Discrete Mathematics 241 (2001) 41–50
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

On the lattice diameter of a convex polygon

Imre Bárány[a,b,*,1], Zoltán Füredi[b,c,2]

[a] Department of Mathematics, University College London, Gower Street, London, WC1E 6BT, UK
[b] Mathematical Institute of the Hungarian Academy of Sciences, P.O. Box 127, Budapest, 1364 Hungary
[c] Department of Mathematics, University of Illinois, 1409 W. Green St., Urbana, IL 61801, USA

Abstract

The lattice diameter, $\ell(P)$, of a convex polygon P in R^2 measures the longest string of integer points on a line contained in P. We relate the lattice diameter to the area and to the lattice width of P, $w_l(P)$. We show, e.g., that $w_l \leqslant \frac{4}{3}\ell + 1$, thus giving a discrete analogue of Blaschke's theorem. © 2001 Elsevier Science B.V. All rights reserved.

MSC: 52C05; 11H06

Keywords: Convexity; Diameter; Lattice; Covering minima

1. The area of lattice polygons

Let P be a convex, closed, non-empty lattice polygon, i.e., $P = \mathrm{conv}(P \cap \mathbb{Z}^2)$. The *lattice diameter*, $\ell(P)$, measures the longest string of integer points on a line contained in P

$$\ell(P) = \max\{|P \cap \mathbb{Z}^2 \cap L| - 1\colon L \text{ is a line}\}.$$

Thus $\ell(P) = 0$, if and only if P consists of a single lattice point, and for the square $Q^1 = [0,\ell] \times [0,\ell]$ and for the special pentagon $Q^2 = \mathrm{conv}(\{(x,y) \in \mathbb{Z}^2\colon 0 \leqslant x, y \leqslant \ell\} \cup \{(\ell+1,\ell+1)\} \setminus \{(0,0)\})$ (for $\ell \in \mathbb{Z}^+$) one has $\ell(Q^1) = \ell(Q^2) = \ell$. (See Fig. 1.) This definition is due to Stolarsky and Corzatt [5] who proved several properties of $\ell(P)$. The lattice diameter is invariant under the group of unimodular affine transformations

* Corresponding author.
E-mail addresses: barany@math-inst.hu (I. Bárány), z-furedi@math.uiuc.edu, furedi@math-inst.hu (Z. Füredi).

[1] Part of this work was done while the author was visiting Cowles Foundation, Yale University. Support from the Program in Discrete Mathematics at Yale University and from Hungarian National Foundation Grant No. 19021 is acknowledged with thanks.
[2] This research was supported in part by the Hungarian National Science Foundation grant OTKA 016389, and by a National Security Agency grant MDA904-98-I-0022.

PII: S0012-365X(01)00145-5

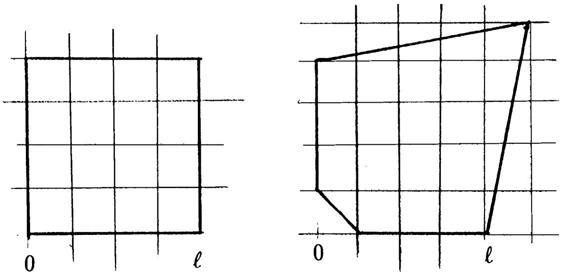

Fig. 1.

$\mathrm{SL}(2,\mathbb{Z})$; these are lattice preserving mappings $R^2 \to R^2$ also preserving parallel lines and area.

A simple consequence of the definition is the following fact on lattice points contained in P which first appeared in the literature in Rabinowitz [10].

$$(P \cap \mathbb{Z}^2) \cap ((\ell(P)+1)z + (P \cap \mathbb{Z}^2)) = \emptyset \quad \text{for every } z \in \mathbb{Z}^2,\ z \neq (0,0). \tag{1}$$

To see this we note that the common point to P, $\mathbb{Z}^2$, and $(\ell(P)+1)z+P$ would be of the form $(\ell(P)+1)z+x$ with $x \in (P \cap \mathbb{Z}^2)$ implying that the string of $\ell(P)+2$ integer points $x, x+z, \ldots, x+(\ell(P)+1)z$ all belong to P contradicting the definition of the lattice diameter. Eq. (1) implies that $\{(\ell(P)+1)z + (P \cap \mathbb{Z}^2)\}_{z \in \mathbb{Z}^2}$ form a "packing" in $\mathbb{Z}^2$ which shows, in turn, that P contains at most $(\ell(P)+1)^2$ lattice points,

$$|P \cap \mathbb{Z}^2| \leqslant (\ell(P)+1)^2. \tag{2}$$

An elementary argument and (1) imply that $(\ell(P)+1)\mathbb{Z}^2 + P$ is a packing in R^2 by translates of P so that

$$\mathrm{area}(P) \leqslant (\ell(P)+1)^2. \tag{3}$$

For higher dimension the volume of P is not bounded by a function of $\ell(P)$; there are empty simplices $S \subset R^d$ (i.e., $S \cap \mathbb{Z}^d = \mathrm{vert}(S)$) having arbitrarily large volume (see [11,4,12]), e.g., one can take (in R^3) $S = \mathrm{conv}(\{(0,0,0),(1,0,0),(0,1,0),(1,1,k)\})$.

Let $\mathrm{a}(k)$ denote the maximal area a convex lattice polygon P with $\ell(P) \leqslant k$ can have. The square, i.e., Example Q^1, implies $\mathrm{a}(k) \geqslant k^2$. Alarcon [1] observed that this is far from being optimal, $\mathrm{area}(Q^2) = k^2+k-1/2$. He also showed $\mathrm{a}(1)=1.5$, $\mathrm{a}(2)=5.5$, $\mathrm{a}(3)=11.5$ and $\mathrm{a}(4)=21$, and improved (3) to $\mathrm{a}(k) \leqslant k^2+2k-2$ for $k \geqslant 5$. Our first result is that $\mathrm{a}(k)$ is very close to the upper bound (3).

Theorem 1. *For $k \geqslant 5$ there exists a convex lattice polygon Q^3 with $\ell(Q^3)=k$ and* $\mathrm{area}(Q^3) = k^2+2k-4$.

The construction $Q^3 = Q^3(k)$ is an octagon with vertices $(-1,0)$, $(0,k-1)$, $(2,k)$, $(k-1,k+1)$, $(k+1,k)$, $(k,1)$, $(k-2,0)$, and $(1,-1)$, see Fig. 2. In fact, for $k>5$ the polygon Q^3 is indeed an octagon with only these eight vertices on its boundary and with $(k+1)^2-8$ interior points. For $k=5$ two of its boundary points, $(2,k)$ and

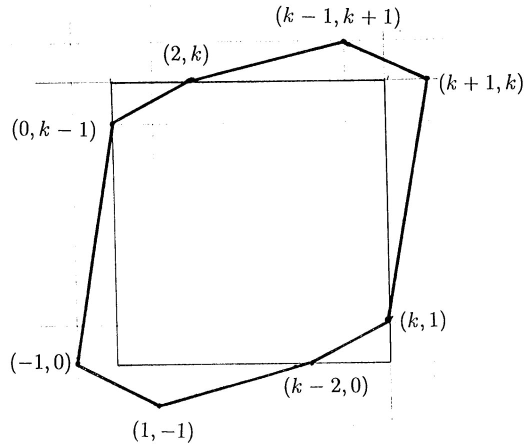

Fig. 2.

$(k-2,0)$, are not vertices, it becomes a hexagon. Thus Pick's theorem [9] on the area of lattice polygons, i.e.,

$$\text{area}(P) = |\text{int}(P) \cap \mathbb{Z}^2| - 1 + \frac{|\partial(P) \cap \mathbb{Z}^2|}{2}$$

implies area$(Q^3) = (k^2 + 2k - 7) - 1 + \frac{8}{2}$, as claimed. (This can be shown directly as well.) Alarcon's improvement of (3) also utilizes Pick's theorem, he shows that a maximal P has at least 4 vertices. We conjecture that Q^3 is extremal, a$(k) = k^2 + 2k - 4$.

2. Slopes of diameters

Bang [2] solved Tarski's plank problem by showing that if a compact convex set in R^2 can be covered by n strips of widths $w_1, w_2, \ldots, w_n$ then it can be covered with one strip of width $\sum_{1 \leqslant i \leqslant n} w_i$. Corzatt [5] conjectured the following discrete analogue. If the set of lattice points contained in the lattice polygon P can be covered by n lines, $(P \cap \mathbb{Z}^2) \subset (L_1 \cup L_2 \cup \cdots \cup L_n)$, then there exists a set of covering lines $\mathscr{L} = \{L'_1, \ldots, L'_n\}$, $(P \cap \mathbb{Z}^2) \subset (L'_1 \cup L'_2 \cup \cdots \cup L'_n)$ such that the lines in $\mathscr{L}'$ have at most four different slopes. This problem motivated Alarcon [1] to ask the maximum number of diameter directions of a lattice polygon.

A non-zero vector $u \in \mathbb{Z}^2$ is a *diameter direction* for the convex lattice polygon P if there is an integer z such that $z, z + u, \ldots, z + \ell(P)u$ all belong to P. Such a u is necessarily a *primitive* vector, i.e., its coordinates are coprime. Write $N(P)$ for the number of diameter directions of P. The triangle with vertices $(-1,-1)$, $(1,0)$, $(0,1)$ and baricenter $(0,0)$ has 6 different diameter directions. Here we prove that

$$N(P) \leqslant 4,$$

for all convex lattice polygons with $\ell(P)>1$. This is done by a good description (Theorem 2 below) of convex lattice polygons P that are maximal to containment with respect to $\ell(P)=\ell$.

Write $\mathcal{M}_\ell$ for the collection of maximal convex lattice polygons, i.e., $P\in\mathcal{M}_\ell$ if $\ell(P)=\ell$, and for any convex lattice polygon P' properly containing P, $\ell(P')>\ell$. One more definition: given primitive vectors u, $b\in\mathbb{Z}^2$ (non-parallel) and $z\in\mathbb{Z}^2$, the *half-open slab* $S(u,b,z)$ is defined as

$$S(u,b,z)=\{z+\alpha u+\beta b\colon 0\leqslant\alpha<\ell+1,\ -\infty<\beta<+\infty\}.$$

Theorem 2. *If $P\in\mathcal{M}_\ell$ then one of the following 3 cases holds.*

(i) *P has exactly two diameter directions, u_1 and u_2, say. They form a basis of $\mathbb{Z}^2$. Further, there are points $z_1,z_2\in\mathbb{Z}^2$ and primitive vectors b_1 and b_2 such that $z_i,z_i+u_i,\dots,z_i+\ell u_i\in P$ and*

$$P=\operatorname{conv}(\mathbb{Z}^2\cap S(u_1,b_1,z_1)\cap S(u_2,b_2,z_2)). \tag{4}$$

(ii) *P has exactly three diameter directions, u_1, u_2, u_3. Any two of them form a basis of $\mathbb{Z}^2$ thus $u_3=\pm u_1\pm u_2$. Further, there are points $z_i\in\mathbb{Z}^2$ and primitive vectors b_i $(i=1,2,3)$ such that $z_i,z_i+u_i,\dots,z_i+\ell u_i\in P$ and*

$$P=\operatorname{conv}\left(\mathbb{Z}^2\cap\bigcap_{1\leqslant i\leqslant 3}S(u_i,b_i,z_i)\right). \tag{5}$$

(iii) *P has exactly four diameter directions. Then* (mod $\mathrm{SL}(2,\mathbb{Z})$, *i.e., up to a lattice preserving affine transformation*) *P is either the square Q^1 or the special pentagon Q^2.* (*See again Fig.* 1.)

The proof is postponed to Section 4.

3. Width and covering radius

The lattice diameter is the natural counterpart of the *lattice width*, $w_l(P)$, which is defined as

$$w_l(P)=\min_{u\in\mathbb{Z}^2 u\neq(0,0)}\left(\max_{x,y\in P}u(x-y)\right).$$

The lattice width is also invariant under the group of unimodular affine transformations $\mathrm{SL}(2,\mathbb{Z})$. Thus $w_l(P)=0$ if and only if P can be covered by a single line. For the square we have $w_l(Q^1)=\ell$ and for the special pentagon Q^2 in Example 1, we have $w_l(Q^2)=\ell+1>\ell(Q^2)=\ell$. In general, in Section 5, we prove the following consequence of Theorem 2.

Theorem 3. $w_l(P)\leqslant\lfloor\frac{4}{3}\ell(P)\rfloor+1$ *and for given ℓ this upper bound is best possible.*

The following example, Q^4, shows that here equality can hold if ℓ is of the form $3t+1$. The polygon $Q^4 = Q^4(t)$ is a triangle with vertices $(0,0)$, $(4t+2, 2t+1)$, and $(2t+1, 4t+2)$; it has lattice diameter $\ell = 3t+1$ and lattice width $w_l(Q^4) = 4t+2$. For the other values of ℓ we obtain equality by considering the triangle $(0,0)$, $(t, 2t+1)$, $(2t+1, t+1)$. Its width is $2t+1$ and its diameter is $\lfloor (3t+1)/2 \rfloor$.

The following example, Q^5, shows that there are other completely different polygons with almost equality in Theorem 3. Let $Q^5 = Q^5(\ell)$ be a hexagon with vertices $(0,0)$, $(\frac{1}{3}\ell, -\frac{1}{3}\ell)$, $(\ell, 0)$, $(\frac{4}{3}\ell, \frac{2}{3}\ell)$, (ℓ, ℓ), and $(\frac{1}{3}\ell, \frac{2}{3}\ell)$. We have $\ell(Q^5) = \ell$, and $w_l(Q^5) = \frac{4}{3}\ell$ for every $\ell \in \mathbb{Z}^+$, ℓ is divisible by 3.

Schnell [13] showed (in a slightly different form) another upper bound for the lattice width of an arbitrary convex, closed planar region C

$$w_l(C) \leqslant \tfrac{4}{3}\,\mathrm{area}(C)\mu_2(C), \tag{6}$$

where $\mu_2 := \mu_2(C)$ is the *covering radius*, i.e., the smallest positive real x such that the union of the regions of the form $z + xC$ for $z \in \mathbb{Z}^2$ covers the plane. For more about covering minima see Kannan and Lovász [8], or the survey of Gritzmann and Wills [7].

Although (6) frequently gives a better bound than Theorem 3, there are several examples, like Q^6 below, when $\ell(P)$ is smaller than $\mathrm{area}(P)\mu_2(P)$. Let $Q^6 = Q^6(t)$ be a tilted square of side length $\sqrt{160}t$ with vertices $(t, -3t)$, $(13t, t)$, $(9t, 13t)$, $(-3t, 9t)$, where $t \in \mathbb{Z}^+$. It contains the inscribed square $(0,0)$, $(10t, 0)$, $(10t, 10t)$, $(0, 10t)$ and its covering radius is $\mu_2 = 1/(10t)$. On the other hand, it is easy to see that $\mathrm{area}(Q^6)\mu_2 = 16t$ is at least 1.2 times larger than $\ell(Q^6) = \lfloor (40/3)t \rfloor$. We conjecture that in general Schnell's bound is at most $(1+\sqrt{2})/2 = 1.207\ldots$ times larger than $\ell(C)$.

Another upper bound for the lattice width is due to Fejes-Tóth and Makai [6]

$$w_l(C) \leqslant \sqrt{\tfrac{8}{3}\mathrm{area}(C)}. \tag{7}$$

This is also sharp for some cases, like for the triangle $(0,t), (t,0), (-t,-t)$, but again Q^6 shows that it could exceed the bound of Theorem 3 by more than 50%.

4. The maximal polygons, the Proof of Theorem 2

We start with a statement that applies to every convex lattice polygon.

Lemma 1. *Assume P is a convex lattice polygon and $u \in \mathbb{Z}^2$, $u \neq (0,0)$. Then there is a longest segment $[z, v]$ contained in P and parallel with u such that z is a vertex of P. Further, for every such longest segment $[z, v]$, v lies on an edge $[v_1 v_2]$ of P so that the line through z and parallel with $[v_1 v_2]$ is tangent to P.*

The *proof* is simple and can be found in [3]. □

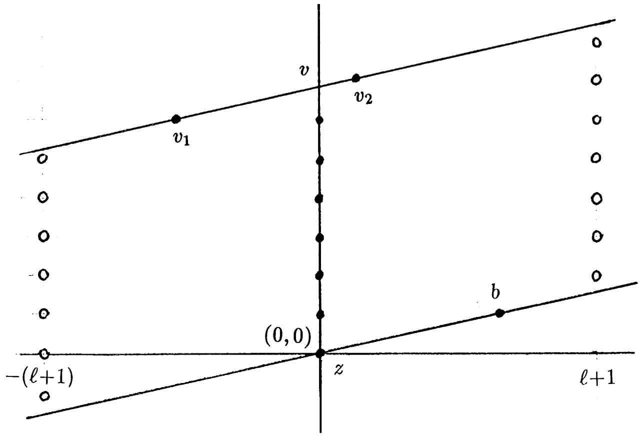

Fig. 3.

Consider now $P \in \mathcal{M}_\ell$ (with $\ell \geqslant 1$) and let u be a diameter direction for P. Apply Lemma 1 to get a longest segment $[z,v]$ with z a vertex. As $[z,v]$ is a longest segment in direction u, $z, z+u, \ldots, z+\ell u \in P \cap \mathbb{Z}^2$. Thus $[z,v]$ contains a lattice diameter.

Applying a suitable lattice preserving affine transformation we may assume $u=(0,1)$, $z=(0,0)$ and $v_2 - v_1 = b = (b_x, b_y)$ with $0 \leqslant 2b_y \leqslant b_x$, here $[v_1, v_2]$ is the edge of P specified by Lemma 1. We conclude that P lies in the half-open slab $S(u,b,z)$, see Fig. 3.

As the area of the z, v_1, v_2 triangle is at most area$(P) \leqslant (\ell+1)^2$ by (3) and the area of the $z+(\ell+1)u, v_1, v_2$ triangle is at least $\frac{1}{2}$, we obtain that P is contained in the slightly narrower half-open slab

$$S'(u,b,z) := \left\{ z + \alpha u + \beta b \colon 0 \leqslant \alpha < \ell + 1 - \frac{1}{2\ell+2},\ -\infty < \beta < +\infty \right\}. \quad (8)$$

It follows from (1) that $(\pm(\ell+1), k) \notin P$ for all $k \in \mathbb{Z}$. Assume now that some $q = (q_x, q_y) \in \mathbb{Z}^2$ with $q_x > \ell + 1$ belongs to P. The triangle $T := \mathrm{conv}\{(0,0),(0,\ell),q\}$ meets the line $x = \ell+1$ in a segment of length $\ell(q_x - \ell - 1)/q_x$. This segment must be lattice point free, so its length is less than 1, implying $q_x < \ell + 3$ for $\ell > 2$. The case $\ell \leqslant 2$ is obvious, so from now on we always suppose $\ell > 2$. A simple computation reveals that T contains a lattice point from the line $x = \ell+1$ unless $q = (\ell+2, \ell+1)$.

We treat first this case $q = (\ell+2, \ell+1) \in P$ (which leads to case (iii) as we shall see soon). First $\mathrm{conv}\{(0,0), v, q\} \subset P$ shows $(0,\ell), (1,\ell), \ldots, (\ell,\ell) \in P$ and $(0,0),(1,1),\ldots,(\ell,\ell) \in P$. So (0,1), (1,0) and (1,1) are diameter directions. As the line $x = \ell+1$ contains no lattice point of P we have $(\ell+1, \ell+1)$ and $(\ell+1, \ell) \notin P \cap \mathbb{Z}^2$. As $(\ell+2, \ell+1) \in P$ this implies that $(k, \ell+1) \notin P$ and $(k, k-1) \notin P$ for all $k \leqslant \ell+1$. We obtain that for all $(x,y) \in P \cap \mathbb{Z}^2$ other than $(\ell+2, \ell+1)$ we have $y \leqslant \ell$ and $x \leqslant y$. Further, $(k,-1) \notin P$ and $(k, \ell+1+k) \notin P$ for all $k \in \{-1,-2,\ldots,-(\ell+1)\}$. Also, $(-\ell, 0) \notin P$ since otherwise $(-\ell,0), (-\ell+2,1), \ldots, (\ell+2, \ell+1)$ all belong to P implying $\ell(P) > \ell$. Fig. 4 shows the room left for P after these restrictions.

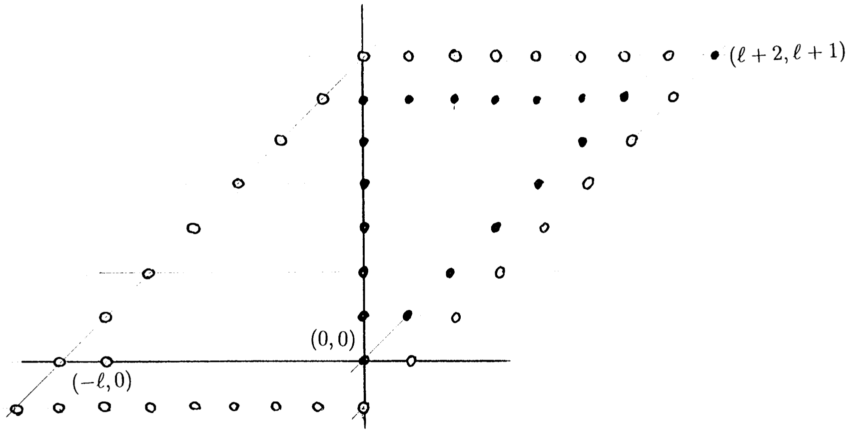

Fig. 4.

The maximality of P implies now that P equals $\mathrm{conv}\{(\ell+2,\ell+1),(0,0),(-\ell+1,0),(-\ell+1,1),(0,\ell)\}$. This is one of the special cases of (iii), the lattice preserving affine transformation $(x,y)\to(x-y+\ell,y)$ carries P to the "almost-square" special pentagon Q^2 of Fig. 1.

From now on we assume that $|x|\leqslant\ell$ for all $(x,y)\in P$. Thus P is confined to the parallelogram of Fig. 3 bounded by the lines $x=\pm\ell$ and two other lines parallel to b. There are only six lattice directions in this parallelogram which can have a chord containing $\ell+1$ integer points. They are $(0,1)$, $(1,0)$, $(1,1)$, $(1,-1)$, $(2,1)$ and $(2,-1)$. To simplify matters we state

Claim 2. *If u_1 and u_2 are diameter directions with $det(u_1,u_2)=2$ of $P\in\mathcal{M}_\ell$ then the diameter segments $[z_1,z_1+\ell u_1]$ and $[z_2,z_2+\ell u_2]$ meet either at their midpoints or one segment is off by u_i. In these cases* (mod $\mathrm{SL}(2,\mathbb{Z}^2)$) *$P$ is either the square Q^1 or the almost square, Q^2, cf. Fig.* 1.

Proof. As we have seen above, we may suppose that $u_1=(0,1)$, $P\subset Q$ as in Fig. 3 and $u_2=(2,1)$ or $u_2=(2,-1)$. The latter case leads to the square with vertices $(-\ell,\ell)$, $(0,0)$, $(\ell,0)$, and $(0,\ell)$. When $u_2=(2,1)$ the diameters are $\{(0,0),(0,1),\dots,(0,\ell)\}$ and $\{(-\ell,i),(-\ell+2,i+1),\dots,(\ell,i+\ell)\}\subset P$. Considering the string of $\ell+2$ lattice points from $(-1,i-1)$ to $(\ell,i+\ell)$ it follows that $(-1,i-1)\notin P$. Since $(-1,1)\in\mathrm{conv}((-\ell,i),(0,0),(0,\ell))\subset P$, it follows $i\leqslant 1$. Using a symmetric argument we obtain that $i\in\{-1,0,1\}$ and can finish the proof as in the case $q=(\ell+2,\ell+1)\in P$ above. □

Assume now that P has exactly k diameter directions, $u_1,\dots,u_k$. Assume that P is not affinely equivalent to Q^1 neither Q^2. Then by the above Claim $\det(u_i,u_j)=\pm 1$ for any two diameter directions. This implies that $k\leqslant 3$. The diameters are $z_i,z_i+u_i,\dots,z_i+\ell u_i$ $(i=1,\dots,k)$ with suitable directions b_i of the edge opposite to z_i of P (see Lemma 1).

Define

$$Q = \bigcap_{1 \leqslant i \leqslant k} S(u_i, b_i, z_i).$$

Clearly $P \subset Q$. We claim $\ell(Q) = \ell$, so again by the maximality of P, $P = \operatorname{conv}(Q \cap \mathbb{Z}^2)$, finishing the proof.

Assume, on the contrary, that there exists a lattice point $q \in (Q \setminus P)$, and suppose that among these points q is one of the closest to P. Add this point to P, consider $P' := \operatorname{conv}(P \cup \{q\})$. So q is the only new lattice point in P', $P' \cap \mathbb{Z}^2 = P \cap \mathbb{Z}^2 \cup \{q\}$. The maximality of P implies that $\ell(P') > \ell(P)$, thus q creates a new longer diameter segment $q, q+u, \ldots, q+(\ell+1)u \in P' \cap \mathbb{Z}^2$ with $u \neq (0,0)$. As $\ell + 1$ of these points belong to P, we obtain that u is a diameter direction of P, too. However $S(u_i, b_i, z_i)$ contains no segments of direction u_i longer than ℓ. Thus u has to be different from $u_1, \ldots, u_k$, contradicting that P has exactly k diameter directions. Evidently, since P is not infinite, there are at least two diameter directions, $k = 2$ or 3. □

5. Bounding the width, the Proof of Theorem 3

As w_l is an integer for a lattice polygon we have to prove only $w_l < (\frac{4}{3})(\ell + 1)$.

We give a sketch for the convex set

$$Q = \bigcap_{1 \leqslant i \leqslant k} S_i,$$

where $S_i = S'(u_i, b_i, z_i)$ are the half-open slabs in (4) and (5) of Theorem 2 modified in (8). Denote the width of the slabs by L. By (8) we have $L = \ell + 1 - 1/(2\ell + 2) < \ell + 1$.

Applying a suitable $\mathrm{SL}(2, \mathbb{Z}^2)$ mapping we may assume that $u_1 = (1,0)$, $u_2 = (0,1)$ and u_3, if exists, is $(1,1)$ or $(-1,-1)$. We will use the fact (which is easy to establish) that the lattice width of Q is realized in one of the directions $(0,1)$, $(1,0)$, $(1,1)$, and $(-1,1)$. The lattice width of Q in direction $q \in \mathbb{Z}^2$ is $w_l(q, Q) := \max_{x,y \in Q} q(x - y)$.

In case (i) of Theorem 2 (see Fig. 5) $x = u$ follows from computing the area of Q in two ways. Similarity of triangles implies $z : x = (L - x) : y$. We get

$$\begin{aligned} &w_l((1,0), Q) = L + y - x, \quad w_l((0,1), Q) = L + z - x, \\ &w_l((-1,1), Q) = 2L + y - 2x - z, \quad w_l((1,1), Q) = 2L + z - 2x - y. \end{aligned} \tag{9}$$

Then

$$w_l(Q) = \min(L + y - x, L + z - x) = L - x + \min\left(y, \frac{(L - x)x}{y}\right)$$

and a simple analysis shows

$$w_l(Q) \leqslant \frac{1 + \sqrt{2}}{2} L \approx 1.207 \ldots L.$$

In case (ii) see Fig. 6.

For the left-hand-side hexagon note that the position of S_3 does not influence the width of Q as long as S_3 cuts off two opposite vertices of the parallelogram $S_1 \cap S_2$. So we may place S_3 so as to contain the isosceles and right angle triangle of Fig. 6. Reflecting inwards the three small triangles and comparing areas gives

$$\tfrac{1}{2}m_1 L + \tfrac{1}{2}m_2 L + \tfrac{1}{2}m_3\sqrt{2}L \leqslant \tfrac{1}{2}L^2$$

implying

$$\min(m_1, m_2, \sqrt{2}m_3) \leqslant \tfrac{1}{3}L.$$

Further, $w_l((1,0),Q) = L + m_2$, $w_l((0,1),Q) = L + m_1$, and $w_l((-1,1),Q) = L + \sqrt{2}m_3$. So $w_l(Q) \leqslant \frac{4}{3}L$.

For the other hexagon of Fig. 6 the computations in (9) can easily be applied.

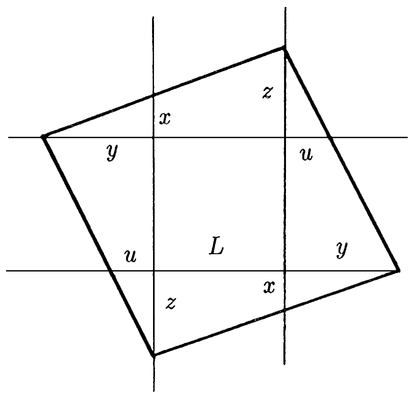

Fig. 5.

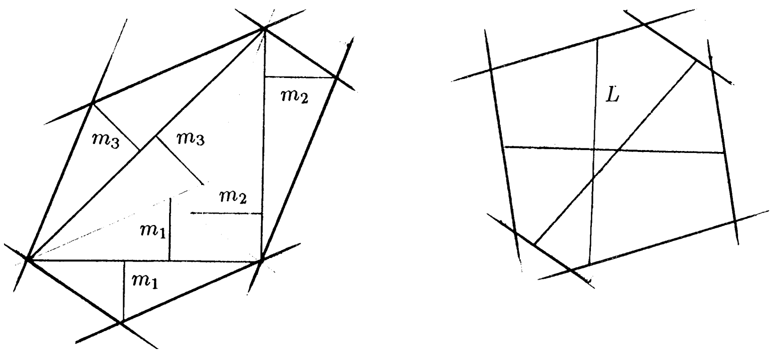

Fig. 6.

References

[1] E. G. Alarcon II, Convex lattice polygons, Ph.D. Dissertation, University of Illinois at Urbana-Champaign, 1995.
[2] T. Bang, A solution of the plank problem, Proc. Amer. Math. Soc. 2 (1951) 990–993.
[3] I. Bárány, J. Pach, On the number of convex lattice polygons, Comb. Probab. Comput. 1 (1992) 205–302.
[4] D.E. Bell, A theorem concerning the integer lattice, Stud. Appl. Math. 56 (1977) 187–188.
[5] C.E. Corzatt, Some extremal problems of number theory and geometry, Ph.D. Dissertation, University of Illinois at Urbana-Champaign, 1974.
[6] L. Fejes-Tóth, E. Makai Jr., On the thinnest non-separable lattice of convex plates, Studia Sci. Math. Hungar. 9 (1974) 191–193.
[7] P. Gritzmann, J.M. Wills, Lattice points, in: P.M. Gruber, J.M. Wills (Eds.), Handbook of Convex Geometry, North-Holland, Amsterdam, 1993, pp. 767–797.
[8] R. Kannan, L. Lovász, Covering minima and lattice-point-free convex bodies, Ann. Math. 128 (1988) 577–602.
[9] G. Pick, Geometrisches zur Zahlenlehre, Naturwiss. Z. Lotos, Prague (1899) 311–319.
[10] S. Rabinowitz, A theorem about collinear lattice points, Utilitas Math. 36 (1986) 93–95.
[11] J.E. Reeve, On the volume of lattice polyhedra, Proc. London Math. Soc. 7 (1957) 387–395.
[12] H.E. Scarf, An observation on the structure of production sets with indivisibilities, Proc. Nat. Acad. Sci. USA 74 (1977) 3637–3641.
[13] U. Schnell, A Minkowski-type theorem for covering minima, Geom. Dedicata 55 (1995) 247–255.