On the Maximum Size of (p, Q)–free Families

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Abstract

Let p be a positive integer and let Q be a subset of $\{0, 1, \dots, p\}$. Call p sets $A_1, A_2, \dots, A_p$ of a ground set X a (p, Q)-system if the number of sets A_i containing x is in Q for every $x \in X$. In hypergraph terminology, a (p, Q)-system is a hypergraph with p edges such that each vertex x has degree $d(x) \in Q$. A family of sets $\mathcal{F}$ with ground set X is called (p, Q)-free if no p sets of $\mathcal{F}$ form a (p, Q)-system on X. We address the Turán type problem for (p, Q)-systems: $f(n, p, Q)$ is defined as $\max |\mathcal{F}|$ over all (p, Q)-free families on the ground set $[n] = \{1, 2, \dots, n\}$.

We study the behavior of $f(n, p, Q)$ when p is fixed (allowing 2^{p+1} choices for Q) while n tends to infinity. The new results of this paper mostly relate to the middle zone where $2^{n-1} \leq f(n, p, Q) \leq (1 - c)2^n$ (in this upper bound c depends only on p). This direction was initiated by Paul Erdős who asked about the behavior of $f(n, 4, \{0, 3\})$. In addition we give a brief survey on results and methods (old and recent) in the low zone (where $f(n, p, Q) = o(2^n)$) and in the high zone (where $2^n - (2 - c)^n < f(n, p, Q)$).

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1 Introduction

The starting point of this paper was a question of Paul Erdős. Unfortunately, we had no opportunity to continue the conversations on the subject with him which started during the Summer of 1996. Following his well-known habit, he started with a special case: "how many subsets of $[n] = \{1, 2, \ldots, n\}$ can you give if no four of them cover their union exactly three times? Can you give more than 2^{n-1} or only significantly less than 2^n?" He explained that it seems to be the first interesting special case of $f(n, p, q)$, the maximum number of sets of $[n]$ such that there are no p sets covering each element of their union exactly q times. He gave some comments as well: "we proved with Vera (T. Sós) that $f(n, 3, 2) = 2^{n-1} + 1$ and $f(n, 5, 3)$ is related to a number theory problem with Vera and Sárközy." Here Paul have had referred to their paper on product representation ([1]).

We shall prove here that $f(n, p, q)/2^n \leq 1 - c_{p,q}$ if $p/2 \leq q$ and give a construction showing that $f(n, p, p-1) \geq 2^{n-1} + cn^{p-3}$. Probably $c_{p,q}$ tends to $1/2$ for $p/2 \leq q$. This seems to be a difficult problem, we could prove it only in a few special cases (In fact this is true for most choices of q in this zone).

We explore what is known and unknown about the following generalization of the initial problem. Assume that Q is an arbitrary subset of $\{0, 1, \ldots, p\}$. Call p sets $A_1, A_2, \ldots, A_p$ of $[n]$ a (p, Q)-system if the number of sets A_i containing x is in Q for every $x \in [n]$. Using hypergraph terminology,

Definition 1 *a (p, Q)-system is a hypergraph with p edges such that each vertex x has degree $d(x) \in Q$. A family of sets $\mathcal{F}$ is (p, Q)-free if it does not contain any (p, Q)-system. Then $f(n, p, Q)$ is defined as* $\max |\mathcal{F}|$ *over all (p, Q)-free families $\mathcal{F}$ on $[n]$.*

Notice that $f(n, p, q) = f(n, p, \{0, q\})$.

Sets with multiplicities are excluded, therefore, $f(n, p, Q) \leq 2^n$. We study the behavior of $f(n, p, Q)$ when p is fixed (allowing 2^{p+1} choices for Q) while n tends to infinity. Many important results and problems of extremal set theory fits into this general Turán type question (for various choices of p and Q). Also observe that, e.g., $f(n, p, \{0, 1, p\})$, is the maximum size of a hypergraph containing no Δ-**system** with p edges, which is a classical (and still unsolved) problem, too.

References

[1] P. Erdős, A. Sárközy, and V. Sós, On product representations of powers I, *European J. Combinatorics* **16** (1995), 567–588.