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Distance graph on $\mathbb{Z}^n$ with ℓ_1 norm

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Abstract

A long-standing open problem in combinatorial geometry is the chromatic number of the unit-distance graph in $\mathbb{R}^n$; here points are adjacent if their distance in the ℓ_2 norm is 1. For $n=2$, we know the answer is between 4 and 7. Little is known about other dimensions. The subgraphs induced by the rational points have been studied with limited success in small dimensions.

We consider the analogous problem on the n-dimensional integer grid with fixed distance in the ℓ_1 norm. That is, we make two integer grid points adjacent if the sum of the absolute differences in their coordinate values is r. Let the chromatic number of this graph be $\chi(\mathbb{Z},r)$.

The main results of this paper are (i) $\chi(\mathbb{Z}^n,2)=2n$ for all n, and (ii) $(1.139)^n \leqslant \chi(\mathbb{Z}^n,r) \leqslant (1/\sqrt{2\pi n})(5e)^n$ for all n and even r. We also give bounds useful for small values of n and r. We also consider the lower and upper bounds on the n-dimensional real space with unit distance under ℓ_p norm for $1 \leqslant p \leqslant \infty$.

Keywords: Unit-distance graph; Coloring metric spaces; Integer grid

1. Introduction

It is a famous problem to determine the chromatic number of unit distance graph on the Euclidean real plane, i.e., $G=(V,E)$ defined by $V=\mathbb{R}^2$ and $\mathbf{xy}\in E(G)$ for $\mathbf{x}=(x_1,x_2)$ and $\mathbf{y}=(y_1,y_2)$ if and only if $\|\mathbf{x}-\mathbf{y}\|_2=1$, where $\|\mathbf{x}-\mathbf{y}\|_2=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$. It is well known that $4\leqslant\chi(G)\leqslant 7$, where $\chi(G)$ is the chromatic number of G. More generally, for given integers n, p with $n\geqslant 2$ and

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$1 \leqslant p \leqslant \infty$, we can consider graphs on n-dimensional real space under the ℓ_p norm. Specifically, we can define the graph $(\mathbb{R}^n_p, 1)$ with vertex set V and edge set E by

$$V = \mathbb{R}^n,$$

$$\mathbf{xy} \in E \text{ if and only if } \|\mathbf{x} - \mathbf{y}\|_p = 1$$

and consider $\chi(\mathbb{R}^n_p, 1)$.

Similarly, we can define $(\mathbb{Q}^n_p, 1)$ and consider $\chi(\mathbb{Q}^n_p, 1)$ for n-dimensional rational space. Woodall [7] proved that $\chi(\mathbb{Q}^2_2, 1) = 2$. Benda and Perles [1] proved that $\chi(\mathbb{Q}^3_2, 1) = 2$ and $\chi(\mathbb{Q}^4_2, 1) = 4$, and Morayne [6] proved that $\chi(\mathbb{Q}^2_p, 1) = 2$ for $p \geqslant 3$ using Fermat's Last Theorem. See [2] for a survey.

In this paper, for given integers n, r with $n \geqslant 2$ and $r \geqslant 1$, we consider graphs on n-dimensional integer grid under the ℓ_1 norm. More precisely, define the graph $(\mathbb{Z}^n, r) := (\mathbb{Z}^n_1, r)$ with vertex set V and edge set E by

$$V = \mathbb{Z}^n$$

$$\mathbf{xy} \in E \text{ if and only if } \|\mathbf{x} - \mathbf{y}\|_1 = r.$$

We seek its chromatic number $\chi(\mathbb{Z}^n, r)$.

Note that $(\mathbb{Z}^n, r)$ is a subgraph of $(\mathbb{R}^n, r) := (\mathbb{R}^n_1, r)$, and, as a graph, $(\mathbb{R}^n, r)$ is isomorphic to $(\mathbb{R}^n, 1)$ for all r by scaling in $\mathbb{R}^n$. Hence we have $\chi(\mathbb{Z}^n, r) \leqslant \chi(\mathbb{R}^n, r) = \chi(\mathbb{R}^n, 1)$.

For an odd number r, it is easy to prove that there is no odd cycle in $(\mathbb{Z}^n, r)$. If $\mathbf{x}_1\mathbf{x}_2 \ldots \mathbf{x}_k$ is an odd cycle in $(\mathbb{Z}^n, r)$, then $\|\mathbf{x}_i - \mathbf{x}_{i+1}\|_1 = \sum_{j=1}^n |x_{i,j} - x_{i+1,j}| = r$, where $\mathbf{x}_i = (x_1, x_2, \ldots, x_n)$ for $1 \leqslant i \leqslant k$ and $\mathbf{x}_{k+1} = \mathbf{x}_1$. Hence $kr = \sum_{i=1}^k \|\mathbf{x}_i - \mathbf{x}_{i+1}\|_1 \equiv \sum_{j=1}^n x_{1,j} + \sum_{j=1}^n x_{2,j} + \sum_{j=1}^n x_{2,j} + \sum_{j=1}^n x_{3,j} + \cdots + \sum_{j=1}^n x_{1,j} = 2\sum_{i=1}^k \sum_{j=1}^n x_{i,j} \equiv 0 \pmod 2$, but $kr \equiv 1 \pmod 2$ because k and r are odd. Hence, we have

Observation 1: $(\mathbb{Z}^n, r)$ *is a bipartite graph for odd* r.

When r is even, determining $\chi(\mathbb{Z}^n, r)$ is not trivial. The graph $(\mathbb{Z}^n, r)$ has cliques of order $2n$ such as that induced by $\{(0, \ldots, 0, \underbrace{\pm r/2}_{i\text{th}}, 0 \ldots, 0) | i = 1, \ldots, n\}$ in $(\mathbb{Z}^n, r)$. Hence $\chi(\mathbb{Z}^n, r) \geqslant 2n$.

In fact, $\chi(\mathbb{Z}^2, r) = 4$. The clique of size 4 gives $\chi(\mathbb{Z}^2, r) \geqslant 4$. Since $\chi(\mathbb{R}^2, 1) \leqslant 4$ using a tiling by diamonds, we obtain $\chi(\mathbb{Z}^2, r) \leqslant \chi(\mathbb{R}^2, r) = \chi(\mathbb{R}^2, 1) \leqslant 4$ for all even r. So the interesting question that remains is finding $\chi(\mathbb{Z}^n, r)$ for $n \geqslant 3$ and even r.

Observe that $(\mathbb{Z}^n, r)$ has two isomorphic components, one induced by $\{\mathbf{x} = (x_1, \ldots, x_n) \mid \sum x_i = \text{even}\}$ and the other induced by $\{\mathbf{x} = (x_1, \ldots, x_n) \mid \sum x_i = \text{odd}\}$. Call these sets $\mathbf{E}^n$ and $\mathbf{O}^n$, respectively. For $\mathbf{x} \in \mathbf{E}^n$, $\mathbf{y} \in \mathbf{O}^n$, it is impossible that $\|\mathbf{x} - \mathbf{y}\|_1 = r$ for even r due to parity. Hence, without loss of generality, we will find the chromatic number of the component induced by $\mathbf{E}^n$.

In this paper, we will show the following results:

Theorem 1. $\chi(\mathbb{Z}^n, 2) = 2n$ *for all* $n \geqslant 3$.

So, the lower bound for $\chi(\mathbb{Z}^n, 2) = 2n$ turns out to be sharp for $r = 2$. We might hope that this result is true for all r, but in Section 5 we will show that $\chi(\mathbb{Z}^n, r) > 2n$ for all $n \geqslant 3$ and even $r \geqslant 4$. In fact, the next two theorems show that $\chi(\mathbb{Z}^n, r)$ is exponential in n.

Theorem 2. *$\chi(\mathbb{Z}^n, r) \geqslant \Omega((1.139)^n)$ for $n = 4q-1$, $r = 2q$, q odd prime. Moreover, $\chi(\mathbb{Z}^n, r) \geqslant \Omega((1.139)^{n/2})$ for all $n \geqslant 11$ and even $r > n/2$.*

Theorem 3. *$\chi(\mathbb{Z}^n, r) \leqslant (1/\sqrt{2\pi n})(5\mathrm{e})^n$ for all n and even r.*

In Section 5, we show that $\chi(\mathbb{Z}^n, r) \geqslant 2n+1$ for all $n \geqslant 3$ and even $r \geqslant 4$, which is better than Theorem 2 for small values of n. The next theorem gives a bound in terms of both n and r, and is better than Theorem 3 when r is small.

Theorem 4. *$\chi(\mathbb{Z}^n, r) \leqslant 3r^{n-2}$ for all $n \geqslant 3$, even $r \geqslant 4$.*

In the proof of the next result, we extend the ideas used in Theorem 2 thus demonstrating how easily this method leads to lower bounds on $\chi(\mathbb{R}^n_p, 1)$ for all n and p. Theorem 3 extends to a corresponding upper bound.

Theorem 5. (i) *$\chi((\mathbb{R}^n_p, 1)) \geqslant 1.139^n$ for $n = 4q-1$, q odd prime. Moreover, $\chi(\mathbb{R}^n_p, 1) \geqslant \Omega((1.139)^{n/2})$ for all $n \geqslant 11$ and even $r > n/2$.*
(ii) *$\chi((\mathbb{R}^n_p, 1)) \leqslant \sqrt{(p/2\pi n)}(5(ep)^{1/p})^n$ for all n, $1 \leqslant p < \infty$.*

In a recent paper [4], the authors give a new upper bound on $\chi(\mathbb{R}^n_p, 1)$, which is independent of p and is based on a special covering of Euclidean space $\mathbb{R}^n$. Recently, Raigorodskii [8] informed us that a more careful calculation gives $\chi((\mathbb{R}^n_p, 1)) \geqslant 1.365^n$ when $p = 1$.

2. Proof of Theorem 1

Consider a coloring $f : \mathbf{E}^n \to \{0, 1, \ldots, 2n-1\}$ defined by

$$f(\mathbf{x}) = \sum_{i=1}^{n} i x_i - \frac{1}{2} \sum_{i=1}^{n} x_i \pmod{2n}.$$

Consider $\mathbf{x}, \mathbf{y} \in \mathbf{E}^n$ with $\|\mathbf{x} - \mathbf{y}\|_1 = 2$. We want to show that $f(\mathbf{x}) - f(\mathbf{y}) = \sum_{i=1}^{n} i(x_i - y_i) - \frac{1}{2}(\sum_{i=1}^{n} (x_i - y_i)) \neq 0 \bmod 2n$.

If there is i such that $|x_i - y_i| = 2$, then $x_j - y_j = 0$ for all $j \neq i$. If $x_i - y_i = 2$, then $f(\mathbf{x}) - f(\mathbf{y}) = 2i - 1$, which is between 1 and $2n-1$. If $x_i - y_i = -2$, then $f(\mathbf{x}) - f(\mathbf{y}) = -2i + 1 \equiv 2(n-i) + 1$, which is between 1 and $2n-1$.

Otherwise, there are $i, j, 1 \leqslant i < j \leqslant n$, such that $|x_i - y_i| = 1, |x_j - y_j| = 1$. Then we have 4 cases.

Case (i): $x_i - y_i = 1, x_j - y_j = 1$: $f(\mathbf{x}) - f(\mathbf{y}) = i + j - \frac{1}{2}(1+1) = i + j - 1$, which is between 1 and $2n-2$.

Case (ii): $x_i - y_i = 1, x_j - y_j = -1$: $f(\mathbf{x}) - f(\mathbf{y}) = i - j - \frac{1}{2}(1-1) \equiv 2n - (j-i)$, which is between $n+1$ and $2n-1$.

Case (iii): $x_i - y_i = -1, x_j - y_j = 1$: $f(\mathbf{x}) - f(\mathbf{y}) = -i + j - \frac{1}{2}(-1+1) = i + j$, which is between 1 and $n-1$.

Case (iv): $x_i - y_i = -1, x_j - y_j = -1$: $f(\mathbf{x}) - f(\mathbf{y}) = -i - j - \frac{1}{2}(-1-1) = -i - j + 1 \equiv 2n - (i+j) + 1$, which is between 2 and $2n-2$.

3. Proof of Theorem 2

We will use the intersection theorem in [3]:

Theorem 6 (Frankl and Wilson [3]). *Let q be a prime number and $\mathscr{F}$ a $(2q-1)$-uniform family of subsets of a set $4q-1$ elements. If no members of $\mathscr{F}$ intersect in precisely $q-1$ elements, then*

$$|\mathscr{F}| \leqslant 2\binom{4q-1}{q-1}.$$

We will use Theorem 6 to construct a finite subgraph of $(\mathbb{Z}^n, r)$ with small independence number and, consequently, large chromatic number.

Lemma 7. $\chi(\mathbb{Z}^n, r) > (1.139)^n$ *for all* $n = 4q-1$ *and* $r = 2q$, *where q is an odd prime.*

Proof. Consider a subgraph $H_q = (V_q, E_q)$ of $(\mathbb{Z}^{4q-1}, 2q)$ induced by

$$V_q = \{\mathbf{x} = (\varepsilon_1, \ldots, \varepsilon_{4q-1}) \in \{0,1\}^{4q-1} | \sum_{i=1}^{n} \varepsilon_i = 2q-1\}. \tag{1}$$

Note that V_q is the set of incidence vectors of the set $\binom{[4q-1]}{2q-1}$. Letting $\mathbf{x}_A$ denote the incidence vector of $A \in \binom{[4q-1]}{2q-1}$, observe that

$$\|\mathbf{x}_A - \mathbf{x}_B\|_1 = 2(2q-1-|A \cap B|). \tag{2}$$

Hence in the language of Theorem 6, $\mathbf{x}_A\mathbf{x}_B \in E_q$ iff $|A \cap B| = q-1$. Thus $\mathscr{F}$ corresponds to an independent set in H_q, hence

$$\alpha(H_q) \leqslant 2\binom{4q-1}{q-1}.$$

Then

$$\chi(\mathbb{Z}^n, r) \geqslant \chi(H_q)$$
$$\geqslant \frac{\binom{4q-1}{2q-1}}{2\binom{4q-1}{q-1}} = \frac{\binom{4q}{2q}}{\binom{4q}{q}} = \frac{q!(3q)!}{(2q)!(2q)!}$$

$$> \frac{1}{e^{1/12q}} \frac{\sqrt{3}}{2} \sqrt[4]{\frac{27}{16}} \left(\sqrt[4]{\frac{27}{16}}\right)^{4q-1}$$
$$= \Omega((1.139)^{4q-1}).$$

For any integer N, there always exists a prime number q such that $N \leqslant q < 2N$. Using this fact, it is easy to show that $\chi(\mathbb{Z}^n, r) \geqslant \Omega((1.139)^{n/2})$ for all $n \geqslant 11$. □

A similar calculation for $\chi(\mathbb{R}_2^n, 1)$ is done in the above-mentioned paper of Frankl and Wilson.

4. Proof of Theorem 3

We will prove $\chi(\mathbb{R}^n, 1) \leqslant (1/\sqrt{2\pi n})(5e)^n$ for all n and even r.

Cut $\mathbb{R}^n$ into regions

$$R(\mathbf{a}) = \{\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n : \left(a_i - \tfrac{1}{2}\right) \frac{1}{n} \leqslant x_i < \left(a_i + \tfrac{1}{2}\right) \frac{1}{n} \text{ for } 1 \leqslant i \leqslant n\} \tag{3}$$

for each $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n$. Each region is nothing but a half-open and half-closed cube centered at $(1/n)\mathbf{a}$ with each side length $1/n$. Observe that (i) $\{R(\mathbf{a}) : \mathbf{a} \in \mathbb{Z}^n\}$ is a tiling of $\mathbb{R}^n$, (ii) $\|1/n\mathbf{a} - \mathbf{x}\|_1 \leqslant \frac{1}{2}$ for all $\mathbf{x} \in R(\mathbf{a})$, and (iii) for each $\mathbf{a} \in \mathbb{Z}^n$ and $\mathbf{x}$, $\mathbf{y} \in R(\mathbf{a})$, we have $\|\mathbf{x} - \mathbf{y}\|_1 < 1$.

Define an auxiliary graph H such that

$$\begin{aligned} V(H) &= \text{cubes} = \{R(\mathbf{a}) : \mathbf{a} \in \mathbb{Z}^n\} \quad \text{and} \\ &R(\mathbf{a})R(\mathbf{a}') \in E(H) \; iff \text{ there are } \mathbf{x} \in R(\mathbf{a}), \mathbf{y} \in R(\mathbf{a}') \\ &\text{such that } \|\mathbf{x} - \mathbf{y}\|_1 = 1. \end{aligned} \tag{4}$$

It is easy to see that a proper coloring of H gives a proper coloring of $(\mathbb{R}^n, 1)$; hence $\chi(\mathbb{R}^n, 1) \leqslant \chi(H)$. We will bound $\chi(H)$ from above by 1 plus its maximum degree.

Lemma 8. *If* $R(\mathbf{a})R(\mathbf{a}') \in E(H)$, *then* $\|(1/n)\mathbf{a} - (1/n)\mathbf{a}'\|_1 \leqslant 2$.

Proof. $R(\mathbf{a})R(\mathbf{a}') \in E(H)$ means that $\|\mathbf{x} - \mathbf{y}\|_1 = 1$ for some $\mathbf{x} \in R(\mathbf{a})$, $\mathbf{y} \in R(\mathbf{a}')$. Then $\|(1/n)\mathbf{a} - \mathbf{x}\|_1 \leqslant \frac{1}{2}$, $\|(1/n)\mathbf{a}' - \mathbf{y}\|_1 \leqslant \frac{1}{2}$. Hence triangle inequality yields $\|(1/n)\mathbf{a} - (1/n)\mathbf{a}'\|_1 \leqslant \|(1/n)\mathbf{a} - \mathbf{x}\|_1 + \|\mathbf{x} - \mathbf{y}\|_1 + \|(1/n)\mathbf{a}' - \mathbf{y}\|_1 \leqslant 2$, as desired. □

Lemma 8 tells us that such $R(\mathbf{a}')$ lies in the ball centered at $R(\mathbf{a})$ of radius $\frac{5}{2}$. In order to bound the maximum degree of H from above, it is enough to count the cubes of side length $1/n$ in a generalized octahedron of radius $\frac{5}{2}$. The number of cubes is at most

$$\begin{aligned}&\frac{\text{Volume of a generalized octahedron of radius } \frac{5}{2}}{\text{Volume of a cube of side length } 1/n}\\&= \frac{\left(\frac{5}{2}\right)^n 2^n/n!}{(1/n)^n}\\&= 5^n \frac{n^n}{n!}\\&< \frac{1}{\sqrt{2\pi n}}(5\mathrm{e})^n\end{aligned}$$

by Stirling's formula

$$\frac{n^n}{n!} \leqslant \frac{1}{\sqrt{2\pi n}}\,\mathrm{e}^n.$$

5. Further results

The next two results are useful for getting non-trivial bounds on $\chi(\mathbb{Z}^n, r)$ for small values of n and r. For example, we can show that $7 \leqslant \chi(\mathbb{Z}^3, 4) \leqslant 12$ for the smallest values of n, r such that $\chi(\mathbb{Z}^n, r)$ is unknown.

Proposition 9. $\chi(\mathbb{Z}^n, r) \geqslant 2n+1$ *for all* $n \geqslant 3$ *and even* $r \geqslant 4$.

Proof. We give a proof for $r = 4$ to keep the notation simple. The proof for general r is identical.

We begin with some definitions and notation.

For $\mathbf{uv} \in E(\mathbf{E}^n, 4)$, where $\mathbf{u}, \mathbf{v} \in \mathbb{Z}^n \subset \mathbb{R}^n$, there can be a point $\mathbf{w} \in \mathbf{E}^n$ lying on a line segment from $\mathbf{u}$ to $\mathbf{v}$ in $\mathbb{R}^n$. Let $\mathbf{u} = (u_1, \ldots, u_n)$ and $\mathbf{v} = (v_1, \ldots, v_n)$. If $|u_i - v_i| = |u_j - v_j| = 2$ for some $i \neq j$ so that $u_t = v_t$ for all $t \neq i, j$, then a point $\mathbf{w} = (w_1, \ldots, w_n)$ with

$$w_t = \begin{cases} \dfrac{u_t + v_t}{2} & \text{if } t = i \text{ or } j, \\ u_t (= v_t) & \text{otherwise} \end{cases} \tag{5}$$

belongs to $\mathbf{E}^n$ and lies on the line segment from $\mathbf{u}$ to $\mathbf{v}$ in $\mathbb{R}^n$. For each edge $\mathbf{uv}$ such that $|u_i - v_i| = |u_j - v_j| = 2$ for some $i \neq j$, we call such $\mathbf{w} \in V(\mathbf{E}^n, 4)$ a *halving vertex of* $\mathbf{uv}$.

For a given $\mathbf{x} \in \mathbf{E}^n$, let $P(\mathbf{x}) = \{\mathbf{x} \pm 2\mathbf{e}_i \mid 1 \leqslant i \leqslant n\}$, where $\mathbf{e}_i$ is the ith row of the $n \times n$ identity matrix. The set $P(\mathbf{x})$ is the set of $2n$ vertices of a generalized octahedron centered at $\mathbf{x}$ with radius 2. As before, $P(\mathbf{x})$ induces a clique of size $2n$ in $(\mathbf{E}^n, 4)$. We call the edges of a generalized octahedron in $\mathbb{R}^n$, *sideedges* to distinguish them from edges of a graph.

When $\mathbf{u}$ and $\mathbf{v}$ are vectors expressed as sums, we write the edge $\mathbf{uv}$ as $\{\mathbf{u}, \mathbf{v}\}$ for clarity. Note that $\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j$ is a halving vertex of $\{\mathbf{x} + 2\mathbf{e}_i,\ \mathbf{x} + 2\mathbf{e}_j\}$. Furthermore, it is not difficult to see that the halving vertex $\mathbf{x}+\mathbf{e}_i+\mathbf{e}_j$ is adjacent to all the vertices of $P(\mathbf{x})$ except $\mathbf{x}+2\mathbf{e}_i$ and $\mathbf{x}+2\mathbf{e}_j$. Similarly, $\mathbf{x}+\mathbf{e}_i-\mathbf{e}_j$, $\mathbf{x}-\mathbf{e}_i+\mathbf{e}_j$, $\mathbf{x}-\mathbf{e}_i-\mathbf{e}_j$, respectively, are halving vertices of $\{\mathbf{x}+2\mathbf{e}_i, \mathbf{x}-2\mathbf{e}_j\}$, $\{\mathbf{x}-2\mathbf{e}_i, \mathbf{x}+2\mathbf{e}_j\}$, $\{\mathbf{x}-2\mathbf{e}_i, \mathbf{x}-2\mathbf{e}_j\}$, respectively. Also, each of the halving vertices $\mathbf{x}+\mathbf{e}_i-\mathbf{e}_j$, $\mathbf{x}-\mathbf{e}_i+\mathbf{e}_j$, $\mathbf{x}-\mathbf{e}_i-\mathbf{e}_j$, respectively, is adjacent to all the vertices of $P(\mathbf{x})$ except for $\mathbf{x}+2\mathbf{e}_i$ and $\mathbf{x}-2\mathbf{e}_j$, $\mathbf{x}-2\mathbf{e}_i$ and $\mathbf{x}+2\mathbf{e}_j$, $\mathbf{x}-2\mathbf{e}_i$ and $\mathbf{x}-2\mathbf{e}_j$, respectively.

Lemma 10. *Suppose that* $(\mathbf{E}^n, r)$ *is* $2n$*-colorable, say under* f *with colors* $\{1,2,\ldots,2n\}$. *If* $\mathbf{w}\in\mathbf{E}^n$ *is a halving vertex of* $\mathbf{uv}$, *then* $f(\mathbf{u})\neq f(\mathbf{v})$, *and* $f(\mathbf{w})\in\{f(\mathbf{u}), f(\mathbf{v})\}$.

Proof. It is immediate that $f(\mathbf{u})\neq f(\mathbf{v})$ by the definition of the halving vertex. Consider a generalized octahedron determined by $P(\mathbf{x})$ with *one side edge passing through* **u,w,v**. For example, if $\mathbf{u}=\mathbf{w}+\mathbf{e}_i+\mathbf{e}_j$ and $\mathbf{v}=\mathbf{w}-\mathbf{e}_i-\mathbf{e}_j$, we can take $\mathbf{x}$ as $\mathbf{w}+\mathbf{e}_i-\mathbf{e}_j$ or $\mathbf{w}-\mathbf{e}_i+\mathbf{e}_j$. Then $P(\mathbf{x})$ consumes all the $2n$ colors. Since $\mathbf{w}$ is non-adjacent only to $\mathbf{u}$ and $\mathbf{v}$ among $P(\mathbf{x})$, $f(\mathbf{w})$ must be one of $f(\mathbf{u})$ or $f(\mathbf{v})$. This completes the proof of the lemma. □

Now, suppose that $(\mathbf{E}^n, 4)$ is $2n$-colorable, say under f with colors $\{1,2,\ldots,2n\}$, and consider a fixed $P(\mathbf{x})$ with $\mathbf{x}\in\mathbf{E}^n$. Without loss of generality, assume $f(\mathbf{x}-2\mathbf{e}_1)=1$, $f(\mathbf{x}+2\mathbf{e}_1)=2$, $f(\mathbf{x}-2\mathbf{e}_2)=3$, $f(\mathbf{x}+2\mathbf{e}_2)=4$, $f(\mathbf{x}-2\mathbf{e}_3)=5$, $f(\mathbf{x}+2\mathbf{e}_3)=6$, etc. By Lemma 10, $f(\mathbf{x}+\mathbf{e}_1+\mathbf{e}_2)\in\{2,4\}$, $f(\mathbf{x}-\mathbf{e}_1+\mathbf{e}_2)\in\{1,4\}$, $f(\mathbf{x}-\mathbf{e}_1-\mathbf{e}_2)\in\{1,3\}$, $f(\mathbf{x}+\mathbf{e}_1-\mathbf{e}_2)\in\{2,3\}$. Observe that $\mathbf{x}$ is a halving vertex of both $\{\mathbf{x}+\mathbf{e}_1+\mathbf{e}_2, \mathbf{x}-\mathbf{e}_1-\mathbf{e}_2\}$ and $\{\mathbf{x}-\mathbf{e}_1+\mathbf{e}_2, \mathbf{x}+\mathbf{e}_1-\mathbf{e}_2\}$. Applying Lemma 10 with $\mathbf{w}=\mathbf{x}$ and $\mathbf{uv}=\{\mathbf{x}+\mathbf{e}_1+\mathbf{e}_2, \mathbf{x}-\mathbf{e}_1-\mathbf{e}_2\}$, and then with $\mathbf{w}=\mathbf{x}$ and $\mathbf{uv}=\{\mathbf{x}-\mathbf{e}_1+\mathbf{e}_2, \mathbf{x}+\mathbf{e}_1-\mathbf{e}_2\}$, we have $f(\mathbf{x})\in\{1,2,3,4\}$. By symmetry, we may assume $f(\mathbf{x})=1$. A similar argument with $\mathbf{x}\pm\mathbf{e}_2\pm\mathbf{e}_3$ yields that $f(\mathbf{x})\in\{3,4,5,6\}$, which contradicts $f(\mathbf{x})=1$. □

Recall that $(\mathbf{E}^n, r)$ and $(\mathbf{O}^n, r)$ are isomorphic induced subgraphs of $(\mathbb{Z}^n, r)$ for even r. Hence we have studied $\chi(\mathbf{E}^n, r)$ for $\chi(\mathbb{Z}^n, r)$. When we say "a proper coloring for $(\mathbb{Z}^n, r)$", we will assume that the vertices of $\mathbf{O}^n$ are colored by translating the color of $\mathbf{E}^n$ along, say, the last coordinate by one, unless some specific coloring is described.

Theorem 4. $\chi(\mathbb{Z}^n, r)\leqslant 3r^{n-2}$ *for all* $n\geqslant 3$, *even* $r\geqslant 4$.

Proof. Let us prove the recurrence relation $\chi(\mathbb{Z}^n, r)\leqslant r\chi(\mathbb{Z}^{n-1}, r)$ for all $n\geqslant 4$ and even $r\geqslant 4$.

For $s\in\mathbb{Z}$ and $0\leqslant i\leqslant r-1$, let $\mathscr{H}_{s,i}$ be the hyperplane defined by $\mathscr{H}_{s,i}=\{\mathbf{x}\in\mathbb{Z}^n \mid x_n = sr+i\}$. Let f be an optimal coloring for $(\mathbb{Z}^{n-1}, r)$, and put $k=\chi(\mathbb{Z}^{n-1}, r)$. We extend f to a proper rk-coloring on $(\mathbb{Z}^n, r)$ by coloring each $\mathscr{H}_{s,i}$ appropriately. The idea is to color $\mathscr{H}_{0,0}$ by f and then translate this coloring to $\mathscr{H}_{0,i}$, for each i, along the vector $\vec{v}:=(0,0,\ldots,0,1,1)$, accompanied by shifting the colors by ik. This coloring of the family of hyperplanes $\{\mathscr{H}_{0,i} \mid 0\leqslant i\leqslant r-1\}$ is extended to all such other families, $\{\mathscr{H}_{s,i} \mid 0\leqslant i\leqslant r-1\}$ for $s\in\mathbb{Z}$ by inductively translating, along the vector $\vec{v}$, the coloring of the family of hyperplanes corresponding to $s\geqslant 0$ ($\leqslant 0$, respectively,) to the family corresponding to $s+1$ ($s-1$, respectively,) along the vector

$\vec{v}$. More precisely, for each i with $0 \leqslant i \leqslant (r-1)$, define g_i on $\mathscr{H}_{0,i}$ by $g_i(x_1,\ldots,x_{n-1},i) = f(x_1,\ldots,x_{n-1}-i)+ik$. Then, for $s=\pm 1, \pm 2, \pm 3, \ldots$, extend g_i to the hyperplane $\mathscr{H}_{s,i}$, inductively by $g_i(x_1,\ldots,x_{n-1},sr+i) = g_i(x_1,\ldots,x_{n-1}-r,(s-1)r+i)$ when $s>0$, and $g_i(x_1,\ldots,x_{n-1},sr+i) = g_i(x_1,\ldots,x_{n-1}+r,(s+1)r+i)$ when $s<0$. Now $\{g_i \mid 0 \leqslant i \leqslant r-1\}$ is a proper rk-coloring for $(\mathbb{Z}^n, r)$ whose vertex set $\mathbb{Z}^n$ is $\bigcup_{0\leqslant i \leqslant r-1} \bigcup_{s\in\mathbb{Z}} \mathscr{H}_{s,i}$. It is enough to show that g_i is a proper k-coloring for the subgraph induced by $\bigcup_{s\in\mathbb{Z}} \mathscr{H}_{s,i}$ for each $i=0,1,\ldots,r-1$ since $ik \leqslant g_i \leqslant ik+(k-1)$, that is, the values for g_i and g_j are disjoint for $i \neq j$. It is easy to see that g_i is a proper coloring on the hyperplane $\mathscr{H}_{s,i}$ for each s because g_i on $\mathscr{H}_{s,i}$ is defined as a translation of a proper coloring f on $(\mathbb{Z}^{n-1}, r)$. So the possible neighbors with the same color in $\bigcup_{s\in\mathbb{Z}} \mathscr{H}_{s,i}$ of an arbitrary vertex $\mathbf{x}=(x_1,\ldots,x_n)$ of the hyperplane $\mathscr{H}_{s,i}$ are $\mathbf{x}' := (x_1,\ldots,x_{n-1},(s-1)r+i)$ in $\mathscr{H}_{s-1,i}$ and $\mathbf{x}'' := (x_1,\ldots,x_{n-1},(s+1)r+i)$ in $\mathscr{H}_{s+1,i}$. Let us assume that $s>0$. (A similar proof applies for $s \leqslant 0$.) The color $g_i(\mathbf{x})$ has been defined inductively to equal $g_i(\mathbf{y}')$, where $\mathbf{y}' := (x_1,\ldots,x_{n-1}-r,(s-1)r+i)$, but $\mathbf{y}'$ is a neighbor of $\mathbf{x}'$ in $\mathscr{H}_{s-1,i}$, and hence $g_i(\mathbf{x}') \neq g_i(\mathbf{y}') = g_i(\mathbf{x})$. Similarly, $g_i(\mathbf{x}'') = g_i(\mathbf{y}'')$, where $\mathbf{y}'' := (x_1,\ldots,x_{n-1}-r,sr+i)$, but $\mathbf{y}''$ is a neighbor of $\mathbf{x}$ in $\mathscr{H}_{s,i}$ hence $g_i(\mathbf{y}'') \neq g_i(\mathbf{x})$. □

The recursive upper bound just proved, along with the fact that $\chi(\mathbb{Z}^2, r)=4$, implies $\chi(\mathbb{Z}^n, r) \leqslant 4r^{n-2}$. We get the stronger bound as stated in the theorem by the following proposition.

Proposition 11. *$\chi(\mathbb{Z}^3, r) \leqslant 3r$ for all even $r \geqslant 4$.*

Proof. To color $(\mathbb{Z}^3, r)$, the idea is similar to the proof of the recurrence above. For $t \in \mathbb{Z}$, let $\mathscr{H}_t$ be the plane defined by $\mathscr{H}_t = \{\mathbf{x} \in \mathbb{Z}^3 \mid x_3 = t\}$. We just give a specific coloring of the planes $\{\mathscr{H}_t \mid t \equiv 0 \,(\mathrm{mod}\,(r/2))\}$ and then translate, along the vector $\vec{v} := (0,1,1)$, these colorings to all other hyperplanes $\{\mathscr{H}_t \mid t \equiv i \,(\mathrm{mod}\,(r/2))\}$ for each i with $1 \leqslant i \leqslant r/2-1$, accompanied by shifting the color by i. More precisely, begin by coloring the vertices $\mathbf{x}$ whose coordinates are all equivalent to 0 modulo $r/2$ with $\sum_{i=1}^{3} x_i/r/2 \equiv 0 \,(\mathrm{mod}\,2)$, i.e., for $\mathbf{x} = (r/2)\mathbf{x}' = (r/2)(x_1', x_2', x_3')$ with $\sum_{i=1}^{3} x_i'$ being even, define $f_0(\mathbf{x}) = \sum_{i=1}^{3} i x_i' - \frac{1}{2}\sum_{i=1}^{3} x_i' \,(\mathrm{mod}\,6)$. Then, using $(x_i-y_i)/r/2 = x_i' - y_i'$ and applying the proof of Theorem 1 with $x_i' - y_i'$, it is easy to check that f_0 is a proper 6-coloring for the subgraph of $(\mathbb{Z}^3, r)$ induced by such $\mathbf{x}$. In the plane $\mathscr{H}_t$ for each $t \equiv 0 \,(\mathrm{mod}\,\frac{r}{2})$, the set of vertices of this type form a diamond tiling for each plane. Extend f_0 to the whole plane $\mathscr{H}_t$ by filling each diamond with the color of the leftmost vertex of the diamond. This yields a proper coloring for the subgraph of $(\mathbb{Z}^3, r)$ induced by $\{\mathscr{H}_t \mid t \equiv 0 \,(\mathrm{mod}\,(r/2))\}$. For each $i = 1,2,\ldots,\frac{r}{2}-1$, consider the subgraph of $(\mathbb{Z}^3, r)$ induced by $\{\mathscr{H}_t \mid t \equiv i \,(\mathrm{mod}\,(r/2))\}$. We will shift the coloring of $\{\mathscr{H}_t \mid t \equiv 0 \,(\mathrm{mod}\,(r/2))\}$ to $\{\mathscr{H}_t \mid t \equiv i \,(\mathrm{mod}\,(r/2))\}$ as described above. That is, for the vertex $\mathbf{x} = (x_1,x_2,x_3) \in \mathscr{H}_t$ where $t \equiv i \,(\mathrm{mod}\,(r/2))$, define $f_i(\mathbf{x}) = f_0(x_1, x_2-i, x_3-i)+i$. Now f_i is a proper 6-coloring for the subgraph induced by the planes $\{\mathscr{H}_t \mid t \equiv i \,(\mathrm{mod}\,(r/2))\}$ since f_i is just a translation of a proper coloring f_0. Distinct $i, j, 0 \leqslant i, j \leqslant r/2-1$, produce disjoint color classes of f_i, f_j; hence $\{f_i \mid 0 \leqslant i \leqslant r/2-1\}$ gives a proper coloring for $(\mathbb{Z}^3, r)$ with $r/2 \times 6 = 3r$ colors. □

It is not possible to use the idea of the proof for $(\mathbb{Z}^3, r)$ to give a better bound on $(\mathbb{Z}^n, r)$. For general n, we can construct a similar $2n$-coloring f_0 on the subgraph induced by the vertices whose coordinates are all congruent to 0 modulo $r/2$. But such vertices form generalized octahedrons in the hyperplane $x_n = t$, $t \equiv 0 \pmod{(r/2)}$, and generalized octahedrons are *not* a tiling for $\mathbb{Z}^n$ (in fact, for $\mathbb{R}^n$). That is, coloring each piece of a generalized octahedron with the color of its leftmost colored vertex under f_0 leaves some uncolored vertices in the hyperplane.

Let us revisit the graph $(\mathbb{R}^n_p, 1)$ for given integers n, p with $n \geqslant 2$, $1 \leqslant p \leqslant \infty$, introduced in Section 1.

It is easy to prove that $\chi(\mathbb{R}^n_\infty, 1) = 2^n$ for all $n \in \mathbf{N}$. The 2^n corner points of hypercube $\{(x_1, \ldots, x_n) \mid x_i = 0 \text{ or } 1 \text{ for } i = 1, \ldots, n\}$ form a clique, hence yield the lower bound, and coloring a vertex $\mathbf{x} = (x_1, \ldots, x_n)$ with $0,1$-strings of length n by $\lfloor x_i \rfloor \pmod 2$ for the ith digit gives us the upper bound.

The ideas similar to Theorems 2 and 3 yield the following theorem for the n-dimensional real space with unit distance under ℓ_p norm for $1 \leqslant p < \infty$.

Theorem 5. (i) $\chi(\mathbb{R}^n_p, 1) > (1.139)^n$ *for* $n = 4q - 1$, q *odd prime. Moreover*, $\chi(\mathbb{R}^n_p, 1) \geqslant \Omega((1.139)^{n/2})$ *for all* $n \geqslant 11$ *and even* $r > n/2$.

(ii) $\chi((\mathbb{R}^n_p, 1)) \leqslant \sqrt{\dfrac{p}{2\pi n}}(5(ep)^{1/p})^n$ *for all* n, $1 \leqslant p < \infty$.

Proof. (i) Note that $(\mathbb{R}^n_p, 1)$ is isomorphic to $(\mathbb{R}^n_p, r)$ for any $r > 0$. Hence it is enough to prove that the value 1.139^n is the lower bound of $\chi(\mathbb{R}^n_p, \sqrt[p]{2q})$.

Let H_q be the subgraph induced by the same V_q in (1). Then substituting

$$\|\mathbf{x}_A - \mathbf{x}_B\|_p^p = 2(2q - 1 - |A \cap B|) \tag{6}$$

for (2) yields exactly the same argument as the proof in the Section 3 and leads to the desired bound.

(ii) We can replace (3), (4) and Lemma 8 by

- $R(\mathbf{a}) = \{\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid (a_i - \frac{1}{2})1/\sqrt[p]{n} \leqslant x_i < (a_i + \frac{1}{2})1/\sqrt[p]{n} \text{ for } 1 \leqslant i \leqslant n\}$
- $R(\mathbf{a})R(\mathbf{a}') \in E(H)$ *iff* there are $\mathbf{x} \in R(\mathbf{a})$, $\mathbf{y} \in R(\mathbf{a}')$ such that $\|\mathbf{x} - \mathbf{y}\|_p = 1$

Lemma 8. *If* $R(\mathbf{a})R(\mathbf{a}') \in E(H)$, $\|(1/n)\mathbf{a} - (1/n)\mathbf{a}'\|_p \leqslant 2$

With the same reason in the proof of Theorem 3, it is enough to count the number of cubes $R(\mathbf{a}')$ in a ball of radius $\frac{5}{2}$ in $\mathbb{R}^n$ under ℓ_p norm, denoted by $B^n_p(\frac{5}{2})$, centered around a cube $R(\mathbf{a})$.

$$\Delta(H) \leqslant \frac{\text{Volume}(B^n_p(\frac{5}{2}))}{\text{Volume}(R(\mathbf{a}))} = \frac{(\frac{5}{2})^n (2\Gamma(1 + 1/p))^n / \Gamma(1 + n/p)}{(1/\sqrt[p]{n})^n}, \tag{7}$$

where Γ is the Gamma function;

$$\Gamma(x) := \int_0^\infty e^{-t} t^{x-1}\, dt.$$

Applying the well-known inequalities

$$\left(\frac{x}{\mathrm{e}}\right)^x \sqrt{2\pi x} \leqslant \Gamma(x) \leqslant \left(\frac{x}{\mathrm{e}}\right)^x \sqrt{2\pi x}\mathrm{e}^{1/12x} \quad \text{for } x>0$$

to (7) gives us the result. □

For the Euclidean norm ($p=2$), a more careful calculation gives an upper bound of $(1/\sqrt{\pi n})(5\pi \mathrm{e})^{n/2}$ for all n.

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