Graphs and Combinatorics (2001) 17:463–471

Graphs and Combinatorics

Ramsey Theory and Bandwidth of Graphs

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Abstract. The bandwidth of a graph is the minimum, over vertex labelings with distinct integers, of the maximum difference between labels on adjacent vertices. Kuang and McDiarmid proved that almost all n-vertex graphs have bandwidth $n - (2 + \sqrt{2} + o(1))\log_2 n$. Thus the sum of the bandwidths of a graph and its complement is almost always at least $2n - (4 + 2\sqrt{2} + o(1))\log_2 n$; we prove that it is always at most $2n - 4\log_2 n + o(\log n)$. The proofs involve improving the bounds on the Ramsey and Turán numbers of the "halfgraph".

Key words. Bandwidth, Ramsey number, Turán number, Halfgraph, Random graph

1. The Problem

When the vertices of a graph are labeled injectively with integers, the *dilation* of an edge is the difference between the labels on its endpoints. The *bandwidth* $B(G)$ of the graph G is the minimum, over all such labelings, of the maximum edge dilation.

Chinn, Chung, Erdős, and Graham [3] investigated the sum $B(G) + B(\overline{G})$, where $\overline{G}$ denotes the complement of G. They proved that $B(G) + B(\overline{G}) \geq n - 2$ whenever G has n vertices (for $n \geq 4$). Equality holds when G is a 4-vertex path. They also established the existence of constants c_1, c_2 such that $B(G) + B(\overline{G}) < 2n - c_1 \log n$ for every n-vertex G and $B(G) + B(\overline{G}) > 2n - c_2 \log n$ for almost every n-vertex G.

Kuang and McDiarmid [17] improved the constant c_2. They proved that for the random graph generated with fixed edge probability p, almost all graphs have bandwidth $n - (2 + \sqrt{2} + o(1)) \log_{1/(1-p)} n$, where n denotes the

* Research supported in part by the Hungarian National Science Foundation under grant OTKA 016389 and by the National Security Agency under grant MDA904-98-I-0022
AMS codes: 05C35, 05C78, 05C80, 05D10

number of vertices. With $p = 1/2$, the complement of a random graph is also a random graph, and we obtain $c_2 < 4 + 2\sqrt{2} + \epsilon$ for any $\epsilon > 0$ and sufficiently large n.

In this paper we increase the constant c_1 in the upper bound, proving that

Theorem 1. *If $f(n)$ is the maximum of $B(G) + B(\overline{G})$ over n-vertex graphs, then*

$$2n - \left\lceil (4 + 2\sqrt{2}) \log_2 n \right\rceil \leq f(n) \leq 2n - 4 \log_2 n + o(\log n).$$

2. Related Extremal Graph Problems

The bandwidth problem can be expressed in terms of other classical extremal graph problems involving the occurrence of fixed subgraphs. We use $V(G)$ and $E(G)$ for the vertex set and edge set of a graph G. We use $[n]$ to denote the set of the first n positive integers and P_n to denote the graph that is an n-vertex path.

The *halfgraph* H_r is the graph with $2r$ vertices defined by $V(H_r) = [2r]$ and $E(H_r) = \{ij\colon\ j - i \geq r\}$. When we let $a_i = i$ for $1 \leq i \leq r$ and $b_j = r + j$ for $1 \leq j \leq r$, we obtain the more common representation $E(H_r) = \{a_i b_j\colon\ i \leq j\}$, which shows that H_r is bipartite.

The first definition of H_r leads to bandwidth via the observation that $\overline{H}_r = P_{2r}^{r-1}$, where P_{2r}^k has vertex set $[2r]$ and edge set $\{ij\colon\ 1 \leq j - i \leq k\}$ (this is the kth *power* of P_{2r}). The bandwidth of P_n^k is k. In general, the bandwidth of an n-vertex graph is at most k if and only if the graph is contained in P_n^k, since the set of edges permitted in an optimal labeling forms P_n^k. For $n \geq 2r \geq 2$, this observation becomes

$$B(G) \leq n - r - 1 \text{ if and only if } H_r \subset \overline{G}. \tag{2.1}$$

Since the appearance of H_r requires $n \geq 2r$, one direction needs no restriction on n.

Thus $B(G)$ is large if and only if $\overline{G}$ contains no large halfgraph, and $B(\overline{G})$ is large if and only if G contains no large halfgraph. Studying the maximum of $B(G) + B(\overline{G})$ amounts to studying what combinations of halfgraphs must appear in G and $\overline{G}$; this is a problem of Ramsey graph theory.

Given graphs A, B, the *Ramsey number* $R(A, B)$ is the smallest integer n such that every red/blue-coloring of the edges of K_n yields a copy of A in red or a copy of B in blue. When $n < R(A, B)$, the clique K_n can be decomposed into a subgraph avoiding A and a subgraph avoiding B. When A and B are cliques, the numbers $R(k, l) = R(K_k, K_l)$ are the classical Ramsey numbers. The general definitions and results we use from Ramsey Theory appear in [14].

Proposition 2. *If $R(H_a, H_b) > n$ with $n \geq max\{2a, 2b\}$, then $f(n) \geq 2n - a - b$.* If $f(n) \geq 2n - k$, then there exist a, b with sum k such that $R(H_a, H_b) > n$.

Proof. If $R(H_a, H_b) > n$, then there is an n-vertex graph G such that $H_a \not\subseteq \overline{G}$ and $H_b \not\subseteq G$. When $n \geq \max\{2a, 2b\}$, (2.1) and the latter statement yield both $B(G) \geq n - a$ and $B(\overline{G}) \geq n - b$. Thus $f(n) \geq 2n - a - b$.

If $f(n) \geq 2n - k$, then there exists an n-vertex graph G such that $B(G) + B(\overline{G}) \geq 2n - k$. This requires the existence of a, b with sum k such that $B(G) \geq n - a$ and $B(\overline{G}) \geq n - b$. By (2.1), this requires both $H_a \not\subseteq \overline{G}$ and $H_b \not\subseteq G$, which yields $R(H_a, H_b) > n$. □

For completeness, we sketch the proof of the lower bound on $f(n)$. We have noted that if $B(G) \geq n - r$ almost always, then $f(n) \geq 2n - 2r$. We have $B(G) \geq n - r$ if and only if $H_r \not\subseteq \overline{G}$. Since H_r has $2r$ vertices and $r(r+1)/2$ edges, the standard Erdős-Rényi [4] random graph model with edge probability $1/2$ yields $P(H_r \subseteq \overline{G}) \leq n^{2r} 2^{-r^2/2}$ (see [3, 18]). When $r > 4 \log_2 n$, the probability is less than $1/2$, and the bound $f(n) \geq 2n - \lceil 4 \log_2 n \rceil$ follows.

Considering the densest subgraph of H_r yields a better bound. Let $H_{r,m}$ be the subgraph obtained by discarding the $2m$ vertices of least degree (m from each partite set). Presence of H_r requires presence of $H_{r,m}$. Since $H_{r,m}$ has $2r - 2m$ vertices and $r(r+1)/2 - m(m+1)$ edges, in the random n-vertex graph G we have

$$P(H_r \subseteq \overline{G}) \leq P(H_{r,m} \subseteq \overline{G}) \leq n^{2r-2m} 2^{-(r^2-2m^2)/2}.$$

The densest subgraph and best bound are obtained by setting $m = (1 - \sqrt{2}/2)r$. When $r > (2 + \sqrt{2}) \log_2 n$, the resulting bound on $P(H_r \subseteq \overline{G})$ is less than $1/2$, which yields the lower bound in Theorem 1. Using the second moment method, Kuang and McDiarmid [17] obtained the precise threshold for the appearance of H_r, thus showing that $B(G) = (2 + \sqrt{2} + o(1)) \log_2 n$ almost always.

In terms of Ramsey numbers, the bound on $P(H_r \subseteq \overline{G})$ yields the following:

Theorem 3. $R(H_r, H_r) > 2^{(r+1)/(2+\sqrt{2})}$. □

Proposition 2 implies that $f(n) = 2n - \min\{a + b: R(H_a, H_b) > n\}$ if $n \geq \max\{2a, 2b\}$ for a pair (a, b) where the minimum occurs. When we set $r = \lceil (2 + \sqrt{2}) \log_2 n \rceil - 1$, Theorem 3 yields $R(H_r, H_r) > n$. When $n \geq 34$, we have $n \geq 2 \lceil (2 + \sqrt{2}) \log_2 n \rceil - 2$, and then Proposition 2 implies that $f(n) \geq 2n - 2\lceil (2 + \sqrt{2}) \log_2 n \rceil + 2$ when $n \geq 34$. With Theorem 3, we also have

Corollary 4. *For* $n \geq 34$, $f(n) = 2n - \min\{a + b: R(H_a, H_b) > n\}$. □

We henceforth assume that $n \geq 34$ (this threshold can be reduced by improving the lower bounds on Ramsey numbers for small halfgraphs).

Upper bounds on $R(H_a, H_b)$ that depend only on $a + b$ yield upper bounds on $f(n)$. An easy bound comes from the classical Erdős-Szekeres (see [14]) upper bound on Ramsey numbers: $R(p, q) \leq \binom{p+q-2}{p-1} < 2^{p+q}$. This yields

$$R(H_a, H_b) \leq R(2a, 2b) < 4^{a+b}. \tag{2.2}$$

By Proposition 2 and (2.2), we have $f(n) \leq 2n - \frac{1}{2} \log_2 n$. We want to increase the coefficient on $\log_2 n$ to 4. To this end, we define the function g by

$$g(k) = \max_{a+b \le k} R(H_a, H_b). \tag{2.3}$$

Corollary 4 tells us that $g(k) \le n$ yields $f(n) < 2n - k$.

Let $\mathrm{ex}(n, F)$ denote the maximum number of edges in an F-free graph with n vertices. This is often called the *Turán number* of F. Kővári, T. Sós and Turán [16] showed that

$$\mathrm{ex}(n, K_{k,k}) \le \frac{1}{2}(k-1)^{1/k} n^{2-1/k} + \frac{1}{2}(k-1)n. \tag{2.4}$$

Let G be an n-vertex graph. If $\mathrm{ex}(n, K_{a,a}) + \mathrm{ex}(n, K_{b,b}) < \binom{n}{2}$, then G contains $K_{a,a}$ or $\overline{G}$ contains $K_{b,b}$. Since $H_r \subset K_{r,r}$, this yields $R(H_a, H_b) \le n$. A careful examination of (2.4) shows that $k = 2\log_2 n - O(\log_2 \log_2 n)$ is small enough to yield $g(k) \le n$, and thus $f(n) < 2n - (2 - o(1))\log_2 n$. This technique of using the Turán number of a bipartite graph to prove an upper bound on its Ramsey number is now standard in extremal graph theory (see [10], for example).

To further improve this upper bound, we need a tighter bound on the Turán number of H_r. Observe that H_r is contained not only in $K_{r,r}$ but even in the subgraph obtained by removing $K_{\lfloor r/2 \rfloor, \lfloor r/2 \rfloor}$ from $K_{r,r}$. Due to its similarity to the halfgraph, we use the notation $H'(r, l)$ to denote the subgraph of the biclique $K_{r,r}$ obtained by deleting the edges of the biclique $K_{l,l}$. In the next section, we obtain an upper bound on $\mathrm{ex}(n, H'(r, l))$ implying that $g(k) \le n$ for k as large as about $4\log_2 n$. This yields the desired upper bound $f(n) < 2n - 4\log_2 n + o(\log_2 n)$.

3. Turán Numbers and the Halfgraph

Before proving our upper bound on $\mathrm{ex}(n, H'(r, l))$ in Theorem 6, we compare it with earlier results.

For every bipartite graph F that is not a forest, there is a positive constant $c(F)$ such that $\Omega(n^{1+c}) \le \mathrm{ex}(n, F) \le O(n^{2-c})$; this was observed by Erdős [unpublished] and appears in [16]. For a given F, the first problem in studying $\mathrm{ex}(n, F)$ is thus to find the right exponent (if such exists). It is conjectured ([7,9]) that $\mathrm{ex}(n, K_{k,k}) = \Theta(n^{2-1/k})$. This was proved for $K_{2,2}$ in Erdős-Rényi-Sós [5] and (simultaneously and independently) in Brown [2]. For $K_{3,3}$ it appears in Brown [2]. For $K_{k,l}$ with $k > l!$, results appear in Kollár-Rónyai-Szabó [15], later improved for $k > (l-1)!$ in Alon-Rónyai-Szabó [1].

Erdős [6] also proved that when $r > l \ge 1$, there exists a constant $c_{r,l}$ such that $\mathrm{ex}(n, H'(r, l)) < c_{r,l} n^{2-1/(r-l)}$. His proof is somewhat complicated and does not give a sufficiently good constant in the range where we need (when both r and l are about $\log_2 n$).

Another upper bound for $\mathrm{ex}(n, H'(r, l))$ follows from the method of Erdős and Simonovits [9]. They proved that $\mathrm{ex}(n, Q) \le O(n^{8/5})$, where Q is the 8-vertex 3-dimensional cube. Similarly, the correct order of growth for $\mathrm{ex}(n, H'(r, l))$ follows from the main result in Füredi [13]. However, in all of these articles the authors concentrated on large values of n compared to k. In the range we need, these results seem not to imply our bound.

In the proof, we extend $\binom{x}{t}$ to nonnegative real x for each nonnegative integer t. When $t = 0$, we take $\binom{x}{t} = 1$ for all real $x \geq 0$. When $t \geq 1$, we take $\binom{x}{t} = 0$ for $0 \leq x < t - 1$, and for $x \geq t - 1$ we view $\binom{x}{t}$ as a real polynomial $x(x-1)\dots(x-t+1)/t!$ of degree t in x. The resulting functions are convex. Thus the mean of values of this function at several points is at least the value at the mean argument. In particular,

$$\sum_{i=1}^{m} \binom{x_i}{t} \geq m \binom{\Sigma x_i/m}{t}. \tag{3.1}$$

We will also use the following simple lemma, which was proved and applied in [12] to a related problem.

Lemma 5 ([12]). *If $v, t \geq 1$ are integers and $c, x_0, x_1, \dots, x_t \geq 0$ are real numbers, then*

$$\sum_{1 \leq i \leq v} \binom{x_i}{t} \leq c \binom{x_0}{t} \quad \text{implies} \quad \sum_{1 \leq i \leq v} x_i \leq x_0 c^{1/t} v^{1-1/t} + (t-1)v. \qquad \square$$

Theorem 6. $\mathrm{ex}(n, H'(r,l)) \leq \frac{1}{2}(r+l-1)^{1/(r-l)} n^{2-1/(r-l)} + \frac{1}{2}(r-l-1)n.$

Proof. Let G be an n-vertex graph not containing $H'(r,l)$; we bound $e = |E(G)|$. Let $d(x)$ denote the degree of a vertex x, and for $A \subset V(G)$ let $d(A)$ denote the number of common neighbors of A.

Let $t = r - l$, and let X be the number of copies of $K(t,t)$ in G. We form such a subgraph by choosing a t-set A and choosing t of its common neighbors. Each copy arises twice. Thus $X = \frac{1}{2}\sum_{|A|=t} \binom{d(A)}{t}$.

We first find a lower bound on X. By (3.1),

$$\frac{1}{2} \sum_{|A|=t} \binom{d(A)}{t} \geq \frac{1}{2} \binom{n}{t} \binom{\sum d(A)/\binom{n}{t}}{t}.$$

Since $d(A)$ counts the stars with leaf set A, the total $\sum d(A)$ is the number of stars with t edges in G. These can alternatively be counted by choosing t neighbors for each choice of the central vertex. Applying (3.1) to the resulting sum yields

$$\sum d(A) = \sum_{x \in V(G)} \binom{d(x)}{t} \geq n \binom{\sum d(x)/n}{t}.$$

Together, these computations yield

$$X \geq \frac{1}{2} \binom{n}{t} \binom{n\binom{2e/n}{t} / \binom{n}{t}}{t}. \tag{3.2}$$

We next find an upper bound on X. Let $\mathscr{A} = \{A \in \binom{V(G)}{t} : d(A) < r\}$. Consider copies of $K_{t,t}$ in which at least one of the partite sets belongs to $\mathscr{A}$. The number of these is at most $|\mathscr{A}|\binom{r-1}{t}$.

Now consider a copy of $K_{t,t}$ with partite sets A, B such that $A, B \notin \mathscr{A}$. Our main observation is that the prohibition of $H'(r,l)$ yields $d(A) \leq 2r - t - 1$. If $d(A) \geq r + r - t = 2l + t$, then A has at least $2l$ common neighbors outside B, and after avoiding (at most) l of the r common neighbors of B there remain at least l common neighbors of A to complete a copy of $H'(r,l)$. The same argument applies to $d(B)$. Thus each such copy of $K_{t,t}$ is generated twice when we arbitrarily choose t common neighbors of a t-set outside $\mathscr{A}$. Thus the number of copies of $K_{t,t}$ with neither partite set in $\mathscr{A}$ is bounded by $\frac{1}{2}\left(\binom{n}{t} - |\mathscr{A}|\right)\binom{2r-t-1}{t}$.

Together, the two upper bounds yield

$$X \leq \binom{n}{t} \max\left\{\binom{r-1}{t}, \frac{1}{2}\binom{2r-t-1}{t}\right\}. \tag{3.3}$$

Since the ratio of $\frac{1}{2}\binom{2r-t-1}{t}$ to $\binom{r-1}{t}$ is exactly $\binom{r-1+(r-t)}{t-1}/\binom{r-1}{t-1}$, always the second term in the maximization is larger.

Comparing (3.2) and (3.3) yields

$$n\binom{2e/n}{t}/\binom{n}{t} \leq 2r - t - 1.$$

Let $v = x_0 = n$, let $x_1 = \ldots = x_v = 2e/n$, and let $c = 2r - t - 1$. Lemma 5 now yields

$$2e \leq (2r - t - 1)^{1/t} n^{2-1/t} + (t-1)n. \qquad \square$$

4. Proof of the Upper Bound

The pair (r,r) is one instance of (a,b) such that $a + b = 2r$, and thus $R(H_r, H_r) \leq g(2r)$, where g is as defined in (2.3). Theorem 6 enables us to prove an upper bound on $g(2r)$ and thus an upper bound on $R(H_r, H_r)$ that differs from the lower bound in Theorem 3 by a factor of less than 2 in the exponent.

Corollary 7. *If r is sufficiently large, then $R(H_r, H_r) \leq g(2r) \leq (3r+1)2^{r/2}$.*

Proof. By the definition of $g(2r)$, there exist a, b with $a + b = 2r$ such that $g(2r) = R(H_a, H_b)$. Let G be an n-vertex graph such that $H_a \not\subseteq G$ and $H_b \not\subseteq \overline{G}$. It suffices to show that $n < (3r+1)2^{r/2}$, when r is sufficiently large.

Because $H_a \not\subseteq G$ and $H_a \subset H'(a, \lfloor a/2 \rfloor)$, Theorem 6 yields

$$e(G) \leq \mathrm{ex}(n, H_a) \leq \mathrm{ex}(n, H'(a, \lfloor a/2 \rfloor)) \leq \frac{n^2}{2}\left(\frac{a + \lfloor \frac{a}{2} \rfloor - 1}{n}\right)^{1/\lceil \frac{a}{2} \rceil} + \frac{n}{2}\left(\left\lceil \frac{a}{2} \right\rceil - 1\right).$$

The right side yields the bound

$$e(G) < \frac{n^2}{2}\left(\frac{3r}{n}\right)^{2/a} + \frac{na}{4}. \tag{4.1}$$

Summing (4.1) and its analogue for $\overline{G}$ yields

$$\binom{n}{2} = e(G) + e(\overline{G}) \le \frac{n^2}{2}\left(\frac{3r}{n}\right)^{2/a} + \frac{na}{4} + \frac{n^2}{2}\left(\frac{3r}{n}\right)^{2/b} + \frac{nb}{4}.$$

This simplifies to

$$1 - \frac{r+1}{n} < c^{2/a} + c^{2/b}, \tag{4.2}$$

where $c = 3r/n$.

We may assume that $r \le 2\log_2(n/3r)$, since otherwise the desired inequality holds. Since $c < 1$, differentiating the function f defined by $f(x) = c^{2/x}$ shows that f is concave for $x > -\ln c$. If $\min\{a,b\} > \ln(n/3r)$, then concavity implies that the upper bound in (4.2) is at most $2c^{4/(a+b)}$. We obtain $1 - \frac{r+1}{n} < 2\left(\frac{3r}{n}\right)^{2/r}$, which rearranges to yield $n < 3r2^{r/2}\left(\frac{n}{n-r-1}\right)^{r/2} < (3r+1)2^{r/2}$.

It remains only to eliminate the case $a = \min\{a,b\} < \ln(n/3r)$. We obtain a contradiction to (4.2) by showing that the right side is too small. Since $c^{2/x}$ is a monotone increasing function, we obtain an upper bound by using $a \le \ln(n/3r)$ and $b < 2r$. We also use our restriction to $r \le 2\log_2(n/3r)$. These yield

$$c^{2/b} < c^{2/2r} = e^{-(1/r)\ln(n/3r)} = e^{-(1/r)\log_2(n/3r)\ln 2} \le e^{-(1/2)\ln 2} = \frac{1}{\sqrt{2}} < .7072$$

$$c^{2/a} < c^{2/\ln(n/3r)} = e^{-2\ln(n/3r)/\ln(n/3r)} = e^{-2} < .1354$$

When $n \ge (3r+1)2^{r/2}$, this contradicts (4.2) for $r \ge 3$, which eliminates this case and completes the proof. □

As observed in Section 2, $g(k) \le n$ yields $f(n) < 2n - k$. Thus Corollary 7 yields the desired upper bound $f(n) < 2n - 4\log_2 n + O(\log_2\log_2 n)$.

5. Conclusion and Open Problems

Our upper and lower bounds for $R(H_r, H_r)$ are much closer together than the best known upper and lower bounds for $R(K_r, K_r)$. One might expect that tighter bounds on $R(H_r, H_r)$ are hopeless without improving the bounds for $R(K_r, K_r)$. It would be interesting to find a direct relationship between these two functions.

Our method can be generalized for decompositions of K_n into k edge-disjoint n-vertex graphs, with k fixed. Over such decompositions, $\max(B(G_1) + B(G_2) + \cdots + B(G_k)) = kn - \Theta(\log n)$. It would be interesting to narrow the gap in the coefficient of $\log n$.

If we can allow the number of pieces in the decomposition to grow arbitrarily with n, then the maximum sum is $\binom{n}{2}$, achieved by decomposition into $\binom{n}{2}$ individual edges. The maximum can reach $O(n^2)$ when the number of pieces is linear, which is not surprising given the result for fixed k. For example, when we decompose K_n into stars of sizes 1 through $n-1$, the sum of the bandwidths is $\lceil \frac{n}{2} \rceil \lceil \frac{n+1}{2} \rceil$. How small can k be in terms of n to achieve various growth rates for the bandwidth sum?

When $l = r-1$, the graph $H'(r,l)$ is a double star with two adjacent vertices of degree r. It is easy to see that $\mathrm{ex}(n, H'(r, r-1)) = (r-1)(n-r+1)$ (for $r > r_0$), so in this case the bound in Theorem 6 is asymptotically optimal within a factor of 2. For $r - l = 2$, Theorem 6 gives

$$\mathrm{ex}(n, H'(r, r-2)) \leq \frac{1}{2}(2r-3)^{1/2} n^{3/2} + O(n).$$

The best lower bound from [13] gives

$$\mathrm{ex}(n, H'(r, r-2)) \geq \mathrm{ex}(n, K_{2,r}) = (1+o(1))\frac{1}{2}(r-1)^{1/2} n^{3/2},$$

so in this case Theorem 6 is asymptotically optimal within a factor of $\sqrt{2}$. It would be interesting to find the correct asymptotic behavior for each $r - l$.

It is easy to see that $\mathrm{ex}(n, H_3) = \mathrm{ex}(n, K_{2,2}) + O(1)$; we expect that equality holds for sufficiently large n. For larger fixed r, we conjecture that $\mathrm{ex}(n, H_r) \sim \mathrm{ex}(n, K_{\lceil r/2 \rceil, \lceil r/2 \rceil})$. In other words, is the Turán number of H_r about the same as the Turán number of its largest complete bipartite subgraph?

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Received: September 2, 1998
Final version received: November 29, 1999