Covering a triangle with homothetic copies

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Abstract Let Δ_0 be a triangle and let $\mathcal{H} = \{\Delta_1, \dots, \Delta_n\}$ be a set of (positive) homothetic copies of Δ_0. We prove a conjecture of Bezdek & Bezdek (1984) that if the total area of $\mathcal{H}$ is at least twice the area of Δ_0, then there exist translates of $\Delta_1, \dots, \Delta_n$ that cover Δ_0.

1. Translation coverings

Let C be a *disk*, i.e., convex, compact set on the Euclidean plane with interior points. Let $\mathcal{H} = \{C_1, \dots, C_i, \dots\}$ be a finite sequence of disks. We say that $\mathcal{H}$ permits a **translation covering** of C if there exist translations τ_i such that $C \subseteq \cup_i \tau_i(C_i)$. Moser and Moon [9] showed that if Q is the unit square and $\mathcal{H}$ is a set of squares of sizes $x_1, x_2, \dots$ with total area $\sum_i x_i^2 \geq 3$ and with sides parallel (or orthogonal) to Q, then $\mathcal{H}$ permits a translation covering of Q. This is the best possible bound as one can see from the example $x_1 = x_2 = x_3 = 1 - \varepsilon$, $x_4 = \cdots = 0$.

L. Fejes Tóth proposed the following question. Suppose that each C_i is a (positive) homothetic copy of C. How large the sum of areas of the C_i's must be, so that C can be covered by translates of the C_i's? Denote the ratio of this minimum (infimum) and

Discrete Geometry, (edited by A. Bezdek) Dekker, New York–Basel, 2003. pp. 435–445.
November 15, 2001 (slightly revised 04/12/2002) `haromsz2bea.tex`
Research supported in part by the Hungarian National Science Foundation under grant OTKA T 032452, and by the National Science Foundation under grant 1-5-29066 NSF DMS 00-70312.

the area of C by $f(C)$. The above cited theorem of Moon and Moser states $f(Q) = 3$. This easily implies $f(C) \leq 12$ for every disks (Bezdek and Bezdek [1]) and it was recently improved to $f(C) \leq 22/3$ by Januszewski [7]. One can observe that for any C one has $f(C) \geq 2$ (two copies of sizes $1-\varepsilon$ can't cover a diameter of C). Bezdek and Bezdek [1] conjectured that this is achievable for any triangle Δ. Here we prove this conjecture and show that $f(\Delta) = 2$.

Theorem 1 *Let Δ_0 be a triangle and let $\mathcal{H}$ be a set of (positive) homothetic copies of Δ_0. Suppose that $\sum_{\Delta \in \mathcal{H}} \mathrm{Area}(\Delta) \geq 2\mathrm{Area}(\Delta_0)$. Then there exist translates of the members of $\mathcal{H}$ that cover Δ_0.*

The minimum density of a covering a (large) convex region with small equal sized homothetic triangles is 3/2 and this can be obtained by a hexagonal lattice arrangement. (See, e.g., in the excellent monograph of Pach and Agarwal [10]). So the constant factor 2 cannot be decreased below this value, even if the triangles are all small. In contrast, in the case of the unit square, Q, Moon and Moser showed that total area

$$> 1 + 2x_1 \tag{1}$$

ensures a translation covering.

However, it seems to be impossible to imitate such hexagonal arrangements if the triangles have different sizes. So we use the method of Moon and Moser, the triangles of $\mathcal{H}$ are ordered in layers with minimum overlaps. This way we are able to keep track the area of the covered region. However, in these arrangements, we only use a small part of most of the triangles $\Delta \in \mathcal{H}$, namely, an inscribed parallelogram, and waste the rest of Δ. As every inscribed parallelogram has area at most $\frac{1}{2}\mathrm{Area}(\Delta)$, we waste a lot and apparently loose a factor of 2 immediately. So proving $f(\Delta_0) \leq 2$ seems impossible by defining layers and the placements of the triangles in this way, but as we can see in the next sections, still can be done with extreme care. (The real improvements comes by adding the triangle Δ_{s+1}, see below in Section 2).

One can generalize Moon and Moser's theorem for every convex disk C. Denote by $\vartheta_T(C)$ the infimum of the densities of coverings the plane by translated congruent copies of C. Let $\mathcal{H}$ be a set of (positive) homothetic copies of C.

Theorem 2 *For every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon, C) > 0$ such that the following holds. If each member of $\mathcal{H}$ is smaller then δC and for the total area we have $\sum_{H \in \mathcal{H}} \mathrm{Area}(H) \geq (\vartheta_T(C) + \varepsilon)\mathrm{Area}(C)$, then there exist translates of the members of $\mathcal{H}$ that cover C.*

The proof of this is straightforward and it is postponed to Section 5.

2. Constructing the covering of Δ_0

The aim of this section is to define a translation of each member of $\mathcal{H}$. In the next section we will show that they really cover Δ_0.

As translation coverings are affine invariant, we may suppose that Δ_0 (and all members of $\mathcal{H}$) are isosceles, right angled triangles, with angles 90°, 45° and 45°. We may also suppose that the coordinates of the vertices of Δ_0 are $(0,0)$, $(1,0)$, and $(0,1)$. Let $a(\Delta)$ denote the length of the sides of the triangle Δ, so we have $\text{Area}(\Delta) = \frac{1}{2}a(\Delta)^2$, and for the total area

$$\sum_{\Delta \in \mathcal{H}} \text{Area}(\Delta) \geq 2\text{Area}(\Delta_0) = 1. \tag{2}$$

We order the triangles in decreasing order by their sizes and form disjoint subgroups $\mathcal{H}_1, \mathcal{H}_2 \dots$ of $\mathcal{H}$, at the same time we define right angled trapezoids $T_1, T_2, \dots$ such that some translations of the members of $\mathcal{H}_k$ cover T_k. The base lengths of T_k are denoted by b_{k-1} and b_k, its heights is h_k, $h_k = b_k - b_{k-1}$ and side lengths h_k and $\sqrt{2}h_k$, with vertices $(0, 1-b_{k-1})$, $(0, 1-b_k)$, $(b_k, 1-b_k)$, and $(b_{k-1}, 1-b_{k-1})$.

The first group, $\mathcal{H}_1$, consists of a single triangle of the largest size from $\mathcal{H}$, let us denote its side length by b_1. Place this largest triangle such that it covers the upper part of Δ_0, i.e., define the (now degenerate) trapezoid T_1 by its three vertices $(0,1)$, $(0, 1-b_1)$, and $(b_1, 1-b_1)$.

Whenever $k \geq 2$ and $\mathcal{H}_1, \dots, \mathcal{H}_{k-1}$ had already been defined together with $T_1, \dots, T_{k-1}$ then we proceed as follows. Consider the rest of the triangles $\mathcal{R}_k := \mathcal{H} \setminus (\mathcal{H}_1 \cup \dots \cup \mathcal{H}_{k-1})$, and denote by $a_1 \geq a_2 \geq \dots$ the lengths of the sides of the members of $\mathcal{R}_k$, with $a_i = a(\Delta_i)$. If $\mathcal{R}_k$ is empty the procedure stops. Also if

$$\sum_{\Delta \in \mathcal{R}_k} a(\Delta) \leq b_{k-1} := b, \tag{3}$$

then our procedure decomposing $\mathcal{H}$ stops. Otherwise let s be the maximum integer, such that

$$\sum_{1 \leq i \leq s} (a_i - \frac{1}{2}a_s) \leq b. \tag{4}$$

Here $s \geq 2$, because $a_1 \leq b$. We have that

$$b < \sum_{1 \leq i \leq s+1} (a_i - \frac{1}{2}a_{s+1}). \tag{5}$$

(If $|\mathcal{R}_k| = s$, then here we introduce $a_{s+1} = 0$, and take Δ_{s+1} as a triangle of side length 0). Define $\mathcal{H}_k$ as the set of these $s+1$ largest triangles of $\mathcal{R}_k$.

Now we are ready to define the trapezoid T_k and its translation covering by $\mathcal{H}_k$. As $b = b_{k-1}$ is already known, the only thing is needed to define T_k its heights $h_k := h$. Define h by the following equation

$$\sum_{1\leq i\leq s} (a_i - h) + \frac{1}{2}a_{s+1} = b. \tag{6}$$

Adding (6) and (5) we obtain (after simplification) that

$$\frac{1}{2}a_{s+1} < h. \tag{7}$$

Similarly, comparing (6) to (4) one obtains that

$$h \leq \frac{1}{2}a_s + \frac{1}{2s}a_{s+1} < a_s. \tag{8}$$

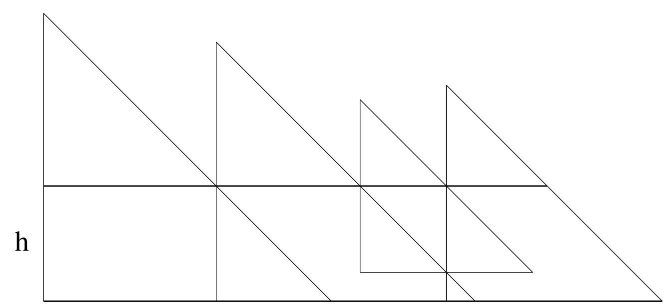

Figure 1: The trapezoid T_k is covered by the members of $\mathcal{H}_k$

For $1 \leq i \leq s$ we place Δ_i such that its apex vertex (the vertex with the right angle) lies on the line $y = 1 - b - h$ $(= 1 - b_k)$. Then the triangle Δ_i meets the line $y = 1 - b$ in an interval I_i of length $a_i - h$ (> 0 by (8)). Place the triangles such a way that $I_1, I_2, \ldots, I_{s-1}$ form a continuous segment starting at the y axis, i.e., the apex vertex of Δ_i is at $(\sum_{1\leq j<i} a_j - (i-1)h, 1 - b - h)$ for $1 \leq i \leq s-1$. These triangles cover T_k except a parallelogram of horizontal side length $a_s - h + \frac{1}{2}a_{s+1}$.

Translate Δ_s such that I_s covers the other end of the upper base $[(0, 1-b), (b, 1-b)]$ of the trapezoid T_k, i.e., the apex vertex of Δ_s is at $(b + h - a_s, 1 - b - h)$. Then the uncovered region, $T_k \setminus \cup_{1\leq i\leq s}\Delta_i$, is a homothetic copy of Δ_{s+1} (with a factor

$-1/2)$, a triangle with vertices $(b-(a_s-h)-\frac{1}{2}a_{s+1}, 1-b)$, $(b-(a_s-h), 1-b)$, and $(b-(a_s-h), 1-b-\frac{1}{2}a_{s+1})$. (These points are in T_k, by (7)). There is a unique translate of Δ_{s+1} covering this region, its apex must go to $(b-(a_s-h)-\frac{1}{2}a_{s+1}, 1-b-\frac{1}{2}a_{s+1})$. We obtained that

$$\bigcup_{\Delta \in \mathcal{H}_k} \Delta \supseteq T_k.$$

3. Area estimates

Suppose that the above procedure stops after ℓ steps, i.e., $\mathcal{H} = \mathcal{H}_1 \cup \cdots \cup \mathcal{H}_\ell \cup \mathcal{R}_{\ell+1}$ and we have obtained the trapezoids $T_1, \ldots, T_\ell$, with heights $h_1, \ldots, h_\ell$ and bases $b_k = \sum_{1 \le i \le k} h_i$. The union of these trapezoids forms an isosceles triangle of side length b_ℓ. We are going to show that (2) implies that $b_\ell \ge 1$, i.e., our procedure results a translation covering of Δ_0.

Let u_k (v_k, respectively) denote the size of a largest (a smallest) triangle in $\mathcal{H}_k$, $(1 \le k \le \ell)$, and $u_{\ell+1}$ is the size of the largest triangle in $\mathcal{R}_{\ell+1}$. If $\mathcal{R}_{\ell+1} = \emptyset$, then we take $u_{\ell+1} = 0$. Using the notations of the previous section $u_k = a_1$ and $v_k = a_{s+1}$.

Elementary calculation shows that (4)–(6) imply that

$$\sum_{1 \le i \le s+1} \frac{1}{2} a_i^2 \le \frac{1}{2} a_1 b + hb + h(b+h) - \frac{1}{2} a_{s+1}(b+h) \tag{9}$$

The proof of this inequality is purely algebraic, it does not use any geometry, therefore it is is postponed to the next section as Lemma 1. The sum of the middle two terms in (9) is, obviously, twice the area of T_k. So reformulating (9) we obtain the following upper estimate for the total area of the members of $\mathcal{H}_k$ $(2 \le k \le \ell)$.

$$\sum_{\Delta \in \mathcal{H}_k} \text{Area}(\Delta) \le \frac{1}{2} u_k b_{k-1} + 2\text{Area}(T_k) - \frac{1}{2} v_k b_k.$$

Since $u_1 = v_1 \ge u_2 \ge v_2 \ge \cdots \ge u_k \ge v_k \ge \cdots \ge v_\ell \ge u_{\ell+1}$, we have that

$$\sum_{\Delta \in \mathcal{H}_k} \text{Area}(\Delta) \le \frac{1}{2} u_k b_{k-1} + 2\text{Area}(T_k) - \frac{1}{2} u_{k+1} b_k. \tag{10}$$

The above inequality holds for $k=1$, too, because $b_1 = u_1$ and we can define $b_0 = 0$.

$$\sum_{\Delta \in \mathcal{H}_1} \text{Area}(\Delta) = \frac{1}{2} b_1^2 \le \frac{1}{2} b_1 (2b_1 - u_2) = \frac{1}{2} u_1 b_0 + 2\text{Area}(T_1) - \frac{1}{2} u_2 b_1.$$

Adding up (10) for all k one obtains that

$$\sum_{1\le k\le \ell}\sum_{\Delta\in\mathcal{H}_k}\text{Area}(\Delta)\le 2\left(\sum_{1\le k\le\ell}\text{Area}(T_k)\right)$$
$$+\left(\frac{1}{2}u_1b_0-\frac{1}{2}u_2b_1\right)+\left(\frac{1}{2}u_2b_1-\frac{1}{2}u_3b_2\right)+\cdots+\left(\frac{1}{2}u_\ell b_{\ell-1}-\frac{1}{2}u_{\ell+1}b_\ell\right)$$
$$=b_\ell^2-\frac{1}{2}u_{\ell+1}b_\ell \qquad (11)$$

Moreover (3) implies that

$$\sum_{\Delta\in\mathcal{R}_{\ell+1}}\text{Area}(\Delta)\le\frac{1}{2}u_{\ell+1}\times\sum_{\Delta\in\mathcal{R}_{\ell+1}}a(\Delta)\le\frac{1}{2}u_{\ell+1}b_\ell.$$

Adding this to (11) we obtain that

$$\sum_{\Delta\in\mathcal{H}}\text{Area}(\Delta)\le b_\ell^2.$$

This and (2) imply $b_\ell\ge 1$, so we obtained that $\mathcal{H}_1\cup\cdots\cup\mathcal{H}_\ell$ really forms a translation covering of Δ_0. □

4. Proof of an inequality

Lemma 1 *Suppose that $b\ge a_1\ge a_2\ge\cdots\ge a_s>0$ and $a_s\ge a_{s+1}\ge 0$ are real numbers and $s\ge 2$ is an integer satisfying* (4) *and* (5)*, i.e.,*

$$\sum_{1\le i\le s}(a_i-\frac{1}{2}a_s)\le b<\sum_{1\le i\le s+1}(a_i-\frac{1}{2}a_{s+1}). \qquad (12)$$

Define h as in (6)*, i.e., by $\sum_{1\le i\le s}(a_i-h)+\frac{1}{2}a_{s+1}=b$. Then*

$$\sum_{1\le i\le s+1}\frac{1}{2}a_i^2\le\frac{1}{2}a_1b+hb+h(b+h)-\frac{1}{2}a_{s+1}(b+h):=R$$

Proof. It is high school algebra. First, observe that the upper bound part of (12) implies that $h > \frac{1}{2}a_{s+1}$. Similarly, the lower bound part of (12) implies that

$$s(\frac{1}{2}a_s - h) + \frac{1}{2}a_{s+1} \geq 0, \tag{13}$$

thus $a_s > \frac{1}{2}a_s + \frac{1}{2s}a_{s+1} \geq h$. We can distinguish two cases.

If $\frac{1}{2}a_s \geq h$, then $(\frac{1}{2}a_i + h)(a_i - h) = \frac{1}{2}a_i^2 + h(\frac{1}{2}a_i - h) \geq \frac{1}{2}a_i^2$ because $\frac{1}{2}a_i \geq \frac{1}{2}a_s \geq h$. We get the following lower bound for the sum of the first two terms of R:

$$\begin{aligned} (\frac{1}{2}a_1 + h)b &= (\frac{1}{2}a_1 + h)\left(\sum_{1\leq i\leq s}(a_i - y)\right) + (\frac{1}{2}a_1 + h)\frac{1}{2}a_{s+1} \\ &\geq \sum_{1\leq i\leq s}(\frac{1}{2}a_i + h)(a_i - h) + (\frac{1}{2}a_1 + \frac{1}{2}a_{s+1})\frac{1}{2}a_{s+1} \geq \sum_{1\leq i\leq s+1}\frac{1}{2}a_i^2. \end{aligned}$$

As the sum of the last two terms of R, i.e., $(h - \frac{1}{2}a_{s+1})(b + h)$ is positive we obtain the desired lower bound for R.

Suppose that $h > \frac{1}{2}a_s$.

$$\begin{aligned} R &> R - h(h - \frac{1}{2}a_{s+1}) = \frac{1}{2}a_1 b + 2hb - \frac{1}{2}a_{s+1}b \geq b(\frac{1}{2}a_1 + 2h - \frac{1}{2}a_s) \\ &= \left(\sum_{1\leq i\leq s}(a_i - h) + \frac{1}{2}a_{s+1}\right) \times \left(\frac{1}{2}a_1 + 2h - \frac{1}{2}a_s\right) \\ &= \left(\sum_{1\leq i\leq s}(a_i - \frac{1}{2}a_s) + s(\frac{1}{2}a_s - h) + \frac{1}{2}a_{s+1}\right) \times \left(\frac{1}{2}(a_1 + a_s) + 2(h - \frac{1}{2}a_s)\right) \\ &= \left(\sum_{1\leq i\leq s}(a_i - \frac{1}{2}a_s)\right) \times \frac{1}{2}(a_1 + a_s) \\ &\quad + \left(s(\frac{1}{2}a_s - h) + \frac{1}{2}a_{s+1}\right) \times \frac{1}{2}(a_1 + a_s) \\ &\quad + \left(\sum_{1\leq i\leq s}(a_i - \frac{1}{2}a_s)\right) \times 2\left(h - \frac{1}{2}a_s\right) \\ &\quad + \left(s(\frac{1}{2}a_s - h) + \frac{1}{2}a_{s+1}\right) \times 2\left(h - \frac{1}{2}a_s\right). \end{aligned}$$

Here the last four lines are positive. The first line is at least $\sum_{1\leq i\leq s}\frac{1}{2}a_i^2$, in the second we can use $\frac{1}{2}(a_1 + a_s) \geq a_s$, in the third $\sum_{1\leq i\leq s}(a_i - \frac{1}{2}a_s) \geq \frac{1}{2}sa_s$ and $h - \frac{1}{2}a_s > 0$

and in the fourth line the first factor is positive by (13). We obtain that

$$R > \sum_{1\leq i\leq s} \frac{1}{2}a_i^2 + \left(s(\frac{1}{2}a_s - h) + \frac{1}{2}a_{s+1}\right) a_s + \frac{1}{2}sa_s \times 2(h - \frac{1}{2}a_s) + 0$$

$$= \sum_{1\leq i\leq s} \frac{1}{2}a_i^2 + \frac{1}{2}a_{s+1}a_s \geq \sum_{1\leq i\leq s+1} \frac{1}{2}a_i^2 \qquad \square$$

5. The case of small copies

Here we sketch the proof of Theorem 2. First, we fix a direction on the plane, say the line L is parallel to the x axis. We may suppose that $\mathrm{Area}(C) = 1$ and its perimeter is bounded (less than 8, say), otherwise we use an affine transformation. We also know that $1 \leq \vartheta \leq 3/2$. (This upper bound is due to Besicovitch [2] and usually attributed to Fáry [3], see again, e.g., [10]. The idea is that every disk contains a large symmetric hexagon).

Next, (a slight generalization of Moon and Moser's proof gives that) there exists a $\delta_1 > 0$ such that the following holds. If $Q_1, Q_2, \ldots$ are squares parallel to L, each of them has size at most δ_1 and total area exceeding

$$\mathrm{Area}(C) + \frac{1}{8}\varepsilon, \tag{14}$$

then these squares permit a translation covering of C.

The definition of ϑ implies that there exists an integer n_0 such that, if $C_1, \ldots, C_n$ are congruent copies of C, $n \geq n_0 = n_0(\varepsilon, C)$ and Q is a square with

$$n\mathrm{Area}(C_1) \geq (\vartheta + \frac{1}{8}\varepsilon)\mathrm{Area}(Q), \tag{15}$$

then $\{C_1, \ldots, C_n\}$ permits a translation covering of Q. Define $\delta =: \min\{\frac{1}{8}\varepsilon, \delta_1\}/n_0$.

Consider $\mathcal{H}$ the set of homothetic copies of C with members smaller than δC. If the size of $H \in \mathcal{H}$ is between $(1 + \frac{1}{8}\varepsilon)^{-k}C$ and $(1 + \frac{1}{8})^{-k+1}C$, then replace it with a homothetic copy of C of size exactly $(1 + \frac{1}{8}\varepsilon)^{-k}C$. Do this with every member of $\mathcal{H}$. Obviously, it is sufficient to make a translation covering of C using the members of the obtained new family, $\mathcal{H}'$. We have that

$$\sum_{H'\in\mathcal{H}'} \mathrm{Area}(H') \geq (1 + \frac{1}{8}\varepsilon)^{-2} \sum_{H\in\mathcal{H}} \mathrm{Area}(H) > \vartheta + \frac{1}{2}\varepsilon.$$

If $\mathcal{H}'$ contains $an_0 + b$ copies of the same size, $C_1, \dots, C_{an_0+b}$, where a and b are integers with $n_0 > b \geq 0$, then leave out b of those and replace the rest with a squares $Q_1, \dots, Q_a$ of the same sizes and parallel to L such that equality holds in (15). Then these squares can be covered by the copies of C they replace so we are done if we show that the obtained new set $\mathcal{H}''$ (consisting only of squares) can cover C.

From each size in $\mathcal{H}'$ we have deleted at most $n_0 - 1$, so their area altogether is at most

$$(n_0 - 1) \times (\max_{H' \in \mathcal{H}'} \text{Area}(H')) \times \sum_k (1 + \frac{1}{8}\varepsilon)^{-k} < \frac{1}{8}\varepsilon.$$

Thus we get for the area of the squares in $\mathcal{H}''$

$$\sum_{H'' \in \mathcal{H}''} \text{Area}(H'') \geq \frac{1}{\vartheta + \frac{1}{8}\varepsilon} \left(\text{Area}(\mathcal{H}') - \frac{1}{8}\varepsilon \right) > \text{Area}(C) + \frac{1}{8}\varepsilon.$$

Then (14) finishes the proof. □

Naturally, we can extend this proof for every dimension.

6. Conclusion, remarks

Note that our Theorem holds for **infinite** sets of triangles satisfying (2). Indeed, almost all of our inequalities are strict (most notable Lemma 1), so after a few steps in the procedure defining the trapezoids, one does not need total area 2, a slightly less will do. This can make $\mathcal{H}$ finite.

One can get a slightly **better** (i.e., thinner) covering of Δ_0 if in the definition of $\mathcal{H}_k$ not only Δ_{s+1}, but $\Delta_{s+2}, \dots, \Delta_{2s-1}$ too are placed between Δ_i and Δ_{i+1} along the upper base line of T_k.. However, the density of the obtained covering will be very very close to 2, even if the triangles are small.

As **every** planar convex set C can be sandwiched between two homothetic triangles $\Delta_1 \subseteq C \subseteq \Delta_2$ with ratio at most $9/4$ (see [4]), Theorem 1 implies $f(C) \leq 2(\Delta_2/\Delta_1)^2 \leq 10.125$. However, we can easily get a better bound from inscribed and circumscribed parallelograms. Let $C_1, \dots,$ be a family of homothetic copies of C with $C_i = x_i C$ and $\sum_i x_i^2 \geq 8$. One can find parallelograms $P' \subseteq C \subseteq P''$ such that $P' = \lambda P''$ and $\lambda \geq 1/2$. Further, $P_i' \subseteq C_i \subseteq P_i''$. Then $1 + 2x_1\lambda \leq \lambda^2(\sum_i x_i^2)$. Thus Moon-Moser's theorem (1) implies that P'' can be covered by some translates of the P_i's giving

$$f(C) \leq \frac{1}{\lambda^2} + \frac{2}{\lambda} \leq 8.$$ □

This is rather close to the best known general upper bound, $f(C) \leq 22/3$ (see [7]).

If $\mathcal{H}$ can contain positive and **negative** homothetic copies of Δ_0 then to ensure a translation covering one needs total area at least $4\text{Area}(\Delta_0)$. This was conjectured by K. Böröczky and proved by Januszewski [6] and for the more special case of $\mathcal{H}$ being a finite sequence purely of homothetic copies of $-\Delta_0$ was proved earlier by Vásárhelyi [11]. She also considered translation coverings of the triangle Δ_0 when $\mathcal{H}$ consist of homothetic copies each of them **rotated** by a certain angle φ, see [12].

Moon and Moser's result was rediscovered by Groemer [5] and Bezdek and Bezdek [1]. They both extended it to **higher dimensions** proving that $f(Q^{(d)}) = 2^d - 1$, where now $Q^{(d)}$ is the d-dimensional cube.

An algorithm for packing or covering a given set K with a sequence of sets $\{C_i\}$ is an **on-line** method if the sets C_i are given in sequence, and C_{i+1} is presented only after C_i has been put in place, without the option of changing the placement afterwards. Januszewski, Lassak, Rote, and Woeginger [8] proved that in Euclidean d-space, every sequence of cubes of total volume greater than or equal to $2^d + 3$ can cover the unit cube in the on-line manner. This volume bound is astoundingly good, considering the best possible bound of $2^d - 1$ for the analogous off-line problem.

7. The case of cubes

Finally, the author cannot resists to add here a lovely short proof for

$$f(Q^{(d)}) \leq 2^d, \tag{16}$$

where $Q^{(d)}$ is the d-dimensional unit cube, a slightly weaker result than the optimal $f(Q^{(d)}) = 2^d - 1$ proved in [1] and [5]. The idea of this proof appears even in the paper of Moon and Moser [9], but not in this explicit form.

Proof. Let $\mathcal{H}$ be a (finite or infinite) set of cubes parallel to $Q^{(d)}$ with total volume at least 2^d. We define the translation covering in two phases, first trimming and then gluing. If the side length a of $H \in \mathcal{H}$ is between $2^\alpha \leq a < 2^{\alpha+1}$ where α is a (usually negative) integer, then replace H with a cube, H', with side length 2^α. We obtain the family $\mathcal{H}'$ of sizes powers of 2. We have that

$$\sum_{H' \in \mathcal{H}'} \text{Vol}(H') > \frac{1}{2^d} \sum_{H \in \mathcal{H}} \text{Vol}(H) \geq 1.$$

Now, whenever we find in our set 2^d cubes of the same size we glue them together to get a cube of double of that size. Finally, we arrive to a set of cubes, $\mathcal{H}''$, all sizes are powers of 2, where every size is represented at most $2^d - 1$ times. If the largest size is at most 2^{-1}, then for the total volume we get

$$\sum_{H' \in \mathcal{H}'} \mathrm{Vol}(H') = \sum_{H'' \in \mathcal{H}''} \mathrm{Vol}(H'') \leq (2^d - 1)\left((\frac{1}{2})^d + (\frac{1}{4})^d + \dots \right) = 1.$$

However, this total volume exceeds 1. So the largest size in $\mathcal{H}''$ is at least $2^0 = 1$, so the cubes of $\mathcal{H}'$ can perfectly cover $Q^{(d)}$ and we are done. □

8. Acknowledgements

The author is greatly thankful to A. Bezdek for thoroughful discussions about the problem of coverings. The author is also indebted to I. Bárány, J. Pach and I. David Berg for helpful conversations.

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