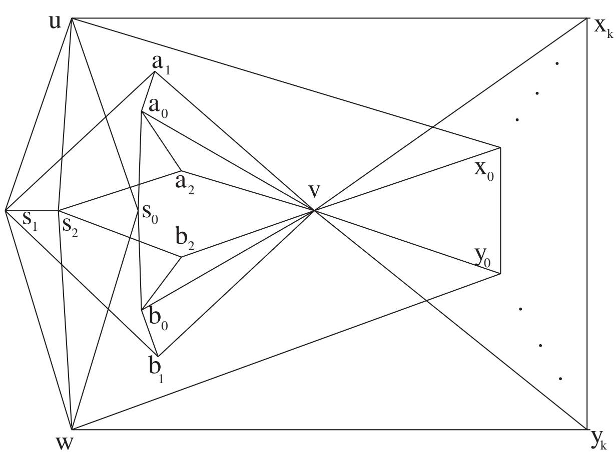

FIGURE 1.

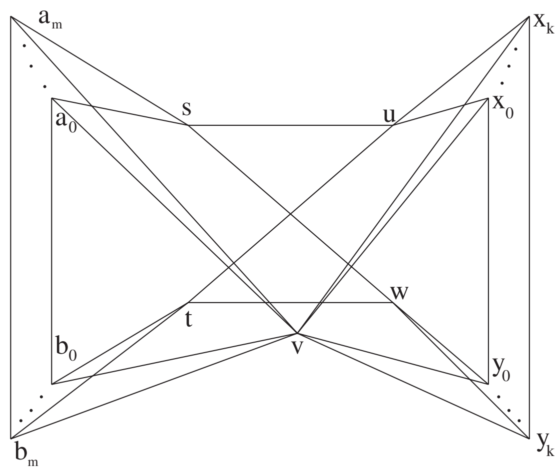

FIGURE 2.

the vertices of G_k^n into five subsets, $C = \{v, u, w, s, t\}$, $A = \{a_0, a_1, \ldots, a_m\}$, $B = \{b_0, b_1, \ldots, b_m\}$, $X = \{x_0, x_1, \ldots, x_k\}$, and $Y = \{y_0, y_1, \ldots, y_k\}$.

For the edge set of G_k^n, we join u to every vertex in X and in $\{s, t\}$; w to every vertex in Y and in $\{s, t\}$; v to all the vertices but C; we join every vertex of A to s, and every vertex of B to t. Finally, we make a matching between X and Y by joining x_i to y_i for $0 \leq i \leq k$; and a matching between A and B by joining a_i to b_i for $0 \leq i \leq m$.

G_k^n is shown in Figure 2. We see that $e(G_k^n) = 14 + 5k + 5m = (5n - 17)/2$. To show that G_k^n is vertex-diameter-2-critical, it is sufficient to check that u, v, x_0 are critical. Deleting u, v, x_0 results in each of $d(s, x_0)$, $d(a_0, x_0)$, $d(u, y_0)$ being greater than 2.

Example 2.3. Suppose that $k \geq 0$, and $m = 0$ or 1 and let $n = 7 + 2k + 2m$. We shall define the vertex-diameter-2-critical graph H_k^n on n vertices. Arrange the vertices of H_k^n into five subsets, $C = \{v, u, w\}$, $X = \{x_0, x_1, \ldots, x_k\}$, $Y = \{y_0, y_1, \ldots, y_k\}$, A, and B, where

$$\begin{cases} A = \{a_0\} \quad \text{and} \quad B = \{b_0\} & \text{if } m = 0, \\ A = \{a_0, a_1\} \quad \text{and} \quad B = \{b_0, b_1\} & \text{if } m = 1. \end{cases}$$

For the edge set of H_k^n, we join u to every vertex in X and in B; w to every vertex in Y and in B; v to every vertex in X, in Y and in A; and we join b_1 to b_0 if $m = 1$. Finally, we make a matching between X and Y by joining x_i to y_i for $0 \leq i \leq k$; and a matching between A and B by joining a_i to b_i for $0 \leq i \leq m$.

H_k^n is shown in Figure 3. We see that $e(H_k^n) = 9 + 5k + 5m = (5n - 17)/2$. To show that H_k^n is vertex-diameter-2-critical, it is sufficient to check that u, v, a_0, b_0, x_0, a_1, b_1 are critical. Deleting these vertices results in each of the distances $d(b_0, x_0)$, $d(a_0, x_0)$, $d(b_0, v)$, $d(a_0, u)$, $d(u, y_0)$, $d(b_1, v)$, $d(a_1, u)$ being greater than 2.

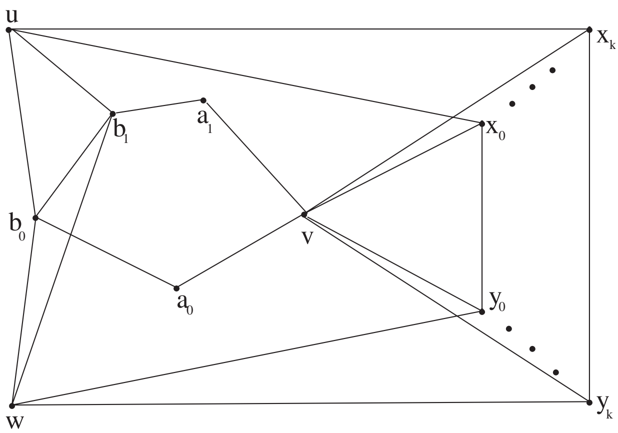

FIGURE 3.

3. STARTING THE PROOF OF THEOREM 1.1

The main idea of the proof, although it is not necessary immediately transparent from the series of Lemmas presented in next four sections, is that there is a deep connection between intersecting hypergraphs and diameter 2 graphs with only linearly many (i.e., $O(n)$) edges. This observation is due to Pach and Surányi [29]. Generally speaking, it means that in a diameter 2 graph G, a set U of size $o(n)$ that usually contains all vertices with high degree can be found such that $N(x) \cap N(y) \cap U \neq \emptyset$ for $x, y \in V \backslash U$. Applying this method to our case, when a vertex-diameter-2-critical graph G has at most $5n/2$ edges, for $n > 1,000$, it can be easily proved that every vertex of small degree must have at least two neighbors in U, which already requires at least $(2 - o(1))n$ edges, and being vertex-critical also implies that every vertex in $V \backslash U$ must have another neighbor in $V \backslash U$, which forces another $n/2 - o(n)$ edges. Thus all but $o(n)$ vertices have degree 3 and have exactly one degree-3 neighbor and two neighbors of high degree. Below we worked out this general argument and replaced it by a series of lemmas to prove Theorem 1.1 except for $n < 23$.

Suppose that G is a vertex-diameter-2-critical graph on $n = n(G)$ vertices, and let $V = V(G)$. We observe that

$$\text{if } u \neq v \in V(G), \text{ then } N(u) \not\subseteq N[v]. \tag{3.1}$$

In particular, if G is a vertex-diameter-2-critical graph, then $\delta(G) \geq 2$.

A tool we frequently use is the following identity, which is obviously true for every graph G and subset $U \subset V = V(G)$.

$$e(G) = e(G[U]) + \sum_{z \in V \backslash U} \left(|N(z) \cap U| + \tfrac{1}{2}|N(z) \cap (V \backslash U)|\right). \tag{3.2}$$

For every vertex $z \in V(G)\backslash U$, we define the *weight* $\omega(z) = \omega_G(z, U)$ of z,

$$w_G(z, U) = |N_G(z) \cap U| + \tfrac{1}{2}\left|N(z) \cap (V(G)\backslash U)\right| = \tfrac{1}{2}(\deg(z) + |N_G(z) \cap U|). \tag{3.3}$$

Usually we use (3.2) in the form $e(G) = e(G[U]) + \sum_{z \in V\backslash U} \omega(z)$.

In the rest of this section, we shall prove the lower bound (1.1) for G when $\delta(G) \geq 4$ and $n \geq 19$. If $\delta(G) \geq 5$, then $e(G) \geq 5n/2$ and we are done. Assume $\delta(G) = 4$. Let x be a vertex of G with $\deg(x) = 4$. We shall apply (3.2) to $U = N[x]$. Since every vertex not in U is adjacent to at least one vertex in U, we have that

$$e(G) = e(G[U]) + \tfrac{5}{2}(n-5) + \tfrac{1}{2}\sum_{z \in V\backslash U}(\deg(z) - 4) + \tfrac{1}{2}\sum_{z \in V\backslash U}(|N(z) \cap U| - 1). \tag{3.4}$$

Now the sum of the first two terms is at least $(5n - 17)/2$ and the last two sums are nonnegative. Thus G satisfies (1.1) if n is odd.

Suppose that n is even and that $e(G) < 5n/2 - 7$. We obtain from (3.4) that $e(G[U]) = 4$ and that all vertices but at most one, say z, in $V\backslash U$ have degree exactly 4 and are joined to U by exactly one edge; moreover for z, we have that $(\deg(z) - 4) + (|N(z) \cap U| - 1) \leq 1$. This implies that $\deg(z) \leq 5$ and that $\sum_{y \in N(x)} \deg(y) \leq (n-1) + (|N(z) \cap U| - 1) \leq n$. Thus there exists a vertex y adjacent to x such that $\deg(y) \leq n/4$. Consider a vertex w adjacent to y other than x and z. Note that $\deg(w) = 4$ and that $N(w) \cap U = \{y\}$. There are at most $\sum_{v \in N(w)} \deg(v) \leq \deg(y) + \deg(z) + 4 + 4 \leq n/4 + 13 < n - 1$ vertices whose distances from w are 1 or 2. This contradicts the fact that $\mathrm{diam}(G) = 2$.

4. TREES IN VERTEX-DIAMETER-2-CRITICAL GRAPHS

In this section, we shall define some subtrees in a vertex-diameter-2-critical graph G. The lemmas and inequalities proved in this section will be used in Sections 5, 6, and 7 to obtain properties of extremal graphs and to prove lower bounds for the number of the edges. Suppose that A, S, and W are nonempty subsets of $V = V(G)$ satisfying the following four conditions.

T1. $A, S \subset W \subseteq V$, and $A \cap S = \emptyset$.

T2. Vertices in A have the same neighbors in $V\backslash W$, that is, $N(a) \cap (V\backslash W) = N(a') \cap (V\backslash W)$ for all $a, a' \in A$; furthermore, vertices in S have the same neighbors in $V\backslash W$, that is, $N(s) \cap (V\backslash W) = N(s') \cap (V\backslash W)$ for all $s, s' \in S$.

T3. Every vertex in A is adjacent to at least one vertex in S.

T4. The farthest distance $d_{G[W]}(A, S)$ between A and S in the subgraph $G[W]$ induced by W is at most 2.

For every $a \in A$, we shall define a tree $T = T(a)$ rooted at the vertex a.

For all $s \in S$ not adjacent to a, pick the middle vertex of one (a, s)-path of length 2 in $G[W]$, call it $t(s)$. The vertex set $V(T)$ of T is $\{a\} \cup S$ together with all the middle vertices $t(s)$ chosen above, that is, $V(T) = \{a\} \cup S \cup \{t(s) : s \in S \backslash N(a)\}$. Let $A_T = A \cap V(T)$. Let L be the set of middle vertices in $W \backslash (A \cup S)$, that is, $L = V(T) \backslash (A \cup S)$.

The edge set $E(T)$ of T contains all the edges between a and S in G, and the edges of the one chosen path of length 2 between a and every vertex in S not adjacent to a, that is, $E(T) = \{as : s \in S \cap N(a)\} \cup \{at(s), t(s)s : s \in S \setminus N(a)\}$.

Lemma 4.1. *Suppose that G is a vertex-diameter-2-critical graph with subsets A, S, W satisfying* **T1**, **T2**, **T3**, **T4**, *that T, $A_T = A \cap V(T)$, and $L = V(T) \backslash (A \cup S)$ are as defined above. Then we have that*

$$|S \backslash N(a)| \geq |A_T \backslash a| + |L| + |\{t(s) : s \in S \backslash N(a)\} \cap S|. \tag{4.1}$$

Suppose further that every vertex in $A \backslash A_T$ has at least two neighbors in $G[W]$. Then we have

$$e(G[W]) \geq \tfrac{3}{2}|A| - \tfrac{1}{2}|A_T| + |S| + |L| - 1 + \tfrac{1}{2} \sum_{z \in W \backslash (A \cup S \cup L)} |N(z) \cap W| + \tfrac{1}{2} \sum_{s \in S} |\{sz \in G[W] : s \in S, z \notin A \setminus A_T, sz \notin T\}|. \tag{4.2}$$

Proof. Since the number of the vertices in S not adjacent to the root a of the tree T is at least as large as the number of the middle vertices chosen, we have that (4.1) holds.

$e(G[W]) = (\sum_{v \in W} |N(v) \cap W|)/2$ counts the terms in **I**, **II**, **III**, and **VI**. In **II**, **III**, and **IV**, every edge contributes $1/2$ to the sum at each of its two endvertices except for the edges between S and $A \backslash A_T$ in **II**, each of which contributes 1 only at its endvertex in $A \backslash A_T$.

I. The edges of T, whose number is $e(T) = |A_T| + |S| + |L| - 1$.

II. The edges incident to the vertices of $A \backslash A_T$ in $G[W]$. Their number is at least

$$\sum_{a \in A \backslash A_T} \left(|N(a) \cap S| + \tfrac{1}{2}|N(a) \cap (W \backslash S)|\right) \geq \tfrac{3}{2}(|A| - |A_T|).$$

Note that the edges between S and $A \backslash A_T$ are counted only in **II**.

III. The edges incident to $W \backslash (A \cup S \cup L)$ in $G[W]$. Their number is $\sum_{z \in W \backslash (A \cup S \cup L)} |N(z) \cap W|/2$.

IV. The edges incident to S in $G[W]$ that we did not count in **I**, **II**, **III**. Their number is at least $(\sum_{s \in S} |\{sz \in G[W]; s \in S, z \notin A \backslash A_T, sz \notin T\}|)/2$. ■

Suppose that G is a vertex-diameter-2-critical graph with subsets A, S, W satisfying **T1**, **T2**, **T3**, **T4**. We shall state the following five properties that are possessed by some graphs we shall discuss in Section 5, 6, and 7.

R1. Every vertex in A is either the only neighbor in A of s for some $s \in S$, or the only common neighbor of nonadjacent vertices a' and s for some $a' \in A$, and $s \in S$.

R2. Every vertex in S is either the only neighbor in S of a for some $a \in A$, or the only common neighbor of nonadjacent vertices s' and a for some $s' \in S$, and $a \in A$.

R3. No vertex in S is adjacent to any vertex in $W\backslash(A \cup S)$.

R4. The farthest distance $d_{G[A\cup S]}(A, S)$ between A and S in the subgraph induced by $A \cup S$ is at most 2.

R5. For all choices of the root $a \in A$ and the tree T, the following inequality holds (where $A_T = A \cap V(T)$ and $L = V(T)\backslash(A \cup S)$ are defined as above) $(|S| - |A_T|) + \sum_{s\in S} |\{sz \in G[W] : s \in S, z \notin A\backslash A_T, sz \notin T\}| \leq 1$.

Lemma 4.2. *Suppose that G is a vertex-diameter-2-critical graph with subsets A, S, W satisfying* **T1**, ..., **T4**, *and* **R1**, ..., **R4**. *Then $|A| + |S|$ is even if $|S| \leq 2$.*

Suppose further that **R5** *also holds and that $|S| \geq 3$. Then we have that $e(G[A \cup S]) \geq 2|A|$ and that $|A| \geq |S|$.*

Proof. First, suppose $S = \{s\}$. For all $a \in A$, a is adjacent to s. If $|A| \geq 2$, then vertices in A violate **R1**. Thus, $|A| + |S| = 2$ is even.

Second, suppose $S = \{s_1, s_2\}$. By **T2**, s_1 and s_2 have the same neighbors in $V\backslash W$. By **R3** and (3.1), we have that $|A| \geq 2$. If $|A| > |S|$, then there is a vertex a in A such that a is not the only neighbor of s_i in A for $i = 1, 2$. By **R1**, without loss of generality, a is the only common neighbor of nonadjacent vertices a' and s_1 for some other vertex a' in A. By **T3**, a' is adjacent to s_2 and $s_1 s_2$ are not adjacent. We observe that a' is not the only neighbor of s_2 in A, otherwise, by **T3** and **R4**, vertices in $A\backslash\{a, a'\}$ are adjacent to s_1 and a', which contradicts the fact that a is the only common neighbor of a' and s_1. Since a' is not the only neighbor of s_2 in A, **R1** implies that a' is the only common neighbor of nonadjacent vertices $a'' \in A$ and s_2. However, $a'' = a$. By **T3**, a'' is joined to s_1, thus $a'' \in N(a') \cap N(s_1)$ whose only member is a. Thus $s_1 a a' s_2$ is an induced path, and a and a' determine each other, $N(a') \cap N(s_1) = \{a\}$ implies $N(a) \cap N(s_2) = \{a'\}$ and vice versa. Thus, A consists of pairs of vertices $\{a_1, a_2\}$ such that a_i is the unique common neighbor of a_{2-i} and s_i such that a_{2-i} and s_i are not adjacent to each other. Hence $|A|$ is even.

Finally, we suppose $|S| \geq 3$. We shall show that S induces no edge in G. Suppose $a \in A$, and let T, A_T be as defined before. Since each edge induced by S but not in T contributes 2 to the sum in **R5**, we observe that all edges in $G[S]$ are in T for every choice of the root a. Consequently, $|E(G[S])| = |S| - |\{s \in S : s \in N(a) \text{ or } (s \notin N(a) \text{ and } t(s) \in A)\}| \leq |S| - |A_T| \leq 1 - \sum_{s\in S} |\{sz \in G[W] : s \in S,$

$z \notin A \backslash A_T, sz \notin T\}|$, by **R5**. This implies that S induces at most one edge in G. In addition, if s_1 is adjacent to s_2 for some $s_1, s_2 \in S$, then by **R2**, a vertex s_3 in $S \backslash \{s_1, s_2\}$ is the only neighbor in S of a^* for some $a^* \in A$, so a^* is adjacent to neither of s_1, s_2. This contradicts the earlier assertion that the edge $s_1 s_2$ has to be in the tree T rooted at a^*, thus justifying the claim that S induces no edge in G.

By **R2**, every vertex in S is the only neighbor in S of a for some $a \in A$, and this implies that $|A| \geq |S|$. By **R4**, the farthest distance $d_{G[A \cup S]}(A, S)$ between A and S in the subgraph induced by $A \cup S$ is at most 2. We partition A into three subsets as follows:

$$A_1 = \{a \in A : |N(a) \cap S| = 1, |N(a) \cap A| \geq 2\},$$
$$A_2 = \{a \in A : |N(a) \cap S| = 1, |N(a) \cap A| = 1\}, \; A_3 = \{a \in A : |N(a) \cap S| \geq 2\}.$$

Note that if $|N(a) \cap S| = 1$, then $|N(a) \cap A| \geq 1$, since $d_{G[A \cup S]}(A, S) \leq 2$, $|S| \geq 3$, and S induces no edge in G.

We observe that for every vertex $a \in A_2$, we have that $N(a) \cap A \subseteq A_3$, since the distances between a and the vertices in S not adjacent to a are at most 2, and there are at least two vertices in S not adjacent to a. We have that

$$\begin{aligned} e(G[A \cup S]) = e(G[A, S]) + e(G[A]) &= \sum_{a \in A} \left(|N(a) \cap S| + \tfrac{1}{2}|N(a) \cap A|\right) \\ &\geq \sum_{a \in A_1} \left(|N(a) \cap S| + \tfrac{1}{2}|N(a) \cap A|\right) + \sum_{a \in A_2} \left(|N(a) \cap S| + |N(a) \cap A_3|\right) \\ &\quad + \sum_{a \in A_3} |N(a) \cap S| \geq 2|A|. \end{aligned}$$

Note that we count the edges between A_2 and A_3 only in the second sum. ■

5. THE PROOF WHEN *G* HAS SEVERAL DEGREE-2 VERTICES

In this section, we present some properties of vertex-diameter-2-critical graphs. This leads to the proof of Theorem 1.1 when the graph G has more than one degree-2 vertices. The following three observations in Lemma 5.1 were discovered by Ando and Egawa.

Lemma 5.1 [1]. *Suppose that G is a vertex-diameter-2-critical graph. Then the following three statements hold:*

1. *If there is an adjacent pair of degree-2 vertices, then G is the 5-cycle C_5.*
2. *If $n(G) > 5$, then there is a unique vertex v^* adjacent to all degree-2 vertices.*
3. *If $n(G) > 5$, then $Y = \bigcup_{x \in V, deg(x)=2} (N(x) \backslash \{v^*\})$ induces a complete subgraph of G.*

Proof. 1. Suppose that x and y are an adjacent pair of degree-2 vertices, that $N(x) = \{y, a\}$, and that $N(y) = \{x, b\}$. Since G is critical, we have that xy, xa, and yb are the only edges induced by $\{x, y, a, b\}$ in G. Since $d(a, b) \leq 2$, there exists a vertex c adjacent to a and b. If there exists another vertex $z \neq c$ in

$V(G)\setminus\{x,y,a,b\}$, then z is not critical, since every vertex in $V(G)\setminus\{x,y,a,b\}$ is adjacent to both a and b. Thus G is isomorphic to the 5-cycle C_5.

2. Suppose that G has $m \geq 2$ degree-2 vertices, $v_1, v_2, \ldots, v_m$. Note that these vertices induce no edge in G. The fact that $\mathrm{diam}(G)=2$ implies that v_1 and v_2 have a common neighbor v^*. Suppose that $N(v_1)=\{v^*, w_1\}$ and that $N(v_2)=\{v^*, w_2\}$. We have that v^* is adjacent to neither w_1 nor w_2, since v_1 and v_2 are critical. Suppose that v_i is another degree-2 vertex. The fact $\mathrm{diam}(G)=2$ implies that v_i is either adjacent to v^*, or adjacent to w_1 and w_2. However, in the latter case, we have that $d(v_i, v^*)>2$, a contradiction.

3. Suppose that $N(v_j)=\{v^*, w_j\}$ for $1 \leq j \leq m$. We have that v^* is adjacent to none of $w_1, w_2, \ldots, w_m$, since v_j is critical for $1 \leq j \leq m$. The fact that $d(v_i, w_j) \leq 2$ implies that w_i is adjacent to w_j for $1 \leq i,j \leq m$, $i \neq j$. ∎

Suppose that G is a vertex-diameter-2-critical graph on $n=n(G)$ vertices, that G has $m \geq 2$ degree-2 vertices, $v_1, v_2, \ldots, v_m$, and that they have common neighbor v^* and the other neighbor $w_1, w_2, \ldots, w_m$, respectively. By Lemma 5.1, $w_1, w_2, \ldots, w_m$ induce a complete subgraph on m vertices in G. Denote $C=\{v^*, v_1, v_2, \ldots, v_m, w_1, w_2, \ldots, w_m\}$, and write $V=V(G)$. Every vertex in $V\setminus C$ is either adjacent to v^* but to none of $w_1, w_2, \ldots, w_m$, or adjacent to all of $w_1, w_2, \ldots, w_m$ but not to v^*, since v_i is the only common neighbor of v^* and w_i for $1 \leq i \leq m$, and by the fact $\mathrm{diam}(G)=2$. Let A be the set of vertices in $V\setminus C$ adjacent to v^* and S be the set of vertices in $V\setminus C$ adjacent to $w_1, w_2, \ldots, w_m$. Note that $V=C\cup A\cup S$.

In the following Claim 1 and Claim 2, we suppose that G is a vertex-diameter-2-critical graph on $n=n(G)\geq 11$ vertices, and that G has at most $5n/2-8$ edges.

Claim 1. *Suppose that G has $m \geq 3$ degree-2 vertices. Then $A \neq \emptyset$ and $|S| \geq 2$.*

First, if $A=\emptyset$, then we have that

$$e(G)=e(G[C])+e(G[C,S])+e(G[S]) \geq 2m+\frac{m(m-1)}{2} + m(n-(2m+1)) \geq \frac{5n-15}{2},$$

if $n \geq 11$ and $m \geq 3$. So, we may suppose $A \neq \emptyset$. Note that vertices in A have distances at most 2 with $w_1, \ldots, w_m$, so $S \neq \emptyset$.

Second, if $S=\{s\}$, then every vertex in A is adjacent to s. If $|A|>1$, then vertices in A are not the only common neighbor of v^* and s, and this is a contradiction. Thus, $|A|=1$. This implies that the vertex in A has degree 2, and this is a contradiction. So, $|S| \geq 2$.

Claim 2. *If G has exactly two degree-2 vertices, then $A \neq \emptyset$ and $S \neq \emptyset$.*

Since $\deg(v^*) \geq 3$, we have that $A \neq \emptyset$. Since vertices in A have distances at most 2 with w_1 and w_2, every vertex in A is adjacent to some vertices in S, in particular, $S \neq \emptyset$.

Lemma 5.2. *Suppose that G is a vertex-diameter-2-critical graph on $n = n(G)$ vertices, that G has at most $5n/2 - 8$ edges, and that $n \geq 11$. Then G has at most two degree-2 vertices.*

Proof. Suppose that A, S, and $W = A \cup S$ are as defined in the paragraph after the proof of Lemma 5.1. It is to see that A, S, and W satisfy all assumptions of Lemma 4.1. Take $a^* \in A$ and consider a tree T rooted at a^* defined in the preceding section. By (4.2), we have that

$$e(G[W]) \geq \frac{3|A|}{2} - \frac{|A_T|}{2} + |S| - 1.$$

Suppose G has m degree-2 vertices. If $m \geq 3$, then we have that

$$\begin{aligned}
e(G) &= e(G[C]) + e(G[C,W]) + e(G[W]) \\
&\geq \left(2m + \frac{m(m-1)}{2}\right) + (|A| + m|S|) + \frac{3|A|}{2} - \frac{|A_T|}{2} + |S| - 1 \\
&= \left(2m + \frac{m(m-1)}{2}\right) + \frac{5|A|}{2} + \frac{5|S|}{2} + \frac{|S| - |A_T|}{2} + (m-2)|S| - 1 \\
&\geq \left(2m + \frac{m(m-1)}{2}\right) + \frac{5(|A| + |S|)}{2} + 1 \geq \frac{5(2m+1) - 15}{2} + \frac{5(|A| + |S|)}{2} \\
&= \frac{5n - 15}{2}.
\end{aligned}$$

This contradicts the fact that $e(G) \leq (5n/2) - 8$, thus we have $m \leq 2$. ■

Lemma 5.3. *Suppose that G is a vertex-diameter-2-critical graph on $n = n(G)$ vertices. If G has exactly two degree-2 vertices and $n \geq 7$, then G satisfies (1.1).*

Proof. Suppose that $e(G) \leq (5n/2) - 8$. We have to show that $e(G) \geq (5n - 17)/2$ and that n is odd. Suppose that A, S, and $W = A \cup S$ are as defined in the paragraph after the proof of Lemma 5.1. It is easy to see that A, S, and W satisfy all assumptions of Lemma 4.1. Take $a^* \in A$ and consider a tree T rooted at a^* defined in the preceding section. By (4.2), we have that

$$e(G[W]) \geq \frac{3|A|}{2} - \frac{|A_T|}{2} + |S| - 1.$$

This implies that

$$\begin{aligned}
e(G) &= e(G[C]) + e(G[C,W]) + e(G[W]) \geq 5 + (|A| + 2|S|) + \frac{3|A|}{2} - \frac{|A_T|}{2} \\
&\quad + |S| - 1 = 4 + \frac{5(|A| + |S|)}{2} + \frac{|S| - |A_T|}{2} \geq 4 + \frac{5(n-5)}{2} + \frac{|S| - |A_T|}{2} \\
&= \frac{5n - 17}{2} + \frac{|S| - |A_T|}{2}.
\end{aligned}$$

Note that by (4.1), $|S|-|A_T|\geq 0$. It is sufficient to show that $|A|+|S|$ is even. Since we assume that $e(G)\leq 5n/2-8$, we have that $|S|-|A_T|\leq 1$, and that **R5** holds. It is easy to see that **R1**, ..., **R4** hold. By Lemma 4.2, we have that $|A|+|S|$ is even if $|S|\leq 2$, and that $e(G[A\cup S])\geq 2|A|$ and $|A|\geq|S|$ if $|S|\geq 3$. Suppose $|S|\geq 3$. We have that

$$\begin{aligned} e(G) &= e(G[C])+e(G[C,W])+e(G[W])\geq 5+(|A|+2|S|)+2|A| \\ &= 5+\frac{5(|A|+|S|)}{2}+\frac{|A|-|S|}{2}\geq 5+\frac{5(n-5)}{2}=\frac{5n-15}{2} \end{aligned}$$

and this is a contradiction. ■

6. EXTREMAL GRAPHS CONTAIN BULLS

Throughout this section, we suppose that G is a vertex-diameter-2-critical graph with at most one degree-2 vertex, that $\delta(G)<4$ and that $e(G)\leq(5n/2)-8$. Define the graph Q with vertex set $\{x, y, u, v, w\}$ and edge set $\{xy, xu, xv, yv, yw\}$. (Sometimes this graph is called the *bull*.) The aim of this section is to prove that if G has at least 23 vertices, then G contains an induced copy of Q. Actually, this is the only part of the proof when the lower bound $n(G)\geq 23$ is used.

Lemma 6.1. *Suppose that the degree-3 vertices are adjacent neither to each other nor to the degree-2 vertex. Then $n(G)\leq 22$.*

Proof. Let x_1 be a vertex of minimum degree, $A_1=N[x_1]$. We have that $|A_1|=\delta+1$, i.e., it is 3 or 4, and all vertices in $V\backslash A_1$ have degree at least 3. Apply (3.2) to $U=N[x_1]$. If every vertex $z\in V\backslash U$ has $\omega(z)\geq 5/2$, then (3.2) implies that $e(G)\geq\delta+(n-\delta-1)5/2=5n/2-(3\delta+5)/2>5n/2-8$, a contradiction. So there exists a vertex $x_2\in V\backslash U$ with $(\deg(x_2)+|N(x_2)\cap U|)/2\leq 2$. Recall that G has at most one degree-2 vertex. As $|N(x_2)\cap U|\geq 1$ and $\deg(x_2)\geq 3$, these imply that here equalities hold.

Let $A_2:=A_1\cup N[x_2]$ and $\{c\}:=N(x_1)\cap N(x_2)$. Suppose that A_i has already been defined, then let x_{i+1} be a degree-3 vertex such that $x_{i+1}\notin A_i$ and $|A_i\cap N(x_{i+1})|=1$. If such a vertex exists then necessarily $N(x_{i+1})\cap A_i=\{c\}$. This process stops after some steps, we obtain the independent vertices $x_1, x_2, \ldots, x_t$ (where $t\geq 2$), all adjacent to c and the set $A:=A_t$ with

$$|A|=3t+(\delta-2)$$

such that every degree-3 vertex not in A is joined to at least two vertices of A. We have the following lower bound for the number of edges $e(A):=e(G[A])$

$$e(A)\geq\begin{cases} 7 & =|A|+1 \text{ if } t=2 \text{ and } \delta=2,\\ 8 & =|A|+1 \text{ if } t=2 \text{ and } \delta=3,\\ 2|A|-2+\frac{1}{2}t(t-5) & \text{for } t\geq 3.\end{cases} \tag{6.1}$$

If here equality holds for $t \geq 4$, then $G[A\backslash\{x_1, \ldots, x_t\}]$ has two components, a star with center c and a complete graph K_t meeting each $N(x_i)$ in one vertex.

To see this, first we have $|A| - t - 1$ edges joining $\{x_1, \ldots, x_t\}$ to $(A\backslash c)\backslash \{x_1, \ldots, x_t\}$. Second, there are $|N[c] \cap A| - 1$ edges joining c to $A\backslash\{c\}$. Third, observe that a vertex $y \in N(x_i)\backslash N[c]$ is joined to at least one vertex in $N(x_j)$ for all $1 \leq j \leq t, j \neq i$), otherwise it was not possible to reach x_j from y by a path of length at most 2. Thus all vertices of $A\backslash N[c]$, has at least $t - 1$ neighbors in $A\backslash N[c]$. We obtain that

$$\begin{aligned} e(A) &\geq (|A| - t - 1) + (|N[c] \cap A| - 1) + \frac{1}{2}(t-1)|A\backslash N[c]| \\ &= 2|A| - t - 2 + \frac{1}{2}(t-3)(|A\backslash N[c]|). \end{aligned}$$

(3.1) implies that $N(x_i) \not\subset N(c)$ thus $t \leq |A\backslash N[c]|$, which implies (6.1).

Let us note that applying (3.2) with $U = A$ and using (6.1), one can obtain $t \leq 6$, hence $|A| \leq 19$. However, we would not use this upper bound (only implicitly). Let M be the set of vertices not in A, which are not adjacent to c, the set L be the set of vertices not in $A \cup M$ whose degree is at least 4, and let $|M| = m$ and $|L| = \ell$.

We shall apply (3.2) to $U = A \cup M$. Since vertices in M are not adjacent to c, thus each of them is adjacent to at least one vertex in $N(x_j)$ for all $1 \leq j \leq t$, we have that $e(G[U]) \geq e(A) + tm$. Every vertex $z \in V\backslash U$ is adjacent to c and by (3.1) it is adjacent to at least another vertex in $V\backslash N(c)$, thus z is adjacent to at least two vertices in U. Then (3.2) implies that

$$\frac{5n}{2} - 8 \geq e(G) \geq e(A) + tm + \frac{5}{2}(n - |A| - m - \ell) + 3\ell.$$

Rearranging we have

$$5|A| - 2e(A) - 16 \geq \ell + (2t-5)m. \tag{6.2}$$

Next we consider $U = A \cup M \cup L$. We have that $e(G[U]) \geq e(A) + tm + 2\ell$. As the vertices in $V\backslash U$ have degree 3, and they are independent, $e(G[U, V\backslash U]) = 3(n - |U|)$. Thus

$$\begin{aligned} \frac{5n}{2} - 8 &\geq e(G) \geq e(G[U]) + e(G[U, V\backslash U]) \\ &\geq e(A) + tm + 2\ell + 3(n - |A| - m - \ell), \end{aligned}$$

hence

$$6|A| - 2e(A) - 16 \geq n - 2\ell + (2t-6)m. \tag{6.3}$$

Multiplying (6.2) by 2 and adding to (6.3) we obtain

$$16|A| - 6e(A) - 48 \geq n + (6t-16)m. \tag{6.4}$$

In case of $t \geq 3$, the left hand side of (6.4) is at most $-3t^2 + 27t - 32 - 4(3-\delta)$ by (6.1). This is at most 22 for $t = 3$ and $t \geq 6$, so we obtain $22 \geq n$ for these cases. For $t = 4$ and 5, the left hand side of (6.4) is at most 28. In these cases, we get $22 \geq n$ again if $e(A)$ exceeds the lower bound (6.1). Suppose now that equality holds in (6.1). As $t \geq 4$, we have that $t = |A \backslash N[c]|$, and that every degree-3 vertex x_i has exactly one neighbor adjacent to c. Consider $y \in N(x_2) \cap N(c)$. (3.1) implies that y has a neighbor z that is not connected to c. However, y is not connected to any other vertex in A, so $z \notin A$. Thus $M \neq \emptyset$, $m \geq 1$. Then (6.4) gives $20 \geq 28 - (6t - 16)m \geq n$ and we are done.

In the remaining case, $t = 2$. Then the left hand side of (6.4) is at most 16 by (6.1). It gives $20 \geq n$ for $m \leq 1$, so from now on we suppose that $m \geq 2$, especially $M \neq \emptyset$. Let k denote the minimum degree of vertices in M, let z_0 be a vertex in M with $\deg(z_0) = k$.

We distinguish three subcases.

I. $|A| = 6$. In this case, $\delta = 2$ and $e(A) \geq 7$.
II. $|A| = 7$ and $e(A) \geq 9$. In this case $\delta = 3$.
III. $|A| = 7$ and $e(A) = 8$. In this case $\delta = 3$.

We discuss cases I and II together. We shall apply (3.2) to $U = A$. Since every vertex b in M has $\deg(b) \geq k$, we know that $\omega(b) \geq (k+2)/2$. Every vertex b' in L has degree at least 4, so that $\omega(b') \geq 5/2$. Finally, every vertex $b'' \in V \backslash (A \cup M \cup L)$ has at least two neighbors in A, so again $w(b'') \geq 5/2$. Thus (3.2) implies that

$$\frac{5n}{2} - 8 \geq e(G) \geq e(A) + \frac{1}{2}(k+2)m + \frac{5}{2}(n - |A| - m).$$

Rearranging we get $5|A| - 2e(A) - 16 \geq (k-3)m$. Here, in cases I and II, the left hand side is at most 1. Thus $m \geq 2$ implies $1 > 1/m \geq k - 3$, and we obtain that $k = 3$.

Consider $B = A \cup N[z_0]$. We know that $|A| + 1 \leq |B| \leq |A| + 2 \leq 9$, since z_0 is adjacent to at least two and at most three vertices in A. The set L' is defined to be the set of vertices not in B whose degree is at least 4, and let $|L'| = \ell'$.

We shall apply (3.2) to $U = B$. Since z_0 is not adjacent to c, every vertex z not in B is adjacent to at least two vertices in B, thus $\omega(z) \geq 5/2$. Also, every vertex z' in L' has $\omega(z') \geq 3$. We have that

$$\frac{5n}{2} - 8 \geq e(G) \geq e(G[B]) + \frac{5}{2}(n - |B| - \ell') + 3\ell',$$

hence

$$5|B| - 2e(B) - 16 \geq \ell'. \tag{6.5}$$

Next we consider $U = B \cup L'$. We have that $e(G[U]) \geq e(B) + 2\ell'$. As the vertices in $V \backslash U$ have degree 3, and they are independent, $e(G[U, V \backslash U]) = 3(n - |U|)$. Thus

$$\frac{5n}{2} - 8 \geq e(G) \geq e(G[U]) + e(G[U, V \backslash U]) \geq e(B) + 2\ell' + 3(n - |B| - \ell'),$$

hence

$$6|B| - 2e(B) - 16 \geq n - 2\ell'. \tag{6.6}$$

Multiplying (6.5) by 2 and adding to (6.6), we obtain

$$16|B| - 6e(B) - 48 \geq n. \tag{6.7}$$

One can easily see, that here the left hand side is at most 18 in both cases I and II. Indeed, there are $e(A)$ edges induced by A, 3 edges incident to z_0. If $N(z_0) \subset A$, then $|B| = |A| + 1$, $e(B) = e(A) + 3 \geq |A| + 4$, so the left hand side of (6.7) is at most $10|A| - 56$, which is at most 14. If $N(z_0) \backslash A \neq \emptyset$, then the vertex in $N(z_0) \backslash A$ is adjacent to at least one vertex in A. So we have that $e(G[B]) \geq e(A) + 4$ and $|B| = |A| + 2$. Then (6.7) gives $n \leq 10|A| - 40 - 6(e(A) - |A|)$, which is at most 14 in the case I, and at most 18 in the case II.

Finally, we turn to case III. Observe that the graph $G[A]$ is unique. As $|A| = 7$ and $e(A) = 8$, the set $N(x_1) \cup N(x_2)$ contains exactly two edges. Both avoid the common vertex c and they are disjoint. Denote them by $y_1 y_2$ and $z_1 z_2$ where $N(x_i) = \{y_i, z_i, c\}$. Thus $N(x_1)$ contains no edge. Since this is the last case, from now on, we may suppose that we get $t = 2$ and the above 8-edge graph by starting our process with any degree-3 vertex x. We may also suppose that $N(x)$ has no edge for any degree-3 vertex x, and $N(x) \cup N(y)$ has exactly 2 edges if $\deg(x) = \deg(y) = 3$ and $|N(x) \cap N(y)| = 1$.

We show that $k \geq 4$ (where $k = \min_{z \in M} \deg(z)$) leads to $22 \geq n$. We shall apply (3.2) again to $U = A$. We already have $w(b) \geq 5/2$ for every $b \in V \backslash A$. Split $V \backslash (A \cup M \cup L)$ into two parts

$H_2 := \{x \in V \backslash A : x \in N(c), \deg(x) = 3 \text{ and } |N(x) \cap A| = 2\}$, and
$H_3 := \{x \in V \backslash A : x \in N(c), \deg(x) = 3 \text{ and } |N(x) \cap A| = 3\}$.

Thus (3.2) implies that

$$\frac{5n}{2} - 8 \geq e(G) \geq e(A) + \frac{1}{2} \sum_{x \in M} (2 + \deg(x)) + \frac{5}{2}\ell + \frac{5}{2}|H_2| + 3|H_3|.$$

Rearranging we get $5|A| - 2e(A) - 16 \geq |H_3| + \sum_{x \in M} (\deg(x) - 3)$, where the left hand side is 3, implying

$$3 \geq |H_3| + \sum_{x \in M} (\deg(x) - 3). \tag{6.8}$$

Hence $3 \geq |H_3| + m \geq m$. The left hand side of (6.2) is also 3, yielding $3 \geq \ell - m$. Thus we obtain the upper bound $6 \geq \ell$. We have $|A \cup (H_3 \cup M) \cup L| \leq 7 + 3 + 6 = 16$. Then $22 \geq n$ follows if we show that $|H_2| \leq 6$.

Consider an arbitrary vertex b in H_2. Note that b is adjacent to c and that it has a neighbor w not in A. As b is a degree-3 vertex and we have supposed that their neighborhoods contain no edge, we obtain that w is not adjacent to c, i.e., $w \in M$. This implies $\sum_{x \in M}(\deg(x) - 2) \geq |H_2|$. Adding this inequality to (6.8), we have $3 + m \geq |H_3| + |H_2|$. Hence $6 \geq |H_2|$, finishing the case $k \geq 4$.

Finally, consider the case $k = 3$. Again let $z_0 \in M$ be a degree-3 vertex. As $N(z_0)$ has no edge, z_0 is connected to at most one of y_1 and y_2, and similarly $|N(z_0) \cap \{z_1, z_2\}| \leq 1$. Equalities must hold, so we may suppose that, say, $N(z_0) \cap A = \{y_1, z_2\}$. Let us denote the third neighbor of z_0 by w, and let $B = A \cup N[z_0]$.

We claim that $G[B]$ is a 9-vertex graph with 12 edges as follows. It consists of a six-cycle $x_1 c x_2 z_2 z_0 y_1$ and three diagonal paths $x_1 z_1 z_2$, $y_1 y_2 x_2$, and $z_0 w c$. We have already described $G[A \cup \{z_0\}]$. We shall show that $N(w) \cap \{y_1, y_2, z_1, z_2\} = \emptyset$. The neighborhood $N(z_0)$ contains no edge, hence $y_1, z_2 \notin N(w)$. Consider the degree-3 vertices x_1 and z_0. They have a single common neighbor, y_1, so by our assumption, their neighbors $N(x_1) \cup N(z_0)$ contain exactly two edges, both avoiding the common neighbor y_1. As $z_1 z_2$ is an edge, the other edge is wc. We also obtained that w is not adjacent to z_1. Considering the degree-3 vertices x_2 and z_0, we obtain in the same way that w is not adjacent to y_2.

We shall apply (3.2) to $U = B$. Every vertex z not in B is adjacent to at least two vertices in B, thus $\omega(z) \geq 5/2$. The set L' is defined again as the set of vertices not in B whose degree is at least 4, and let $|L'| = \ell'$. Now (3.2) implies that

$$\frac{5n}{2} - 8 \geq e(G) \geq e\left(G[B]) + \frac{5}{2}(n - |B|\right) + \sum_{x \notin B}\left(w(x) - \frac{5}{2}\right),$$

hence

$$5 = 5|B| - 2e(B) - 16 \geq \sum_{x \in L'}(\deg(x) - 3) + \sum_{x \notin B}(|N(x) \cap B| - 2). \tag{6.9}$$

Here the first sum is at least ℓ', and we shall show that the second sum is at least 1, thus (6.9) gives $4 \geq \ell'$. Indeed, there is a vertex x in the common neighborhood of z_1 and y_2, since $d(z_1, y_2) \leq 2$. This x has a common neighbor with $N(z_0)$, too, implying $|N(x) \cap B| \geq 3$.

The left hand side of (6.6) is 14 so $4 \geq \ell'$ implies $22 \geq 14 + 2\ell' \geq n$. ■

Lemma 6.1 implies that for $n(G) \geq 23$, the graph G contains an adjacent pair of vertices x, y with $\deg(x), \deg(y) \leq 3$.

Lemma 6.2. *Suppose that $n(G) \geq 10$, and that G has a single degree-2 vertex x adjacent to a degree-3 vertex y. Then G contains an adjacent pair of degree-3 vertices having exactly one common neighbor.*

Proof. Suppose that $N(x) = \{y, x_1\}$ and that $N(y) = \{x, y_1, y_2\}$. We shall apply (3.2) to $U = N[x] \cup N[y]$. Since every vertex z not in U is adjacent to x_1 and at least one of y_1, y_2, and $\deg(z) \geq 3$, we have that

$$e(G) = 4 + \frac{5}{2}(n-5) + \frac{1}{2}\sum_{z \in V\setminus U}(\deg(z) - 3) + \frac{1}{2}\sum_{z \in V\setminus U}(|N(z) \cap U| - 2) \geq \frac{5n-17}{2}.$$

This implies that all vertices but at most one in $V\setminus U$ have degree 3 and are adjacent to exactly two vertices in U, and that G has at least $(n-5-1-2)/2$ adjacent pairs of degree-3 vertices in $(V\setminus U) \cap N(x_1)$ one adjacent to y_1, and the other adjacent to y_2. Namely, G has adjacent pairs of degree-3 vertices having exactly one common neighbor. ■

Lemma 6.3. *Suppose that $n(G) \geq 23$, that G has an adjacent pair of degree-3 vertices, x, y. Then G contains an adjacent pair of degree-3 vertices having exactly one common neighbor.*

Proof. If x and y have one common neighbor, then we are done. Suppose that x is adjacent to a_1, a_2, y, and that y is adjacent to x, b_1, b_2. First, we consider the case $\delta(G) = 3$. We shall prove that G has more than one adjacent pairs of degree-3 vertices. We shall apply (3.2) to $U = N[x] \cup N[y]$. Every vertex z not in U is adjacent to at least one of a_1, a_2 and at least one of b_1, b_2, so that $\omega(z) \geq 5/2$. Define $r = \sum_{z \in V\setminus U}(\omega(z) - 5/2)$. Now (3.2) implies that $5n/2 - 8 \geq e(G) \geq 5 + (n-6)5/2 + r$, hence

(i) $r \leq 2$;
(ii) at most four vertices in $V\setminus U$ have degree at least 4;
(iii) at most eight degree-3 vertices in $V\setminus U$ are adjacent to vertices of $V\setminus U$ whose degree is at least 4.

These imply that there are at least $(n-6) - 4 - 8 \geq 5$ degree-3 vertices in $V\setminus U$, which are adjacent to two vertices of U and to one degree-3 vertex of $V\setminus U$. Namely, there are at least three adjacent pairs of degree-3 vertices each adjacent to one of a_1, a_2 and to one of b_1, b_2.

Now we shall prove that G has an adjacent pair of degree-3 vertices having exactly one common neighbor. Suppose on the contrary that all adjacent pairs of degree-3 vertices have no common neighbor. Without loss of generality, z, w is an adjacent pair of degree-3 vertices in $V\setminus U$, where z is adjacent to a_1, b_1, w, and w is adjacent to a_2, b_2, z.

In addition to z, w, there are at least two adjacent pairs of degree-3 vertices in $V\setminus U$ each adjacent to two vertices in U. If s, t is another adjacent pair of degree-3 vertices in $V\setminus U$, then $\mathrm{diam}(G) = 2$ implies that each of s and t is adjacent to one of a_1, a_2, to one of b_1, b_2, to one of a_1, b_1, and to one of a_2, b_2. This implies that s, t are adjacent to a_1, b_2 and a_2, b_1, respectively. But, in addition to s, t, there are at least one adjacent pair of degree-3 vertices each adjacent to two vertices of U. The fact $\mathrm{diam}(G) = 2$ implies that each of them is adjacent to one of a_1, a_2, to one of b_1, b_2, to one of a_1, b_1, to one of a_2, b_2, to one of a_1, b_2, and to one of a_2, b_1. This is a contradiction since two vertices cannot meet all the six pairs. Thus, we have finished the case $\delta = 3$.

In the case $\delta = 2$, we shall prove Lemma 6.3 for all $n \geq 11$. There is a unique vertex c with $\deg(c) = 2$. If c is a neighbor of x or y, then Lemma 6.2 yields the desired pair of degree-3 vertices. So we may suppose that $c \notin N[x] \cup N[y]$, and $N(c) = \{a_2, b_1\}$. The vertices a_1 and b_2 are neighbors of $N(c)$, so $N[x] \cup N[y]$ induces at least 7 edges. Applying (3.2) again to $U = N[x] \cup N[y] \cup \{c\}$, we obtain that (i) $r \leq 1/2$, (ii) at most 1 vertex in $V \backslash U$ has degree at least 4, and (iii) at most two degree-3 vertices in $V \backslash U$ are adjacent to vertices of $V \backslash U$ whose degree is at least 4.

These imply that there are at least $(n - 6 - 1) - 1 - 2 \geq 1$ degree-3 vertices in $V \backslash U$, which are adjacent to two vertices of U and to one degree-3 vertex of $V \backslash U$. As above, we obtain a pair of adjacent degree-3 vertices $z, w \in V \backslash U$ where z is adjacent to a_1, b_1, w, and w is adjacent to a_2, b_2, z.

Let $U = N[x] \cup N[y] \cup \{c\} \cup N[z] \cup N[w]$. Then $|U| = 9$ and it contains at least 14 edges. Moreover, every vertex $u \notin U$ is joined to at least 3 vertices of $\{a_1, a_2, b_1, b_2\}$. Thus (3.2) gives $(5/2)n - 8 \geq e(G) \geq 14 + 3(n - 9)$, implying $5 \geq n/2$. ■

Lemma 6.4. *If G contains an adjacent pair of degree-*3 *vertices having exactly one common neighbor, then G contains a spanned copy of the bull Q.*

Proof. Let x and y be an adjacent pair of degree-3 vertices having exactly one common neighbor. Suppose that $N(x) = \{u, v, y\}$, and $N(y) = \{w, v, x\}$ and let $I = N[x] \cup N[y]$. By (3.1), the vertex u is not adjacent to v, and w is not adjacent to v. We shall show that u is not adjacent to w, i.e., $G[I] = Q$. Suppose on the contrary that u is adjacent to ω.

Let A be the set of vertices not in I, which are adjacent to only v in I, and let S be the set of vertices not in I, which are adjacent to exactly the two vertices u and w in I. Note that $S = V \backslash (A \cup I)$, since x is the only common neighbor of u, v, and y is the only common neighbor of v, w. We observe that $A \neq \emptyset$, since $\deg(v) \geq 3$. The fact $N(a) \nsubseteq N[v]$ implies that a is adjacent to some vertices in S for all $a \in A$, in particular, $S \neq \emptyset$.

Let H be the subgraph induced by $V \backslash I$, that is, $H = G[A \cup S]$. Then $\mathrm{diam}(G) = 2$ implies that for every vertex $a \in A$, $s \in S$, $d_H(a, s) \leq 2$. Note that if G has a degree-2 vertex, then the degree-2 vertex is in A. It is easy to see that A, S and $W = A \cup S$ satisfy **T1**, . . . ,**T4**. Take $a \in A$ and consider a tree T rooted at a defined in Section 5. If G has a degree-2 vertex, then we take the degree-2 vertex to be the root of the tree. So, every vertex in $A \backslash A_T$ has at least two neighbors in W. By (4.1) and (4.2), we have that $|S| \geq |A_T|$ and that $e(G[W]) \geq 3|A|/2 - |A_T|/2 + |S| - 1$, hence $e(G) = e(G[I]) + e(G[I,W]) + e(G[W])$

$$\geq 6 + (|A| + 2|S|) + \frac{3|A|}{2} - \frac{|A_T|}{2} + |S| - 1 = 5 + \frac{5(|A| + |S|)}{2} + \frac{|S| - |A_T|}{2}$$
$$\geq 5 + \frac{5(n-5)}{2}.$$

This is a contradiction, since $e(G) \leq 5n/2 - 8$. ∎

7. PROOF OF THEOREM 1.1

Let G be a vertex-diameter-2-critical graph on $n \geq 23$ vertices with $e(G) \leq 5n/2 - 8$ edges. To prove Theorem 1.1, we shall show that $e(G) \geq (5n - 17)/2$ and that n is odd. By the results of Section 3, we can assume that $\delta(G) < 4$. By Lemma 5.2 and 5.3, we may assume that G has at most one degree-2 vertex. By Lemma 6.1, G contains an adjacent pair of vertices with degrees at most 3. By Lemma 6.2 and 6.3, G contains an adjacent pair of degree-3 vertices having exactly one common neighbor. Then Lemma 6.4 implies that G contains the bull, Q, with vertex set $I = \{x, y, u, v, w\}$ as an induced subgraph. Let $V = V(G)$.

We partition $V \backslash I$ into five subsets A, B, C, S, and F according to their adjacency with $\{u, v, w\}$. The sets of neighbors in I of vertices in A, B, C, S, F are $\{v\}$, $\{u, v\}$, $\{w, v\}$, $\{w, u\}$, and $\{u, v, w\}$, respectively. Note that $V = I \cup A \cup B \cup C \cup S \cup F$. We denote the subgraph of G induced by $V \backslash I$ by H, that is, $H = G[V \backslash I]$. Note that if G has a degree-2 vertex, then the degree-2 vertex is in A.

We make a number of observations.

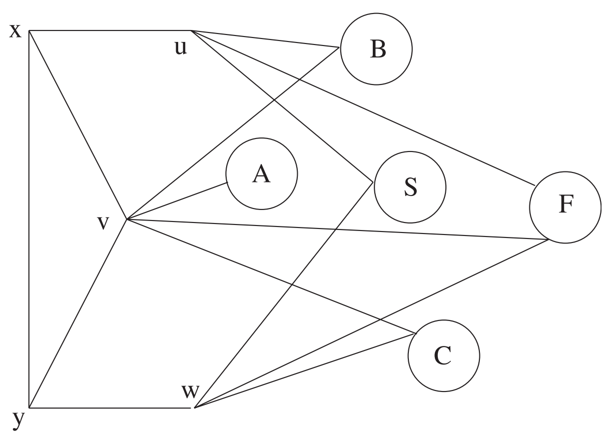

FIGURE 4.

Observation 1. $A \neq \emptyset$.

Note that $e(G[I]) = 5$ and that $e(G[I, V\backslash I]) = |A| + 2|B| + 2|C| + 2|S| + 3|F|$. Since every vertex in $V\backslash(A \cup I)$ has degree at least 3, if $A = \emptyset$, then $e(H) \geq (|B| + |C| + |S|)/2$, hence $e(G) = e(G[I]) + e(G[I, V\backslash I]) + e(H) \geq 5 + (2(n-5) + |F|) + (n - 5 - |F|)/2 = (5n - 15 + |F|)/2$. This is a contradiction.

Observation 2. *Every vertex $a \in A$ is the only common neighbor of nonadjacent pairs $\{a', s\}$ or $\{v, s'\}$, for some $a' \in A$, and s, $s' \in S$, since a is critical; in particular, every vertex in A is adjacent to some vertices in S, and $S \neq \emptyset$.*

Observation 3. *Every vertex f in F has $deg(f) \geq 4$.*

Since f is critical, f must be the only common neighbor of some pairs of its neighbors. But f is not the only common neighbor of u and w, since $S \neq \emptyset$.

It is easy to see that A, S and $W = V\backslash I$ satisfy **T1**, ..., **T4**. Take $a \in A$ and consider a tree T rooted at a defined in Section 5. If G has a degree-2 vertex, then we take the degree-2 vertex to be the root of the tree. So, every vertex in $A\backslash A_T$ has at least two neighbors in W. By (4.2), we have that

$$e(G[W]) \geq \frac{3|A|}{2} - \frac{|A_T|}{2} + |S| + |L| - 1 + \frac{1}{2} \sum_{z \in W\backslash(A\cup S\cup L)} |N(z) \cap W|$$
$$+ \frac{1}{2} \sum_{s \in S} |\{sz \in G[W] : s \in S, z \notin A\backslash A_T, sz \notin T\}|,$$

where

$$\sum_{z \in W\backslash(A\cup S\cup L)} |N(z) \cap W| = (|B| + |C| + |F| - |L|) + \sum_{z \in B\cup C\backslash L} (\deg(z) - 3)$$
$$+ \sum_{z \in F\backslash L} (\deg(z) - 4).$$

Note that $e(G[I]) = 5$, and that $e(G[I, V\backslash I]) = |A| + 2|B| + 2|C| + 2|S| + 3|F|$.

Observation 4. $e(G) = e(G[I]) + e(G[I, V\backslash I]) + e(H)$

$$\geq \frac{1}{2}(5n - 17) + \frac{1}{2}(|S| - |A_T|) + |F| + \frac{1}{2}|L| + \frac{1}{2} \sum_{z \in B\cup C\backslash L} (\deg(z) - 3)$$
$$+ \frac{1}{2} \sum_{z \in F\backslash L} (\deg(z) - 4)$$
$$+ \frac{1}{2} \sum_{s \in S} |\{sz \in G[W] : s \in S, z \notin A\backslash A_T, sz \notin T\}| \geq \frac{5n - 17}{2}.$$

Observation 5. $F = \emptyset$ *and* $L = \emptyset$.

If $L \neq \emptyset$, then, by (4.1), $|S| - |A_T| \geq |L| \geq 1$. This implies that $e(G) \geq (5n - 15)/2$.

Observation 6.

$$(|S|-|A_T|)+\sum_{z\in B\cup C}(\deg(z)-3)+\sum_{s\in S}|\{sz\in G[W]: s\in S, z\notin A\backslash A_T, sz\notin T\}|\leq 1.$$

Observation 7. *There is no edge between $B\cup C$ and S. In particular, vertices in S are adjacent only to u,w, to vertices in S and in A. That is, $N(S)\subseteq S\cup A\cup\{u,w\}$.*

Suppose on the contrary that there is an edge $zs^*\in E(G)$, where $z\in B\cup C$ and $s^*\in S$. Then by Observation 5, $L=\emptyset$, we have that $zs^*\in\{sz\in G[W]: s\in S, z\notin A\backslash A_T, sz\notin T\}$. By Observation 6, we have that $\deg(z)=3$ and that $|S|=|A_T|$. Since z is critical, z is the only common neighbor of v and s^*. This implies that $N(s^*)\cap A=\emptyset$. On the other hand, the equality $|S|=|A_T|$ implies that $t(s^*)\in A$ which is a contradiction.

Observation 8. *Every vertex in B is adjacent to exactly one vertex in C, and every vertex in C is adjacent to exactly one vertex in B. Thus, $|B|=|C|$.*

By Observations 5 and 7, every vertex b in B is the only common neighbor of c and u for some $c\in C\cup A$. By Observation 2, $c\notin A$, so b is adjacent to at least one vertex in C. Similarly, every vertex in C is adjacent to at least one vertex in B. Suppose $b\in B$ is adjacent to c_1, $c_2\in C$. By Observation 6, $\deg(c)=3$ for all $c\in C$. This implies that $N(c_1)=N(c_2)$, a contradiction.

Observation 9. *If G has a degree-2 vertex a^*, then $|S|\leq 2$.*

Indeed, in this case, we have that $A_T=\{a^*\}$, and then Observation 6 implies $|S|\leq 2$.

Note that $|I|=5$, that $F=\emptyset$, and that $|B|=|C|$. It is sufficient to show that $|A|+|S|$ is even. By Observation 2 and 7, it is easy to see that G satisfies **R1**, ..., **R4**. Suppose, first, that $|S|\leq 2$. Then Lemma 4.2 implies that $|A|+|S|$ is even and we are done. By Observation 9, this includes also the case if G has a degree-2 vertex.

Finally, suppose that $|S|\geq 3$ and so G has no degree-2 vertex. Then by Observation 6, **R5** holds. Again Lemma 4.2 implies that $e(G[A\cup S])\geq 2|A|$ and $|A|\geq|S|$. We have now

$$\begin{aligned} e(G) &= e(G[I])+e(G[I,V\backslash I])+e(G[A\cup S])+e(G[B,C])\\ &\geq 5+(|A|+2|B|+2|C|+2|S|)+2|A|+\frac{|B|+|C|}{2}=\frac{5n-15}{2}\\ &+\frac{|A|-|S|}{2}\geq\frac{5n-15}{2}, \end{aligned}$$

and this is a contradiction, thus we have justified Theorem 1.1. ∎

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