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Triple Systems Not Containing a Fano Configuration

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A Fano configuration is the hypergraph of 7 vertices and 7 triplets defined by the points and lines of the finite projective plane of order 2. Proving a conjecture of T. Sós, the largest triple system on n vertices containing no Fano configuration is determined (for $n > n_1$). It is 2-chromatic with $\binom{n}{3} - \binom{\lfloor n/2 \rfloor}{3} - \binom{\lceil n/2 \rceil}{3}$ triples. This is one of the very few nontrivial exact results for hypergraph extremal problems.

1. Turán's problem

Given a 3-uniform hypergraph $\mathscr{F}$, let $\mathrm{ex}_3(n, \mathscr{F})$ denote the maximum possible size of a 3-uniform hypergraph of order n that does not contain any subhypergraph isomorphic to $\mathscr{F}$. Our terminology follows that of [16] and [10], which are comprehensive survey articles of Turán-type extremal graph and hypergraph problems, respectively. Also see the monograph of Bollobás [2].

There is an extensive literature on extremal graph problems. Nevertheless, we know much less about the hypergraph extremal problems and we have even fewer *exact* results on hypergraphs. One of the main contributions of this paper is that we improve an earlier result of de Caen and Füredi [5], providing the exact solution of the Fano hypergraph extremal problem.

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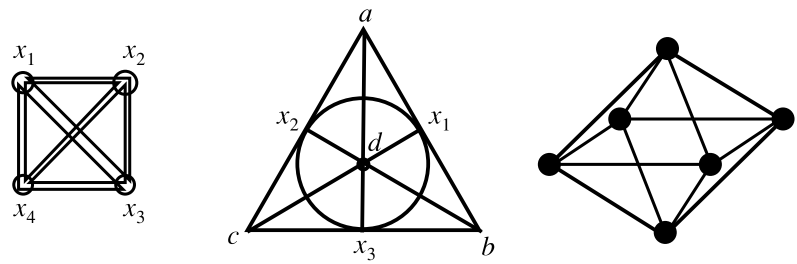

Figure 1. The complete 4-graph, the Fano hypergraph, and the octahedron

The *tetrahedron*, $K_4^{(3)}$, *i.e.*, a complete 3-uniform hypergraph on four vertices, has four triples $\{x_1,x_2,x_3\}$, $\{x_1,x_2,x_4\}$, $\{x_1,x_3,x_4\}$, $\{x_2,x_3,x_4\}$. The *complete* 3-*partite* triple system $K^{(3)}(V_1,V_2,V_3)$ consists of $|V_1||V_2||V_3|$ triples meeting all the three V_is. We also use the simpler notation $K^{(3)}(n_1,n_2,n_3)$ if $|V_i|=n_i$. $K^{(3)}(2,2,2)$ is sometimes called the *octahedron*. The *Fano configuration* $\mathbb{F}$ (or Fano plane, or finite projective plane of order 2, or Steiner triple system, $STS(7)$, or blockdesign $S_2(7,3,2)$) is a hypergraph on 7 elements, say $\{x_1,x_2,x_3,a,b,c,d\}$, with 7 edges $\{x_1,x_2,x_3\}$, $\{x_1,a,b\}$, $\{x_1,c,d\}$, $\{x_2,a,c\}$, $\{x_2,b,d\}$, $\{x_3,a,d\}$, $\{x_3,b,c\}$.

An averaging argument shows [12] that the ratio $\mathrm{ex}_3(n,\mathscr{F})/\binom{n}{3}$ is a non-increasing sequence. Therefore

$$\pi(\mathscr{F}) := \lim_{n\to\infty} \mathrm{ex}_3(n,\mathscr{F})/\binom{n}{3}$$

exists. This monotonicity implies that $\mathrm{ex}_3(5,K_4^{(3)}) \leqslant \lfloor \binom{5}{3}\mathrm{ex}_3(4,K_4^{(3)})/\binom{4}{3}\rfloor = 7$, thus

$$\mathrm{ex}_3(n,K_4^{(3)}) \leqslant 0.7\binom{n}{3} \quad \text{holds for every} \quad n \geqslant 5. \tag{1.1}$$

We note that the determination of $\pi(K_4^{(3)})$ is one of the oldest problems of this field, due to Turán [18], who published a conjecture in 1961 that this limit value is 5/9, and Erdős [8] offered \$1000 for a proof. The best upper bound, 0.5935..., is due to Fan Chung and Linyuan Lu [6].

Concerning the octahedron, a very special case of an important theorem of Erdős [7] states that

$$\mathrm{ex}_3(n,K^{(3)}(2,2,2)) = O(n^{3-(1/4)}), \tag{1.2}$$

i.e., in this case the limit $\pi = 0$.

The limit $\pi(\mathscr{H})$ is known only for very few cases when it is nonzero. De Caen and Füredi [5] proved the following.

Theorem A.

$$\mathrm{ex}_3(n,\mathbb{F}) = \frac{3}{4}\binom{n}{3} + O(n^2).$$

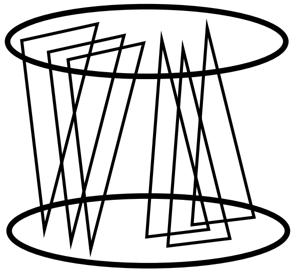

Figure 2. The conjectured extremal graph

This was conjectured by Vera T. Sós [17]. She also conjectured that the following hypergraph, $\mathscr{H}^n$, gives the exact value of $\mathrm{ex}_3(n, \mathbb{F})$. Let $\mathscr{H}(X, \overline{X})$ be the hypergraph obtained by taking the union of two disjoint sets X and $\overline{X}$ as the set of vertices and define the edge set as the set of all triples meeting both X and $\overline{X}$. For $\mathscr{H}^n$ we take $|X| = \lceil n/2 \rceil$ and $|\overline{X}| = \lfloor n/2 \rfloor$, (*i.e.*, they have nearly equal sizes). Then

$$e(\mathscr{H}^n) = \binom{n}{3} - \binom{\lfloor n/2 \rfloor}{3} - \binom{\lceil n/2 \rceil}{3},$$

which is $\frac{3}{4}\binom{n}{3} + O(n^2)$.

The chromatic number of a hypergraph $\mathscr{H}$ is the minimum p such that its vertex set can be decomposed into p parts with no edge contained entirely in a single part. It is well known and easy to check that the Fano plane is not two-colourable, its chromatic number is 3. Therefore $\mathbb{F} \nsubseteq \mathscr{H}(X, \overline{X})$. Thus $\mathscr{H}^n$ supplies the lower bound for $\mathrm{ex}_3(n, \mathbb{F})$ in Theorem A, implying that $\pi(\mathbb{F}) \geqslant \frac{3}{4}$.

In this paper we prove the exact version of T. Sós's conjecture, even in a stronger form, describing the extremal hypergraph as well.

Theorem 1.1. *There exists an n_1 such that the following holds. If $\mathscr{H}$ is a triple system on $n > n_1$ vertices not containing the Fano configuration $\mathbb{F}$ and of maximum cardinality, then it is 2-colourable. Thus $\mathscr{H} = \mathscr{H}^n$ and*

$$\mathrm{ex}_3(n, \mathbb{F}) = \binom{n}{3} - \binom{\lfloor n/2 \rfloor}{3} - \binom{\lceil n/2 \rceil}{3}.$$

This is an easy consequence of the following structure theorem.

Theorem 1.2. *There exist a $\gamma_2 > 0$ and an n_2 such that the following holds. If $\mathscr{H}$ is a triple system on $n > n_2$ vertices not containing the Fano configuration $\mathbb{F}$ and*

$$\deg(x) > \left(\frac{3}{4} - \gamma_2\right)\binom{n}{2}$$

holds for every $x \in V(\mathscr{H})$, then $\mathscr{H}$ is bipartite, $\mathscr{H} \subseteq \mathscr{H}(X, \overline{X})$ for some $X \subseteq V(\mathscr{H})$.

This result is a distant relative of the following classical theorem of Andrásfai, Erdős and T. Sós [1]. Let G be a triangle-free graph on n vertices with minimum degree $\delta(G)$.

$$\text{If } \delta(G) > \frac{2}{5}n, \text{ then } G \text{ is bipartite.} \tag{1.3}$$

The blow-up of a five-cycle C_5 shows that this bound is the best possible. They further determined

$$\delta(n,F) := \max\{\delta(G) : |V(G)| = n, \quad G \text{ is } F\text{-free } \chi(G) \geqslant \chi(F)\}$$

for $F = K_p$. The general case is still open, although Erdős and Simonovits [9] determined a number of cases and showed, *e.g.*, that K_p behaves uniquely: in the case $\chi(F) = p$, $F \neq K_p$ one has $\delta(n,F) - \delta(n,K_p) \geqslant n/(6p^2) - o(n)$.

Using the method of [5], Mubayi and Rödl [14] determined the limit π for a few more 3-uniform hypergraphs, obtaining $\pi = 3/4$ for all of them. It is very likely that the extremal hypergraphs are 2-colourable in those cases, too.

Turán [18] also conjectured that the 2-colourable triple system $\mathscr{H}^n$ is the largest $K_5^{(3)}$-free hypergraph. Sidorenko [15] disproved this conjecture, in this sharp form, for odd values $n \geqslant 9$. But it is still conjectured that it is true for all even values and it seems that $\pi(K_5^{(3)}) = 3/4$ holds as well. However, this question seems to be extremely difficult.

The main idea of the proof

The proof of Theorem A in [5] had the potential to prove our Theorem 1.1, but had to be improved in several places. One of these improvements was to introduce coloured multigraphs instead of multigraphs.

Earlier Brown, Erdős and Simonovits proved several results on multigraph extremal problems, but the excluded graphs in [11] had special form (as in Bondy–Tuza [3]): see, *e.g.*, the survey paper [4]. A method called 'augmentation' was developed there which is used implicitly here as well.

De Caen and Füredi [5] applied some multigraph extremal results of Füredi and Kündgen [11]. Now we shall use coloured multigraph extremal results.

Theorem 1.1 was proved independently and in a fairly similar way by Keevash and Sudakov [13]. Our Theorem 1.2 easily implies Theorem 1.1. Theorems 2.2 and 2.3 in the next section deal with new type of problems.

2. Fano plane and the links

First we describe how we can find a Fano plane in a triple system, using multigraphs. This will lead us to further investigation of multigraphs and coloured multigraphs.

Definition. The graphs G_1, $G_2, \ldots$ (with the common vertex set V) have 3 *pairwise crossing pairs* if there are four vertices $\{a,b,c,d\} \subseteq V$ and three graphs G_{i_j} such that $ad, bc \in E(G_{i_1})$, $ac, bd \in E(G_{i_2})$, and $ab, cd \in E(G_{i_3})$.

Notation. If $G_1, \ldots, G_p$ are (simple) graphs with the same vertex set V, then $G_{1,\ldots,p}$ denotes a coloured multigraph on V in which we join two vertices $a, b \in V$ by an edge of colour i if $ab \in E(G_i)$. Thus, $\deg_{G_{1,\ldots,p}}(x) = \sum_{1\leqslant i\leqslant p} \deg_{G_i}(x)$.

The *multiplicity* of a pair $\{a,b\} \subset V$ is denoted by $\mu(ab)$ and it is the number of graphs among the G_is containing ab as an edge. We have $0 \leqslant m \leqslant p$. Also, the set $\{i : ab \in E(G_i)\}$ is called the set of *colours* of the pair ab.

As usual, $e(G)$ stands for the number of edges of G (for multigraphs it is counted with multiplicity). $G[X]$ denotes the induced (multi)graph of G spanned by the subset of vertices X. When possible, we shall use simplified notations, discarding parentheses and commas.

Given a triple system $\mathscr{H}$ with vertex set V and a vertex $x \in V$, the *link graph* $G(\mathscr{H},x)$ is defined as the set of pairs $\{y,z\}$ such that $\{x,y,z\}$ is a hyperedge of $\mathscr{H}$. Two of our simple but crucial observations are as follows.

Claim 2.1. *Assume that* $\mathbb{F} \not\subseteq \mathscr{H}$.

(a) *If* $\{x_1,x_2,x_3\}$ *is a hyperedge of* $\mathscr{H}$ *and* $S = \{a,b,c,d\}$ *is disjoint from* $\{x_1,x_2,x_3\}$, *then consider the three link graphs* $G_i := G(\mathscr{H},x_i)$. *We have*

$$G_1,\ G_2 \text{ and } G_3 \text{ have no 3 pairwise crossing pairs on } S. \tag{2.1}$$

(b) *Consider a vertex* x *and suppose that the link graph* $G = G(\mathscr{H},x)$ *contains three vertex-disjoint complete graphs with vertex sets* U_i, *i.e.,* $G[U_i] \equiv K(U_i)$. *Then the triples of* $\mathscr{H}$ *meeting each* U_i *can not form an octahedron:*

$$\mathscr{H} \cap K^{(3)}(U_1,U_2,U_3) \text{ contains no } K^{(3)}(2,2,2). \tag{2.2}$$

As a matter of fact, in the last statement four triples of appropriate position in $\mathscr{H}$ would already yield a Fano configuration.

However, the main idea in the proof of Theorem A from [5] was to consider a $K_4^{(3)}$ on $\{x_1,x_2,x_3,x_4\}$ in $\mathscr{H}$ and to show that, for the four links,

$$\sum_{i\leqslant 4} e(G(\mathscr{H},x_i)) \leqslant 3\binom{n}{2} + O(n).$$

In this paper our primary aim is to prove an exact form of this and describe the corresponding extremal structures. Besides this we shall also prove some coloured multigraph extremal theorems.

Let $B(X,\overline{X})$ denote the coloured multigraph on the n-element vertex set V with a partition $V = X \cup \overline{X}$, coloured in 1, 2, 3, and 4. Assume also that all edges in X have colours 1 and 2, all edges in $\overline{X}$ have colours 3 and 4, and all edges joining X and $\overline{X}$ have all the four colours.

Theorem 2.2. *Let* $G_1,\ldots,G_4$ *be four graphs on the common* n*-element vertex set* V, *for* $n \geqslant 4$. *If they do not contain 3 pairwise crossing pairs, then*

$$\sum_{i\leqslant 4} e(G_i) \leqslant 2\binom{n}{2} + 2\left\lfloor \frac{n^2}{4} \right\rfloor. \tag{2.3}$$

Further, for $n > 7$, *equality holds in* (2.3) *if and only if their union* $G_{1,2,3,4}$ *is isomorphic* (*up to permuting the colours*) *to* $B(X,\overline{X})$ *with* $||X| - |\overline{X}|| \leqslant 1$.

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For $n = 4, 5, 6$ there are other extremal configurations: for example, one can add all the four colours to every edge of the 3-partite Turán graph $T_{n,3}$. The case $n = 7$ remains open (concerning the uniqueness of extremal configurations).

Again, Theorem 2.2 is strongly related to a structure theorem, the strongest result in this paper.

Theorem 2.3. *There exists a $\gamma_4 > 0$ such that the following holds. Let $G_1, \dots, G_4$ be four graphs on the common n-element vertex set V, and let $G = G_{1,2,3,4}$. Suppose that they do not contain three pairwise crossing pairs and*

$$\deg_G(x) > (3 - \gamma_4)n$$

holds for every vertex $x \in V$. Then $G_{1,2,3,4}$ is a submultigraph of some $B(X, \overline{X})$ (up to permuting the colours $1, 2, 3, 4$).

We prove this theorem with $\gamma_4 = 1/5$. It can probably be sharpened. However, we cannot take a $\gamma_4 > 1/3$ as shown by the coloured multigraph $4T_{n,3}$. If we colour each edge of the 3-partite Turán graph by all the 4 colours we get an example with $\delta = 4\lfloor 2n/3 \rfloor$.

3. Extremal noncrossing graphs

In this section we prove some lemmas, and then Theorem 2.2. As in [5], we first investigate the 4-element subsets of V.

Lemma 3.1. *Let $G = G_{1,2,3,4}$ be a coloured multigraph without 3 pairwise crossing pairs. Suppose that $n = 4$, $V = \{a, b, c, d\}$. Then:*

- (i) *$e(G) \leqslant 20$, with equality if V can be split into two pairs, $V = X \cup \overline{X}$, so that the 4 edges of the complete bipartite graph $K(X, \overline{X})$ (in fact it is a C_4) belong to all the four graphs; however, X and $\overline{X}$ do not belong to the same $E(G_i)$,*
- (ii) *if $e(G) = 19$, then G is obtained from the above example by deleting an edge,*
- (iii) *if $\mu(ac) + \mu(ad) + \mu(bc) + \mu(bd) \geqslant 14$ then ab and cd get different colours,*
- (iv) *$e(G_1) + e(G_2) + e(G_3) \leqslant 15$.*

Proof. There are only finitely many configurations to check. A quick way to do it is as follows. Consider the 3 perfect matchings of $V = \{a, b, c, d\}$, $M^1 = \{ab, cd\}, M^2 = \{ac, bd\}$ and $M^3 = \{ad, bc\}$ and arrange the 24 possible edges of G into a 4×3 array of 'cells'. Namely, the cell in the ith row and the jth column contains the intersection $E(G_i) \cap M^j$. A cell is called *full* if it contains 2 edges, otherwise it is *incomplete*. In this setting 3 crossing pairs correspond to 3 full cells in different rows and columns.

A very special case of Frobenius's theorem (in other words, the König–Egerváry theorem) states that if there are no 3 such cells, then all the full cells can be covered by 2 rows or 2 columns or by 1 column and 1 row. (We use these deep theorems only to make the proof more transparent: for 4 vertices we do not really need this heavy artillery.)

Suppose $e(G) \geqslant 19$. Since $e(G)$ is the sum of the number of the edges in the 12 cells, there must be at least seven full cells. By the Frobenius theorem, they are in 2 columns.

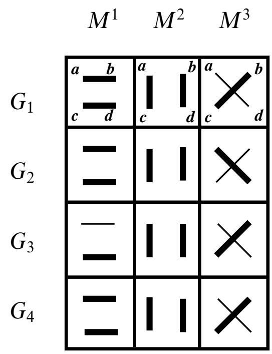

Figure 3. The matching table

So we may assume that all the four cells in the third column are incomplete. We have arrived at the structure of G claimed in (i) and (ii).

Concerning (iii), to get the 14 edges, the columns of M^2 and M^3 must contain at least 6 full cells. Then the Frobenius theorem gives that the first column, M^1, has only incomplete cells. This is exactly the assertion of (iii).

Finally, the proof of (iv) is similar to the proof of (i), but simpler. □

We will frequently use the following obvious estimate for the degrees in a set of vertices $U \subset V$. (In fact it is an identity, but we use it as an upper bound.)

$$\sum_{u\in U} \deg(u) \leqslant 2 \times e(U) + \sum_{x\in V\setminus U} \left(\sum_{u\in U} \mu(xu) \right). \tag{3.1}$$

The *type* of a triangle (triple) $\{a,b,c\}$ is the *list* of the multiplicities of its pairs, $(\mu(ab), \mu(bc), \mu(ca))$. (Note that these triangles have nothing to do with our 3-uniform hypergraphs: this section is about graphs, not hypergraphs.)

Lemma 3.2. *Let $G = G_{1,2,3,4}$ be a coloured multigraph without 3 pairwise crossing pairs. Suppose that $\delta(G) > (8/3)n$. Then G has no triangle of types $(4,4,4)$, $(4,4,3)$, $(4,3,3)$.*

Proof. Suppose that $\mu(ab) = \mu(bc) = \mu(ca) = 4$. Consider an $x \in V \setminus \{a,b,c\}$. Then $abcx$ contains at most 20 edges (by Lemma 3.1(i)), so $\mu(ax) + \mu(bx) + \mu(cx) \leqslant 8$. Adding up these inequalities for every x (more exactly, applying (3.1) to $U = \{a,b,c\}$), we obtain

$$\deg(a) + \deg(b) + \deg(c) \leqslant 2 \times 12 + 8(n-3) = 8n.$$

This contradicts our condition $\delta(G) > (8/3)n$. So from now on, we may suppose that there is no triangle of type $(4,4,4)$.

Suppose that $\mu(ab) = \mu(ac) = 4$, $\mu(bc) = 3$. Consider $V \setminus \{a,b,c\}$ and classify its vertices according to their sum of multiplicities:

$$V_{\leqslant 7} := \{x \in V \setminus \{a,b,c\} : \mu(ax) + \mu(bx) + \mu(cx) \leqslant 7\},$$
$$V_{\geqslant 8} := \{x \in V \setminus \{a,b,c\} : \mu(ax) + \mu(bx) + \mu(cx) \geqslant 8\}.$$

If $x \in V_{\geqslant 8}$ then $\{a, b, c, x\}$ contains at least 19 edges. So Lemma 3.1(ii) gives that the edges of multiplicities 4 are contained in the 4-cycle a–b–x–c–a. Thus the colours of ax and bc are distinct. Hence $\mu(ax) + \mu(bc) \leqslant 4$, implying $\mu(ax) \leqslant 1$. Thus

$$\deg(a) \leqslant 8 + 4|V_{\leqslant 7}| + |V_{\geqslant 8}|. \tag{3.2}$$

This inequality, together with the lower bound on $\delta(G)$, implies that $|V_{\leqslant 7}|$ is large. However, then there are too few edges going to $\{a, b, c\}$, a contradiction. More formally, Lemma 3.1(i) implies that $\mu(ax) + \mu(bx) + \mu(cx) \leqslant 9$. Apply (3.1) to $U = \{a, b, c\}$:

$$\deg(a) + \deg(b) + \deg(c) \leqslant 2 \times 11 + 7|V_{\leqslant 7}| + 9|V_{\geqslant 8}|. \tag{3.3}$$

Adding the double of (3.2) to the triple of (3.3) and using $|V_{\leqslant 7}| + |V_{\geqslant 8}| = n - 3$, we get

$$11 \times (8/3)n < 5\deg(a) + 3\deg(b) + 3\deg(c) \leqslant 82 + 29(|V_{\leqslant 7}| + |V_{\geqslant 8}|) < 29n.$$

This contradiction implies that there is no triangle of type $(4, 4, 3)$ either.

Finally, suppose that $\mu(ac) = \mu(bc) = 3$, $\mu(ab) = 4$. We show that $\mu(ax) + \mu(bx) + \mu(cx) \leqslant 8$ for every $x \in V \setminus \{a, b, c\}$. Consider an $x \in V \setminus \{a, b, c\}$. Suppose that $\mu(ax) + \mu(bx) + \mu(cx) \geqslant 9$. Then Lemma 3.1(ii) can be applied. Thus there are (exactly) 3 edges of multiplicities 4 in $\{a, b, c, x\}$ forming a path; ab could not be its middle edge, so the path is, say, a–b–x–c. Then the triangle bxc is of type $(4, 4, 3)$, contradicting our earlier observations. Apply (3.1) to $U = \{a, b, c\}$:

$$3\delta(G) \leqslant \deg(a) + \deg(b) + \deg(c) \leqslant 2 \times 10 + 8(n - 3) = 8n - 4.$$

This contradicts our condition $\delta(G) > (8/3)n$, completing the proof of Lemma 3.2. □

Define

$$f(n) := 2\binom{n}{2} + 2\left\lfloor \frac{n^2}{4} \right\rfloor.$$

Let $\mathcal{M}_n$ be a multigraph obtained by taking four times the edges of a complete bipartite graph $K(X, \overline{X})$ on n vertices with an equipartition $(X, \overline{X})$ and by taking the other edges of K_n twice. Obviously, $e(\mathcal{M}_n) = f(n)$.

Lemma 3.3. *Let $\mathcal{M}$ be a multigraph with maximum edge-multiplicity at most 4. If $\mathcal{M}$ has no triangle of types $(4, 4, 4)$, $(4, 4, 3)$, $(4, 3, 3)$, then $e(\mathcal{M}) \leqslant f(n)$. Here equality holds only if $\mathcal{M} \equiv \mathcal{M}_n$.*

This is the part where we do not use colours.

Proof. We just copy the proof of the Turán–Mantel theorem using induction from $n - 2$ to n. The cases $n = 1, 2, 3$ are obvious.

If there is no edge of multiplicity 4, then $e(\mathcal{M}) \leqslant 3\binom{n}{2} \leqslant f(n)$ (for $n \geqslant 3$) and we are done. Now suppose that $\mu(ab) = 4$ and let $A := \{x : \mu(bx) \geqslant 3\}$ and $B := \{y : \mu(ay) \geqslant 3\}$.

Our condition implies that $A \cap B = \emptyset$, thus

$$\begin{aligned}\deg(a) + \deg(b) &\leqslant 4|A| + 2(n-1-|A|) + 4|B| + 2(n-1-|B|) \\ &= 4n - 4 + 2(|A|+|B|) \leqslant 6n-4. \qquad (3.4)\end{aligned}$$

Use induction for $\mathscr{M} \setminus \{a,b\}$. We have

$$e(\mathscr{M}) = e(\mathscr{M} \setminus \{a,b\}) + \deg(a) + \deg(b) - 4 \leqslant f(n-2) + 6n - 8 = f(n).$$

If equality holds here then it holds in (3.4) too. This implies that $A \cup B = V(\mathscr{M})$ and every edge of the form bx with $x \in A$ has multiplicity 4. Thus every edge in A has multiplicity at most 2. The same holds for B, implying

$$e(\mathscr{M}) \leqslant 2\binom{|A|}{2} + 2\binom{|B|}{2} + 4|A||B| \leqslant f(n).$$

In the case of equality every A–B edge must have multiplicity four, thus $\mathscr{M}$ is isomorphic to $\mathscr{M}_n$. □

Proof of Theorem 2.2. Let $e(n)$ be the maximum of $e(G)$, where $G := G_{1,2,3,4}$. We prove by induction that $e(n) = f(n)$ for every $n \geqslant 4$. By Lemma 3.1(i) we have $e(4) = 20$.

A standard averaging argument shows that the sequence $e(n)/\binom{n}{2}$ is monotone decreasing (non-increasing). This gives that $e(5) \leqslant \binom{5}{2}e(4)/\binom{4}{2} = 33.33\ldots$. We claim that $e(5) = 32$. Suppose, on the contrary, that $V = \{a,b,c,d,e\}$ and $e(G) = 33$ with no three crossing pairs. Since $e(4) = 20$, we have that every degree of G is at least $33 - e(4) = 13$, so the degree sequence of G is $(13,13,13,13,14)$. Thus every four-element subset of V contains at least 19 edges. Then Lemma 3.1(ii) implies that every four-element set $X \subset V$ contains a unique disjoint pair of edges $P_1(X)$, $P_2(X)$ with total multiplicities at most 4. Suppose that $\mu(P_1) \leqslant \mu(P_2)$. Suppose that $P_1(X) = \{a,b\}$ for $X = \{a,b,c,d\}$ with $\mu(ab) := \mu \leqslant 2$. Consider the sets $X = V \setminus \{c\}$, $V \setminus \{d\}$, and $V \setminus \{e\}$, we get that $P_2(X) = de$, ce, and cd, respectively. We get $\mu(de)$, $\mu(ce)$, and $\mu(cd) \leqslant 4 - \mu$. Hence $e(G) \leqslant \mu + 3(4-\mu) + 6 \times 4 = 36 - 2\mu$. This is at most 32 for $\mu = 2$, a contradiction. So the last case to consider is when $\mu(P_1(X)) \leqslant 1$ for every X. In this case every $X \subset V$, $|X| = 4$ contains a *unique* pair with multiplicity at most 1. However, this is impossible.

From now on we suppose that $n \geqslant 6$ and that $e(G)$ is maximal, *i.e.*, $e(G) = e(n)$. Consider, first, the case when G has no triangle of types $(4,4,4)$, $(4,4,3)$, $(4,3,3)$. Then Lemma 3.3 implies that $e(G) \leqslant f(n)$ and in case of equality the edges of multiplicity 4 form a complete bipartite graph. Then Lemma 3.1(i) implies that G is isomorphic to a $B(X, \overline{X})$.

Consider the other case, when G has a triangle of edge-multiplicities at least 4, 3 and 3. Lemma 3.2 gives that there exists a vertex x of small degree

$$\deg(x) \leqslant \left\lfloor \frac{8}{3}n \right\rfloor \leqslant 2n - 2 + 2\left\lfloor \frac{n}{2} \right\rfloor. \qquad (3.5)$$

Applying induction to $G \setminus \{x\}$, we have

$$e(G) \leqslant e(n-1) + \deg(x) \leqslant f(n-1) + 2(n-1) + 2\left\lfloor \frac{n}{2} \right\rfloor = f(n), \qquad (3.6)$$

completing the proof of $e(n) = f(n)$.

Now suppose that $e(G) = e(n)$ and $n \geqslant 8$. For $n \geqslant 10$ and $n = 8$ inequality (3.5) is sharp, so (3.6) gives $e(G) < f(n)$. Thus, in these cases $e(G) = e(n)$ implies that G is isomorphic to a $B(X, \overline{X})$.

Finally, when $n = 9$, $e(G) = e(n)$, $\delta(G) = 2(n-1) + 2\lfloor n/2 \rfloor = (8/3)n$ we can return to the proof of Lemma 3.2. One can sharpen it in the following way: if $\delta(G) = (8/3)n$ and it contains a triangle of types $(4, \geqslant 3, \geqslant 3)$, then G is isomorphic to $4T_{n,3}$. The details are omitted. □

4. The structure of 4 noncrossing graphs

In this section we first prove two lemmas, and then Theorem 2.3.

Lemma 4.1. *Let $G = G_{1,2,3,4}$ be a coloured multigraph without 3 pairwise crossing pairs. If $n \geqslant 5$ then there is an edge of multiplicity at most 2.*

Proof. There are only finitely many configurations to check. A quick way to do it is as follows. Suppose, on the contrary, that every pair has multiplicity at least 3. We may also suppose that each edge has multiplicity exactly 3 (if not, delete some extra multiplicities) and that $n = 5$, $V = \{a, b, c, d, e\}$. Consider the restriction of G to $\{a, b, c, d\}$ and the 4×3 cells we can form from its homogeneous matchings (*i.e.*, on both of its edges the set of colours is the same). (See Figure 3.) The number of the edges in a column is the sum of the multiplicities of the two corresponding edges, so each column contains exactly 6 edges. Thus each column contains at least two full cells. As we have seen, the Frobenius theorem implies that the full cells can be covered by 2 *rows*; two columns or a column and a row would not suffice. The possibility of an empty cell is also excluded. Thus in two rows we have the 6 full cells and in the other two rows we have 1 edge in each cell. Then two of the G_is are K_4s, a third one is a triangle, and the fourth is the complementary star of 3 edges. We have, *e.g.*, that all the 6 edges of the K_4 generated by $\{a, b, c, d\}$ have colours 1 and 2, and ab, ac, bc have colour 3 and ad, bd, cd have colour 4.

Consider *abce*. Its colours form the same structure that we have seen on *abcd*. The triangle *abc* has colours 1, 2 and 3, so *ae*, *be*, *ce* must form a star of colour 4. Then, in *abde* the edges *ad*, *db*, *be* and *ea* have colour 4, but *ab* does not, contradicting the fact that each colour class is a K_4, a triangle, or a star of 3 edges. □

Lemma 4.2. *Let $G = G_{1,2,3,4}$ be a coloured multigraph without 3 pairwise crossing pairs. Suppose that $\delta(G) > (11n - 8)/4$. If G has no triangle of type $(4, 4, 3)$ or $(4, 3, 3)$, then it also has no triangle of type $(3, 3, 3)$.*

Proof. Suppose that $\mu(ab) = \mu(ac) = \mu(bc) = 3$. If for all $x \in V \setminus \{a, b, c\}$ we have $\mu(ax) + \mu(bx) + \mu(cx) \leqslant 8$, then (3.1) leads to

$$3\delta(G) \leqslant \deg(a) + \deg(b) + \deg(c) \leqslant 8n - 6,$$

a contradiction.

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So there exists a vertex d joined with at least 9 edges to abc. If $\mu(ad) = 4$, then consider the abd triangle. Our condition implies that its third side, bd has multiplicity at most 2. Considering acd we obtain $\mu(cd) \leqslant 2$. Thus $\mu(ad) + \mu(bd) + \mu(cd) \leqslant 8$, a contradiction.

Thus $\mu(ad) \leqslant 3$, implying $\mu(ad) = \mu(bd) = \mu(cd) = 3$. Now we repeat the above argument. If every $x \in V \setminus \{a, b, c, d\}$ is joined to $abcd$ by at most 11 edges, then applying (3.1) to $U = \{a, b, c, d\}$ we get the contradiction

$$\begin{aligned} 11n - 8 < 4\delta(G) &\leqslant \deg(a) + \deg(b) + \deg(c) + \deg(d) \\ &\leqslant 2 \times 18 + 11(n-4) = 11n - 8. \end{aligned}$$

Thus there exists an $e \in V$ with $\mu(ae) + \mu(be) + \mu(ce) + \mu(de) \geqslant 12$. Our condition implies again that the multiplicities of these edges are 3. So all edges of $abcde$ have multiplicities exactly 3. However, this contradicts Lemma 4.1, completing the proof of Lemma 4.2. □

Proof of Theorem 2.3. Let $G^{3,4}$ be the graph formed by the edges with multiplicities 3 and 4. Let $d^{(i)}(x)$ be the number of pairs xy with multiplicities i (in G) and $d^{3,4}(x) := d^{(3)}(x) + d^{(4)}(x)$. In this proof we abbreviate $\deg_G(x)$ to $\deg(x)$. For any x we have

$$\begin{aligned} \deg(x) &= \sum_{i \leqslant 4} i d^{(i)}(x) \\ &\leqslant 4(d^{(4)}(x) + d^{(3)}(x)) + 2(d^{(2)}(x) + d^{(1)}(x) + d^{(0)}(x)) \\ &= 2d^{3,4}(x) + 2(n-1). \end{aligned}$$

Thus

$$d^{3,4}(x) \geqslant \frac{1}{2}\deg(x) - (n-1) > \frac{1-\gamma_4}{2} n \geqslant \frac{2}{5} n. \tag{4.1}$$

Here in the last step we used that $\gamma_4 = 1/5$.

Lemmas 3.2 and 4.2 give that $G^{3,4}$ is triangle-free, and (4.1) gives that its minimum degree exceeds $2n/5$. Then the result of Andrásfai, Erdős and T. Sós [1], *i.e.*, (1.3) can be applied. Hence this graph is bipartite.

Let $X, \overline{X}$ be the parts of the bipartite graph $G^{3,4}$. We may suppose that $|\overline{X}| \leqslant n/2$. Then (4.1) gives

$$\frac{1}{2}\delta(G) - n < |\overline{X}| \leqslant \frac{n}{2} \leqslant |X| < 2n - \frac{1}{2}\delta(G). \tag{4.2}$$

Let Q be the induced subgraph $G^2[X]$, the subgraph induced by the edges of multiplicity 2 in X. We claim that this graph is connected; moreover, it has diameter 2.

Claim 4.3. *For every $a, b \in X$ there exists a vertex $x \in X$ with $ax, bx \in E(Q)$.*

Proof. Let

$$N := \{c : c \in X, \quad \mu(ac) + \mu(bc) = 4\}.$$

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Apply (3.1) to $U = \{a, b\}$:

$$\begin{aligned}2\delta(G) &\leqslant \deg(a) + \deg(b) \leqslant 2 \times 2 + 3(|X| - 2) + |N| + 8|\overline{X}| \\ &< 3(|X| + |\overline{X}|) + 5|\overline{X}| + |N| \leqslant 5.5n + |N|.\end{aligned}$$

Then $\delta(G) > (11/4)n$ implies that $N \neq \emptyset$. □

Claim 4.4. *Suppose that* $a, b, c \in X$ *and suppose that* ab *has colours* 1 *and* 2. *Then* bc *cannot have colour* 3 (*or colour* 4).

Proof. Suppose on the contrary, that bc has colour 3, and let

$$N := \{x : x \in \overline{X}, \quad \mu(ax) + \mu(bx) + \mu(cx) \geqslant 11\}.$$

Apply (3.1) to $U = \{a, b, c\}$:

$$\begin{aligned}\deg(a) + \deg(b) + \deg(c) &\leqslant 2 \times 6 + 6(|X| - 3) + 12|N| + 10(|\overline{X}| - |N|) \\ &= 6n + 2|N| + 4|\overline{X}| - 6. \end{aligned} \tag{4.3}$$

Now $\delta > (8/3)n$ and $|\overline{X}| \leqslant n/2$ imply that $|N| > 3$. Fix a vertex $x \in N$ and let y be another arbitrary vertex in N. We have

$$\mu(ax) + \mu(bx) = (\mu(ax) + \mu(bx) + \mu(cx)) - \mu(cx) \geqslant 11 - 4 = 7,$$

and similarly, $\mu(ay) + \mu(by) \geqslant 7$. Apply Lemma 3.1(iii) to the a–x–b–y–a cycle. It gives that the colours of xy are different from the colours of ab. Similarly, we get that $\mu(xb) + \mu(xc) \geqslant 7$, and also $\mu(yb) + \mu(yc) \geqslant 7$. Applying Lemma 3.1(iii) again to x–b–y–c–x, we obtain that the colours of xy are different from the colours of bc, too. Thus xy can have at most one colour. We obtain

$$\deg(x) \leqslant 4|X| + (|N| - 1) + 2(|\overline{X}| - |N|) < 2n + 2|X| - |N|.$$

Adding the double of this to (4.3) we get

$$\begin{aligned}5\delta(G) &\leqslant 2\deg(x) + \deg(a) + \deg(b) + \deg(c) \\ &< 2(2n + 2|X| - |N|) + 6n + 2|N| + 4|\overline{X}| = 14n.\end{aligned}$$

This contradicts $\delta > (14/5)n$, completing the proof of Claim 4.4. □

By Claim 4.3 Q is connected, so the above Claim 4.4 implies that it is homogeneous, *i.e.*, all of its edges get the same pair of colours, say colours 1 and 2. Then the claim also implies that all pairs of $G[X]$ can have only colours 1 and 2.

The only thing left to show is that the edges of $G[\overline{X}]$ do not have colour 1 (or 2). Suppose on the contrary that $x, y \in \overline{X}$ and xy has colour 1. Consider

$$N := \{a : a \in X, \quad \mu(ax) + \mu(ay) \geqslant 7\}.$$

Apply (3.1) to $U = \{x, y\}$:

$$\deg(x) + \deg(y) \leqslant 4|\overline{X}| - 4 + 6|X| + 2|N| < 4n + 2|X| + 2|N|. \tag{4.4}$$

Then the upper bound (4.2) on $|X|$ implies that $|N| > \frac{3}{2}\delta(G) - 4n$, so $|N| > n/5 \geqslant 2$.

Fix a vertex $a \in N$ and apply Lemma 3.1(iii) to a–x–b–y–a with $b \in N$. We get that ab cannot have colour 1, so it has only at most one colour (namely 2). Thus

$$\deg(a) \leqslant 4|\overline{X}| + 2|X| - 2 - (|N| - 1) < 2n + 2|\overline{X}| - |N|.$$

Adding the double of this to (4.4), we get the contradiction

$$4\delta(G) \leqslant 2\deg(a) + \deg(x) + \deg(y) < 8n + 2(|X| + |\overline{X}|) + 2|\overline{X}| \leqslant 11n. \qquad \square$$

5. The structure of Fano-free triple systems

In this section we prove Theorem 1.2 and then Theorem 1.1.

To avoid the use of $o(1), o(n)$, we define $\gamma_2, \gamma_5, \gamma_6, \gamma_7$ and $n_2, \ldots, n_7$. Each of these $\gamma_i = \gamma(\gamma_1, \ldots, \gamma_{i-1})$ and $n_i = n(\gamma_i)$ are explicitly computable so that $\gamma_i \to 0$ whenever all previous $\gamma_j \to 0$.

Proof of Theorem 1.2. Let V be the set of vertices of $\mathscr{H}$. Add up the degrees of $\mathscr{H}$ for all $x \in V$. We obtain

$$e(\mathscr{H}) = \frac{1}{3}\sum_{x \in V} \deg_{\mathscr{H}}(x) > \left(\frac{3}{4} - \gamma_2\right)\binom{n}{3}.$$

Here the right-hand side is at least $0.7\binom{n}{3}$ for $\gamma_2 \leqslant 1/20$. Thus (1.1) implies that $\mathscr{H}$ contains a four-element set $W_1 := \{x_1, x_2, x_3, x_4\}$ with a complete subhypergraph $K_4^{(3)}$ on it. Consider the link graphs $G(\mathscr{H}, x_i)$ and restrict them to $V_1 := V \setminus W_1$, $L_i := G(\mathscr{H}, x_i)[V_1]$, $L = L_{1,2,3,4}$. Thus we have deleted some edges corresponding to the triples meeting W_1 in at least 2 vertices, so $e(L_i) \geqslant \deg_{\mathscr{H}}(x_i) - 3(n-4) - 3$. Altogether

$$e(L) = \sum_{i \leqslant 4} e(L_i) \geqslant \sum_{i \leqslant 4} \deg_{\mathscr{H}}(x_i) - 12(n-4) - 12 > (3 - 5\gamma_2)\binom{|V_1|}{2}. \qquad (5.1)$$

Here the last inequality holds for every $n > 24/\gamma_2$.

Let $\gamma_5 \gg \gamma_2$, say $\gamma_5 = \sqrt{10\gamma_2}$ (we suppose that γ_2 is sufficiently small).

Claim 5.1. *There exists a subset $V_2 \subseteq V_1$ with $|V_2| \geqslant (1 - \gamma_5)n$, such that*

$$\deg_G(x) > (3 - \gamma_5)|V_2| \qquad (5.2)$$

holds for every $x \in V_2$, where $G_i := L_i[V_2]$ and $G := G_{1,2,3,4}$.

Proof. Let $V^0 := V_1$. Define a procedure for $k = 0, 1, 2\ldots$ to obtain the sets V^k and graphs $L^k := L[V^k]$ as follows. If one can find a vertex $v^k \in V^k$ such that

$$\deg_{L^k}(v^k) \leqslant (3 - \gamma_5)|V^k|,$$

then let $V^{k+1} := V^k \setminus \{v^k\}$. If no such vertex exists then the procedure stops. Suppose the last set defined was V^p and call it V_2. By (2.1) the graphs G_i do not have 3 crossing pairs,

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so Theorem 2.2 implies (for $\gamma_5 \leqslant 1/5$) that

$$\sum_{i\leqslant 4} e(G_i) \leqslant 3\binom{|V_2|}{2} + \frac{1}{2}|V_2|, \tag{5.3}$$

Using the notation $q := |V_1|$ $(= n-4)$ we obtain from (5.1) and (5.3) that

$$\begin{aligned}(3-5\gamma_2)\binom{q}{2} < \sum_{i\leqslant 4} e(L_i) &\leqslant \sum_{q\geqslant k>q-p} (3-\gamma_5)k + e(G)\\ &\leqslant (3-\gamma_5)\left(\binom{q+1}{2} - \binom{q-p+1}{2}\right) + 3\binom{q-p}{2} + \frac{1}{2}(q-p).\end{aligned}$$

Rearranging, we get

$$\gamma_5 p(2q-p+1) < \frac{1}{2}(q+5p) + 5\gamma_2\binom{q}{2}.$$

This gives for $n > n_0(\gamma_2)$ that $\gamma_5 pq < 5\gamma_2 q^2$, *i.e.*, $p < (5\gamma_2/\gamma_5)q = \frac{1}{2}\gamma_5 q$. This implies $|V_2| = q - p > (1-\gamma_5)(q+4) = (1-\gamma_5)n$ for $n > 10/\gamma_2$. □

By (5.2) we can apply Theorem 2.3 to G. We obtain the disjoint sets X and $\overline{X}$ such that $G \subseteq B(X, \overline{X})$. We also have, as in (4.2), that

$$\frac{1-\gamma_5}{2}|V_2| \leqslant |X|,\, |\overline{X}| \leqslant \frac{1+\gamma_5}{2}|V_2|. \tag{5.4}$$

Without loss of generality we may suppose that X contains only edges of colours 1 and 2 (that is, no edges of either G_3 or G_4), while $G[\overline{X}]$ has edges only of colours 3 and 4.

Let Q be the graph on X formed by the edges of G with two colours. We claim that for every $x \in X$

$$\deg_Q(x) > (1-5\gamma_5)|X|. \tag{5.5}$$

Indeed, we have a lower bound (5.2) for $\deg_G(x)$. On the other hand, x has exactly $\deg_Q(x)$ neighbours in X joined by an edge of multiplicity 2; the other vertices of X are joined by edges with multiplicities at most 1. Thus

$$(3-\gamma_5)(|X| + |\overline{X}|) < \deg_G(x) \leqslant 2\deg_Q(x) + (|X| - \deg_Q(x)) + 4|\overline{X}|.$$

Rearranging, we get

$$\deg_Q(x) > (1-2\gamma_5)|X| - (1+\gamma_5)(|\overline{X}| - |X|).$$

This and $||\overline{X}| - |X|| \leqslant \frac{2\gamma_5}{1-\gamma_5}|X|$ (a corollary of (5.4)) give (5.5).

We will prove that $\mathscr{H}[X]$ and $\mathscr{H}[\overline{X}]$ contain almost no triples. Later we shall see that they have no triples at all. First we show the following.

Claim 5.2. *There exists a $\gamma_6 = O((\gamma_5)^{1/8})$ and a subset $X_1 \subset X$ such that $|X_1| > (1-\gamma_6)n/2$ and $\mathscr{H}$ has at most $\gamma_6 n^3$ triples in X_1.*

Proof. Let $k := \lceil 1/\sqrt{5\gamma_5}\rceil$. Let $Y \subset X$, $|Y| \geqslant 5k\gamma_5|X|$ and consider $Q[Y]$. Inequality (5.5) implies that every vertex of $Q[Y]$ has degree at least $|Y| - 5\gamma_5|X| \geqslant \frac{k-1}{k}|Y|$. This implies

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(say, via Turán's theorem) that Y contains a k-set $U_1 \subset Y$ inducing a complete subgraph of Q. Applying this to another Y disjoint from U_1, we get U_2. Iterating this procedure one can cover a 'large' part of X by disjoint k-sets $U_1, \ldots, U_m$ such that, for $X_1 = \cup_{i \leqslant m} U_i$, we have $|X - X_1| \leqslant 5k\gamma_5 |X|$. Moreover, the complete graphs $K[U_1], K[U_2], \ldots, K[U_m]$ are all subgraphs of Q.

Let $1 \leqslant a < b < c \leqslant m$ and consider $\mathscr{H}[U_a, U_b, U_c]$, the set of hyperedges of $\mathscr{H}$ meeting all U_a, U_b and U_c in 1 element. According to our earlier observation (2.2) we have that this hypergraph is $K^{(3)}(2,2,2)$-free. We can apply Erdős's theorem to it, *i.e.*, (1.2) implies that

$$e(\mathscr{H}[U_a, U_b, U_c]) \leqslant O(k^{11/4}).$$

Altogether we have that

$$\begin{aligned} e(\mathscr{H}[X_1]) &= \sum_{1 \leqslant a<b<c \leqslant m} e(\mathscr{H}[U_a, U_b, U_c]) + \sum_{1 \leqslant a<b \leqslant m} e(\mathscr{H}[U_a, U_a, U_b]) + \sum_a e(\mathscr{H}[U_a]) \\ &\leqslant \binom{m}{3} O(k^{11/4}) + m(m-1)\binom{k}{2}k + m\binom{k}{3} \\ &= O(|X_1|^3/k^{1/4}) = O(n^3 \gamma_5^{1/8}). \end{aligned}$$

□

A procedure (similar to the one leading to (5.2) and to Claim 5.1, but here we have to delete vertices of 'large' degrees) gives the following.

Claim 5.3. *There exists a $\gamma_7 = O((\gamma_6)^{1/2})$ and a subset $X_2 \subset X_1$ such that $|X_2| > (1 - \gamma_7)n/2$ and for every $x \in X_2$ the degree of x in $\mathscr{H}[X_2]$ is at most $\gamma_7 n^2$.* □

Claim 5.4. *X_2 contains no triple from $\mathscr{H}$.*

Proof. Suppose, on the contrary, that $\{y_1, y_2, y_3\} \in \mathscr{H}$, $y_1, y_2, y_3 \in X_2$. Consider the link graphs $L_i := G(\mathscr{H}, y_i)$, and let G_i be the restriction of L_i to $V \setminus X_2$.

Let Z be an arbitrary 4-tuple of vertices in $V \setminus X_2$. Consider $G_1[Z]$, $G_2[Z]$ and $G_3[Z]$. These 3 graphs do not contain 3 pairwise crossing pairs, by (2.1). Then Lemma 3.1(iv) implies that $e(G_1[Z]) + e(G_2[Z]) + e(G_3[Z]) \leqslant 15$ instead of the maximum possible $3 \times \binom{4}{2}$. There are $\binom{n-|X_2|-2}{2}$ 4-tuples $Z \subset X_2$ containing any edge, hence

$$\begin{aligned} \binom{n-|X_2|-2}{2} &\times \big(e(G_1) + e(G_2) + e(G_3)\big) \\ &= \sum_{Z \subseteq V \setminus X_2} \big(e(G_1[Z]) + e(G_2[Z]) + e(G_3[Z])\big) \\ &\leqslant 15 \times \binom{n-|X_2|}{4}. \end{aligned}$$

Therefore

$$e(G_1) + e(G_2) + e(G_3) \leqslant \frac{5}{2} \times \binom{n-|X_2|}{2}.$$

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There are at most $(|X_2|-1)(n-|X_2|)$ edges of L_i joining X_2 to its complement. By Claim 5.3 we also have that L_i has at most $\gamma_7 n^2$ edges in X_2. Altogether we get

$$\deg_{\mathscr{H}}(y_1)+\deg_{\mathscr{H}}(y_2)+\deg_{\mathscr{H}}(y_3) < 3\gamma_7 n^2 + 3|X_2|(n-|X_2|) + \frac{5}{2}\binom{n-|X_2|}{2}.$$

Here the right-hand side is at most $\left(\frac{17}{16}+O(\gamma_7)\right)n^2$ (because $|X_2| > (1-\gamma_7)\frac{n}{2}$), while for the left-hand side we have the lower bound condition $3\times\left(\frac{3}{8}+O(\gamma_2)\right)n^2$. This contradiction verifies our claim, that X_2 contains no hyperedges. □

Analogously, there exists an $X_3 \subseteq \overline{X}$ containing no hyperedges such that $|X_3| > (1-\gamma_7)\frac{n}{2}$.

Claim 5.5. *For an arbitrary $x \notin (X_2 \cup X_3)$ consider the linkgraph $L := G(\mathscr{H}, x)$. Either $L[X_2]$ or $L[X_3]$ contains no edge.*

If, say, $L[X_2]$ has no edge, then we can add x to X_3 and continue applying Claim 5.5 until no vertex is left. This claim will complete the proof of Theorem 1.2.

Proof. Suppose, on the contrary, that L has edges in both X_2 and X_3. Since $e(L) > (\frac{3}{4}-\gamma_2)\binom{n}{2}$ and there are at most $O(\gamma_7 n^2)$ edges of L not contained in $X_2 \cup X_3$ and, further, there are at most $n^2/4$ edges meeting both X_2 and X_3, we obtain that there are at least

$$\frac{1}{2}\left(\left(\frac{3}{4}-\gamma_2\right)\binom{n}{2} - O(\gamma_7 n^2) - \frac{1}{4}n^2\right) = \left(\frac{1}{16}-O(\gamma_7)\right)n^2$$

edges contained in one of the sides, say in X_3. Then $L[X_3]$ also contains a matching $a_1b_1, a_2b_2, \ldots, a_mb_m$ of size

$$m > \left(\frac{1}{8}-O(\gamma_7)\right)n.$$

Let $cdx \in \mathscr{H}$, $c, d \in X_2$. Consider the three-element sets meeting cd and two of the matching edges, a_ib_i, a_jb_j. If all of these 8 triples belong to $\mathscr{H}$, then by (2.2) one can extend to a Fano plane the triples xcd, xa_ib_i, xa_jb_j. In fact, not more than 6 of these 8 triples can belong to $\mathscr{H}$. Thus at least $2\binom{m}{2}$ such triples are missing from $\mathscr{H}$, specifically missing from those containing c or d. We obtain

$$2\times\left(\frac{3}{4}-\gamma_2\right)\binom{n}{2} < \deg_{\mathscr{H}}(c)+\deg_{\mathscr{H}}(d) < 2\times\left(\binom{n}{2}-\binom{|X_2|-1}{2}\right) - 2\binom{m}{2}.$$

Here the right-hand side is at most $2\times\left(\frac{47}{64}\right)\binom{n}{2}+O(\gamma_7)n^2$, a contradiction if γ_7 is sufficiently small. □

Since $\gamma_2 = O(\gamma_5^2) = O(\gamma_6^{2\times 8}) = O(\gamma_7^{2\times 8\times 2})$, Theorem 1.2 is true for all sufficiently small γ_2 (and $n > n_0(\gamma_2)$).

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Proof of Theorem 1.1. Knowing Theorem 1.2, it is a standard calculation. Let $g(n) := e(\mathscr{H}^n) = \binom{n}{3} - \binom{\lfloor n/2 \rfloor}{3} - \binom{\lceil n/2 \rceil}{3}$. First, we prove by induction that, for every n,

$$\mathrm{ex}_3(n, \mathbb{F}) \leqslant g(n) + \binom{n_2}{3}. \tag{5.6}$$

(Here n_2 is a constant from Theorem 1.2.) Indeed, this inequality obviously holds for $n \leqslant n_2$. For $n > n_2$ suppose that $e(\mathscr{H}) = \mathrm{ex}_3(n, \mathbb{F})$. If $\min\deg(\mathscr{H}) > (\frac{3}{4} - \gamma_2)\binom{n}{2}$ then we can apply Theorem 1.2. In this case $\mathscr{H}$ is 2-colourable, and $e(\mathscr{H}) \leqslant g(n)$. Otherwise, there exists a vertex x of small degree:

$$\deg_{\mathscr{H}}(x) \leqslant \left(\frac{3}{4} - \gamma_2\right)\binom{n}{2} < g(n) - g(n-1) = \frac{3}{4}\binom{n}{2} + O(n).$$

Applying induction to $e(\mathscr{H} \setminus \{x\})$, we get

$$e(\mathscr{H}) \leqslant g(n-1) + \binom{n_2}{3} + \deg_{\mathscr{H}}(x) \leqslant g(n) + \binom{n_2}{3},$$

verifying (5.6) for all n.

Now suppose that $n > n_1$, where $n_1 = (n_2)^2/\gamma_2$. If $\min\deg(\mathscr{H}) > (\frac{3}{4} - \gamma_2)\binom{n}{2}$ then, as we have seen, Theorem 1.2 completes the proof. Otherwise, there exists a vertex x of small degree:

$$\deg_{\mathscr{H}}(x) \leqslant \left(\frac{3}{4} - \gamma_2\right)\binom{n}{2}.$$

Applying (5.6) to $e(\mathscr{H} \setminus \{x\})$, we get

$$\begin{aligned} e(\mathscr{H}) &\leqslant g(n-1) + \binom{n_2}{3} + \deg_{\mathscr{H}}(x) \\ &\leqslant g(n-1) + \binom{n_2}{3} + \left(\frac{3}{4} - \gamma_2\right)\binom{n}{2} < g(n-1) + \frac{3}{4}\binom{n}{2} < g(n). \end{aligned}$$

Thus the extremal $\mathscr{H}$ is 2-colourable. □

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