The 9-point circle touches the incircle and the escribed circles

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Using inversion we give a short proof for the above theorem of Feuerbach.

Standard notations. Let Δ be a triangle with vertices A, B, C. The side lengths are $a = |BC|$, $b = |AC|$, $c = |AB|$; the lines determined by the sides are ℓ_a, ℓ_b and ℓ_c; the midpoints of the sides AB, BC, and CA are H_c, H_a, and H_b; the semiperimeter is s. Let $\mathcal{C}_0$ be the incircle (the inscribed circle) of Δ, it touches a at A_0. Let $\mathcal{C}_a$ be the escribed circle touching the side a at A_1, by definition it also touches ℓ_b and ℓ_c. In this note we define the **9-point circle** $\mathcal{C}_F$ as the circle thru H_a, H_b, H_c.

The fourth tangent line. The disjoint disks $\mathcal{C}_0$ and $\mathcal{C}_a$ have four common tangents, namely ℓ_a, ℓ_b, ℓ_c and a line ℓ'_a which is the mirror image of the line ℓ_a to the angle bisector f going thru A and the centers of the circles. Let B' and C' on ℓ'_a be the images of B and C mirrored to f.

The inversion. Knowing that two tangents from any point to any circle have equal lengths it is easy to calculate that $|CA_0| = s - c$ and that $|BA_1| = s - c$. Thus the length of the segment A_0A_1 is $|a - 2(s-c)| = |c - b|$, and its midpoint is H_a. Let $\mathcal{K}$ be the circle with center H_a and diameter A_0A_1. To avoid vacuous statements we suppose that $b \neq c$. Consider the inversion i to the circle $\mathcal{K}$. We have $i(A_0) = A_0$, $i(A_1) = A_1$, $i(\ell_a) = \ell_a$.

Claim. $i(\mathcal{C}_0) = \mathcal{C}_0$, $i(\mathcal{C}_a) = \mathcal{C}_a$ and $i(\ell'_a) = \mathcal{C}_F$.

Proof. The inversion keeps tangency so $i(\mathcal{C}_0)$ is a circle touching $i(\ell_a) = \ell_a$ at $i(A_0) = A_0$. We obtain that the image of $\mathcal{C}_0$ is itself. Similarly, $i(\mathcal{C}_a) = \mathcal{C}_a$, too.

To prove that $i(\ell'_a)$ is the 9-point circle it is enough to show that $i(\mathcal{C}_F) = \ell'_a$. Since $\mathcal{C}_F$ goes thru the center of the inversion its image is a line. We only need that the images of H_b and H_c lie on ℓ'_a. Consider H_b, the case of H_c is similar. Let X be the intersection point of the lines H_aH_b and ℓ'_a. Considering the similar triangles $B'AC'$ and $B'H_bX$ we obtain

$$|H_bX| = |AC'|\frac{|H_bB'|}{|AB'|} = |AC'|\frac{|AB'| - |AH_b|}{|AB'|} = b\frac{c - b/2}{c}.$$

If $c - b/2$ is negative, then X is outside the segment $[H_bH_a]$. We obtain $|H_bX| < c/2$ so X

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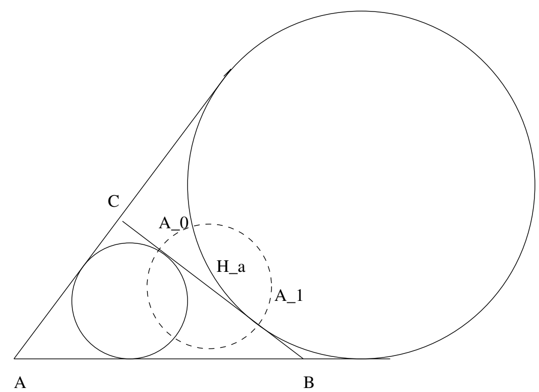
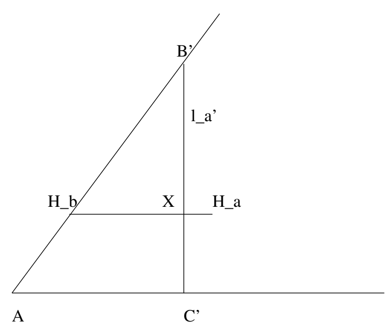
lies on the ray $[H_aH_b)$. Moreover

$$|H_bH_a||XH_a| = |H_bH_a|(|H_bH_a| - |H_bX|) = \frac{c}{2}\left(\frac{c}{2} - b\frac{c-b/2}{c}\right) = \frac{1}{4}(c-b)^2.$$

Thus $i(H_b) = X$ and $i(H_b) \in \ell'_a$. □

Finally, as ℓ'_a is a common tangent to $\mathcal{C}_0$ and $\mathcal{C}_a$ its inversion image $\mathcal{C}_F$ is touching the images of these circles. Since $\mathcal{C}_a$ was chosen arbitrarily we get that the 9-point circle touches the incircle and all the three escribed circles.





Appendix, the properties of inversions

The *inversion* i to a circle $\mathcal{C}(O, r)$ (center O, radius r) is a bijection of $\mathbf{R}^2 \setminus \{O\}$ to itself, such that $i(P)$ lies on the open half ray emanating from O thru P and $|OP| \times |Oi(P)| = r^2$. It is an involution, $i(i(P)) = P$.

— The image of a straight line ℓ thru O is itself.

— The image of ℓ with $O \notin \ell$ is a circle thru O. (More precisely, for $O \in \ell$ we have $i(\ell \setminus \{O\}) = \ell \setminus \{O\}$, and for $O \notin \ell$ the image of the line is a circle minus the point O).

— The image of a circle $\mathcal{C}$ with $O \in \mathcal{C}$ is a line avoiding the center.

— The image of a circle avoiding O is another circle with homothety center O.

— The inversion keeps tangency, touching lines and circles become touching lines and circles (actually it keeps all angles).