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Covering a graph with cuts of minimum total size

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Abstract

A *cut* in a graph G is the set of all edges between some set of vertices S and its complement $\bar{S} = V(G) - S$. A *cut-cover* of G is a collection of cuts whose union is $E(G)$ and the *total size* of a cut-cover is the sum of the number of edges of the cuts in the cover. The cut-cover size of a graph G, denoted by $\mathrm{cs}(G)$, is the minimum total size of a cut-cover of G. We give general bounds on $\mathrm{cs}(G)$, find sharp bounds for classes of graphs such as 4-colorable graphs and random graphs. We also address algorithmic aspects and give sharp bounds for the sum of the cut-cover sizes of a graph and its complement. We close with a list of open problems.

Keywords: Minimum cover; Cut; Random graphs; Nordhaus–Gaddum; Geometric representation; Average distance

1. The cut-cover problem

Covering the edges of a graph by subgraphs from a given family of graphs, like cliques, matchings, trees, or cycles, is one of the basic themes in graph theory (see [23] for a survey of results). Erdős, Goodman and Pósa [7] showed that the edges of every graph on n vertices can be covered by $\lfloor n^2/4 \rfloor$ cliques, and the balanced complete bipartite graph shows that this is best possible. It can also be desirable to minimize parameters other than the number of subgraphs used in the cover. Győri and Kostochka [14], Chung [5] and Kahn [20] independently proved the stronger result that

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every graph has a decomposition into cliques whose *order-sum* (sum of the number of vertices of the cliques in the cover) is at most $\lfloor n^2/2 \rfloor$.

It is well known that the minimum number of bipartite subgraphs (or equivalently cuts) needed to cover the edges of a graph G with chromatic number $\chi(G)$ is $\lceil \lg \chi(G) \rceil$ (see, e.g., [11,16,22]), where lg denotes the base 2 logarithm. To obtain such a covering we label the vertices in the jth color class by the binary expansion of $j-1$, thus associating with each vertex a $\{0,1\}$-vector of length $\lceil \lg \chi(G) \rceil$. From this labeling we can construct the desired cut-cover by letting the ith cut consist of all the edges between vertices whose labels differ in the ith coordinate. These $\lceil \lg \chi(G) \rceil$ cuts cover all the edges of G, since adjacent vertices have different colors, and therefore different labels. To see that this way of covering the graph with cuts is best possible, notice that we can extract a labeling of the vertices with binary vectors of length k from a cover with k cuts. Adjacent vertices must receive different labels, so that the labeling is a proper coloring with at most 2^k colors.

The *size* of a graph is the number of its edges. In the cover by $\lceil \lg \chi(G) \rceil$ cuts the sum of the sizes of the cuts could be as big as $\lfloor n^2/4 \rfloor \lceil \lg \chi(G) \rceil$, but will usually be much smaller. When minimizing the total size, however, other ways of cutting the graph can be more efficient. It is the aim of this paper to give upper and lower bounds on the minimum total size of a cut-cover.

2. Definitions and main results

Throughout this paper G will be a graph with vertex set $V = V(G)$, and edge set $E = E(G)$. For a given graph G we will define its *order* by $n = n(G) = |V(G)|$, its *size* by $\mathrm{e}(G) = |E(G)|$ and denote its chromatic number by $\chi(G)$. For a partition of the vertex set $V = S \cup \bar{S}$ we will define the *cut* induced by S to be the set of edges between S and $\bar{S}$,

$$[S, \bar{S}] := \{uv \in E(G)\colon u \in S,\ v \in \bar{S}\}.$$

A *cut-cover* of a graph G is a collection $\mathscr{C} = \{[S_1, \bar{S}_1], [S_2, \bar{S}_2], \dots, [S_k, \bar{S}_k]\}$ of cuts whose union is $E(G)$. The *total size* of $\mathscr{C}$ is the sum of the sizes $|[S_i, \bar{S}_i]|$ of the cuts in $\mathscr{C}$. The *cut-cover size* of G, denoted by $\mathrm{cs}(G)$ is the minimum total size of a cover of $E(G)$ with cuts.

We immediately get the trivial bounds that $\mathrm{e}(G) \leqslant \mathrm{cs}(G) \leqslant \mathrm{cs}(K_n)$, where equality in the lower bound holds for all bipartite graphs. The cut-cover size of the complete graph has been determined in [17,18,21,30]:

$$\mathrm{cs}(K_n) = \begin{cases} (n-1)^2, & n \neq 4, 8, \\ (n-1)^2 - 1, & n = 4, 8. \end{cases} \tag{1}$$

For complete graphs with at least 8 vertices the optimal cut-cover is unique, up to isomorphism. For $n > 8$, cover K_n with stars by taking $n-1$ cuts so that the S_i are distinct sets of size one. For $n = 8$, cover K_8 with $K_{4,4}$'s by taking 3 cuts such that $|S_i| = 4$, $|S_i \cap S_j| = 2$ for $i \neq j$ and $|S_1 \cap S_2 \cap S_3| = 1$.

For odd cycles we have $\mathrm{cs}(C_{2k+1}) = 2k + 2$. Indeed, every cut in a cycle has even size, so $\mathrm{cs}(C_{2k+1})$ must also be even. Together with the trivial lower bound, this fact yields the lower bound. There are many different covers achieving this value.

This observation implies that when G has c disjoint odd cycles, the trivial lower bound can be improved to to $\mathrm{e}(G) + c$. Hence, in a sense, $\mathrm{cs}(G)$ measures how 'non-bipartite' a graph is. To state the improved bounds that are the main subject of investigation in this paper, we need to define the following parameters:

$$\mathrm{Cut}(G) := \max\{|[S, \overline{S}]|: S \subset V\}, \tag{2}$$

$$\mathrm{Cut}'(G) := \max\{|[S, \overline{S}]|: S \subset V,\ S \text{ stable set}\}. \tag{3}$$

A vertex set S is *stable*, or *independent*, if the subgraph induced by S has no edges. A *stable cut* is a cut in which one of the partition sets forms a stable set. We denote the size of a maximum stable set in G by $\alpha = \alpha(G)$. If we denote the minimum and maximum degree in G by $\delta(G)$ and $\Delta(G)$, then we get

$$\alpha(G)\delta(G) \leqslant \mathrm{Cut}'(G) \leqslant \alpha(G)\Delta(G), \tag{4}$$

although a stable set achieving $\mathrm{Cut}'(G)$ need not be maximum.

Theorem 1.

$$2\mathrm{e}(G) - \mathrm{Cut}(G) \leqslant \mathrm{cs}(G), \tag{5}$$

$$\mathrm{cs}(G) \leqslant 2\mathrm{e}(G) - \mathrm{Cut}'(G). \tag{6}$$

The proof of Theorem 1 and the following results are postponed to the next sections. There are a number of graphs for which Theorem 1 suffices to compute $\mathrm{cs}(G)$. For example, in the case of the typically uncooperative Petersen graph P, we get $\mathrm{Cut}(P) = \mathrm{Cut}'(P) = 12$, so that $\mathrm{cs}(P) = 18$. Note that every graph can be embedded in a slightly bigger graph for which $\mathrm{Cut} = \mathrm{Cut}'$.

Definition 2. Given disjoint graphs G and H, we define their join $G \vee H$ to be the graph obtained by making every vertex from $V(G)$ adjacent to every vertex in $V(H)$. We have $n(G \vee H) = n(G) + n(H)$, $\mathrm{e}(G \vee H) = \mathrm{e}(G) + \mathrm{e}(H) + n(G)n(H)$ and $\chi(G \vee H) = \chi(G) + \chi(H)$.

Proposition 3. *For every graph* G *and* $m \geqslant \max\{\Delta(G), 1\}$, $G \vee \overline{K}_m$ *contains* G, $\chi(G \vee \overline{K}_m) = \chi(G) + 1$, $\mathrm{Cut}'(G \vee \overline{K}_m) = \mathrm{Cut}(G \vee \overline{K}_m) = nm$, *and* $\mathrm{cs}(G \vee \overline{K}_m) = 2\mathrm{e}(G) + nm$.

Thus, it is unlikely that specific subgraphs, other than large cuts, play an important role in determining $\mathrm{cs}(G)$. There are, however, many cases when equality holds in (5) or (6). A graph is type A if equality holds in (5), otherwise it is type A'. Similarly, it is type B if equality holds in (6) and type B' otherwise. A graph is type AB if it is both type A and B and so on. Bipartite graphs are trivially type AB.

Theorem 4. *If G is 4-colorable, then it is of type A, that is*

$$\mathrm{cs}(G) = 2\mathrm{e}(G) - \mathrm{Cut}(G).$$

In Section 4, we will show that Theorem 4 implies that determining $\mathrm{cs}(G)$ is NP-complete, but can be determined in polynomial time when G is planar. In Proposition 17 of Section 7, we will see that Theorem 4 is best possible in the sense that for every fixed number $k > 4$ and every one of the four possible types AB, $A'B$, AB' and $A'B'$ there are infinitely many graphs with chromatic number k and the specified type. However, the next rather technical result will imply that almost all graphs are of type $A'B$. Theorem 5 is also used in the proof of Proposition 8.

Theorem 5. *Let G be a graph with edge-density $d = \mathrm{e}(G)/\binom{n}{2}$. If G satisfies the conditions given below, then it is of type $A'B$.*

$$\alpha(G) \leqslant d^2 n/100, \tag{7}$$

$$\text{If } |S_1|, |S_2| \geqslant dn/10, \text{ then } |[S_1, S_2]| \geqslant \tfrac{5}{6} d |S_1||S_2|, \tag{8}$$

$$\text{If } |S| \leqslant n/2, \text{ then } |[S, \bar{S}]| \geqslant \tfrac{20}{43} dn|S|. \tag{9}$$

Almost all random graphs fulfill these three requirements. The probability space $\mathscr{G}(n, p)$ is defined for $0 \leqslant p \leqslant 1$, where p may depend on n. The ground set of $\mathscr{G}(n, p)$ is the set of all $2^{\binom{n}{2}}$ graphs with $V(G) = \{1, \ldots, n\}$, and the probability of a graph G is given by $\mathrm{Prob}(G_p = G\colon G_p \in \mathscr{G}(n, p)) = p^{\mathrm{e}(G)}(1 - p)^{\mathrm{e}(\bar{G})}$. We say that for a graph property Q and a sequence of probabilities $p(n)$ *almost every graph* in $\mathscr{G}(n, p)$ has property Q if

$$\mathrm{Prob}(G_p \text{ has property } Q : G_p \in \mathscr{G}(n, p)) \to 1 \quad \text{as } n \to \infty.$$

Theorem 6. *Almost every graph $G \in \mathscr{G}(n, p)$ is of type $A'B$ for $p = p(n) \geqslant 6((\log n)/n)^{1/3}$, and of type AB for $p = p(n) \leqslant (1 - \varepsilon)/n$.*

The *Turán graph* $T(n, k)$ is the complete k-partite graph on n vertices with part sizes as equal as possible, i.e. size $\lfloor n/k \rfloor$ or $\lceil n/k \rceil$. Note that $t(n, k) = \mathrm{e}(T(n, k))$ is the maximum number of edges among all k-colorable graphs on n vertices. In Section 6, we prove the following bounds on the sum of the cut-cover size of a graph and its complement:

Theorem 7. *For every n-vertex graph G and its complement $\bar{G}$*

$$2\binom{n}{2} - t(n, 4) \leqslant \mathrm{cs}(G) + \mathrm{cs}(\overline{G}) \leqslant \mathrm{cs}(K_n),$$

and the bounds are best possible.

In Section 7, we will prove some further bounds and exact results for special types of graphs.

Proposition 8. *If G is a complete k-partite graph, then*

$$\mathrm{cs}(K_{k-1} \vee \overline{K}_{n-k+1}) \leqslant \mathrm{cs}(G) \leqslant \mathrm{cs}(T(n,k)). \tag{10}$$

Furthermore,

$$\mathrm{cs}(K_{k-1} \vee \overline{K}_{n-k+1}) = (k-1)(n-1), \tag{11}$$

except that $\mathrm{cs}(K_4) = 8$ *and* $\mathrm{cs}(K_8) = 48$. *Also for all but finitely many pairs* (n,k) *with* $n \geqslant k > 8$

$$\mathrm{cs}(T(n,k)) = 2t(n,k) - \left\lceil \frac{n}{k} \right\rceil \left(n - \left\lceil \frac{n}{k} \right\rceil \right). \tag{12}$$

However, $\mathrm{cs}(T(n,4)) = 2t(n,4) - \lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor$.

Remark 9. Specifically, we will show that (12) holds if

- $k \leqslant 3$,
- $k > 200$, or
- $k > 8$ and $n > 2(k-1)^3$ or $k|n$.

In Section 8, we will give a geometric formulation for $\mathrm{cs}(G)$ that is similar to the bandwidth-sum of G. We will close by posing several open questions in Section 9.

3. Upper and lower bounds

Proof of Theorem 1. For the upper bound we need an efficient covering. Given S as in the definition of $\mathrm{Cut}'(G)$, we simply take a covering of $E(G)$ by stars centered in $\overline{S}$. Since S is a stable set this covers all edges in $E(G)$, and furthermore the edges in $[S, \overline{S}]$ are all covered exactly once, while all edges within $\overline{S}$ are covered exactly twice.

For the lower bound, we consider the labeling mentioned in the introduction. That is with a given optimal covering by k cuts we identify a labeling of the vertex set with binary vectors of length k as follows: let the ith entry of the label of v be 1 if $v \in S_i$ and 0 otherwise. The number of times an edge is cut is exactly the number of coordinates in which the two labels differ. Let the *weight* of a label be the number of ones it contains. If we now define S_{odd} to be the set of vertices with odd weight, then the edges that are covered once must be contained in $[S_{\mathrm{odd}}, \bar{S}_{\mathrm{odd}}]$, so that

$$\mathrm{cs}(G) \geqslant 2\mathrm{e}(G) - |[S_{\mathrm{odd}}, \overline{S}_{\mathrm{odd}}]| \geqslant 2\mathrm{e}(G) - \mathrm{Cut}(G). \qquad \square$$

We denote the neighborhood of a vertex v by

$$N(v) := \{u \in V : uv \in E\}.$$

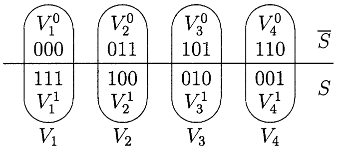

Fig. 1.

Remark 10. If $N(u) = N(v)$, so in particular u and v are not adjacent, then

(1) in a maximum cut u and v can be assumed to be on the same side of the cut, since otherwise the vertex with the fewer crossing edges can be moved to the other,
(2) in an optimal cut-cover u and v can be assumed to be on the same side in every cut, since otherwise the vertex with the bigger total number of crossing edges can be moved to the side of the other vertex in every cut.

Proof of Proposition 3. Since vertices in $V(\bar{K}_m)$ all have the same neighborhood, we can assume that in a maximum cut they are on the same side, say $V(\overline{K}_m) \subset S$. Furthermore, all vertices in $V(G)$ have at least as many neighbors in $V(\overline{K}_m)$ as in $\bar{S}$, since $m \geqslant \Delta$, so that we can move them to $\bar{S}$ without decreasing the size of the cut. Thus, $[V(\overline{K}_m), V(G)]$ is a maximum cut with size nm. It is a stable cut, since $V(\overline{K}_m)$ is a stable set. □

Proof of Theorem 4. To see that the lower bound can be achieved, it suffices to construct a covering such that all the edges in the maximum cut $[S, \overline{S}]$ are covered once and all other edges are covered twice. Denote the color classes by V_i $(1 \leqslant i \leqslant 4)$ and let $V_i \cap S = V_i^1$ and $V_i \cap \overline{S} = V_i^0$. We define the required covering by defining the equivalent labeling as suggested in Fig. 1, that is, we give all vertices in V_i^j the label as indicated. Then $E(G)$ is covered by 3 cuts, determined by $S_1 = V_1^1 \cup V_2^1 \cup V_3^0 \cup V_4^0$, $S_2 = V_1^1 \cup V_2^0 \cup V_3^1 \cup V_4^0$ and $S_3 = V_1^1 \cup V_2^0 \cup V_3^0 \cup V_4^1$, respectively.

Edges within S (or $\overline{S}$) are covered twice, since the labels on the vertices of these edges all have odd (respectively even) weights and must thus differ by 2. Edges between S and $\overline{S}$ are covered once, because the weights of the vertices involved have different parity (and hence are covered 1 or 3 times), but since each V_i is a stable set no edge will be covered three times. □

4. Algorithmic aspects

It is well known that the problem of determining $\mathrm{Cut}(G)$ is NP-complete [12]. This has been sharpened by Yannakakis [26] who showed that determining $\mathrm{Cut}(G)$ is NP-complete for graphs of maximum degree $\Delta \leqslant 3$. Thus, Theorem 4 implies that determining $\mathrm{cs}(G)$ is NP-complete, even for graphs with maximum degree $\Delta(G) \leqslant 3$, because graphs with $\Delta(G) \leqslant 3$ are clearly 4-colorable.

On the other hand,

Theorem 11. *Every planar graph G is of type A and a minimum size cut-cover can be found in polynomial time.*

Proof 1. If we assume the 4-Color Theorem (every loopless planar graph is 4-colorable), then Theorem 4 implies that every planar graph is type A. For planar graphs, Hadlock [15,2] proved that a maximum (weighted) cut can be found in polynomial time. Robertson, Sanders, Seymour and Thomas [24] observe that their proof of the 4-Color Theorem could be turned into an $O(n^2)$ algorithm for finding a proper 4-coloring. □

The following proof and the observation that the problem is connected to T-joins were also provided by Kostochka:

Proof 2. We can also prove Theorem 11 without the help of the 4-Color Theorem, by considering the dual graph. If G is a (loopless) plane graph, then the dual graph G^* has no cut-edge and every cut in G corresponds to a disjoint union of cycles in G^*. Thus, determining $\mathrm{cs}(G)$ is equivalent to determining the length of a shortest cycle cover of G^*, a problem that is in turn (for planar graphs) equivalent to finding a shortest postman tour, that is a shortest closed walk covering $E(G^*)$, denoted by $\ell(G^*)$ (see, for example, [1]). So using a result of Edmonds and Johnson [6], $\mathrm{cs}(G) = \ell(G^*)$ can be computed in polynomial time (see also [3,13]).

Furthermore, note that if H is an edge-disjoint union of cycles, then in $H' = G - E(H)$ the degree of every vertex will have the same parity as in G, i.e. H' is a *parity subgraph* of G. Thus,

$$\begin{aligned} &2e(G) - \mathrm{Cut}(G) \\ &\quad = 2e(G^*) - \max\{e(H)\colon H \text{ is a disjoint union of cycles in } G^*\} \\ &\quad = e(G^*) + \min\{e(H')\colon H' \text{ is a parity subgraph of } G^*\} \\ &\quad = \ell(G^*), \end{aligned}$$

where the last equality is well known, see [29, 8.1.4]. Thus, G is type A. □

The algorithms in [15,6] are based on the idea that it basically suffices to find a smallest parity subgraph in G^*. This can be done by finding a minimum weight perfect matching in an auxiliary graph: The graph is the complete graph whose vertex set consists of the odd degree vertices in G^* with the weight of each edge being the distance between the two vertices in G^*. The fastest algorithm known for this problem requires $O(n^{5/2}(\log n)^{3/2}\alpha(n^2, n)^{1/2})$ steps [10], where α denotes the (very slowly growing) inverse of the Ackerman function. This essentially determines the running time for the algorithms obtained by either approach. One way to obtain a shortest cycle cover in G^* in polynomial time from the parity subgraph is by an algorithm of Fleischner and Frank [9].

5. Cut-covers in random graphs

In this section, we prove Theorems 5 and 6 and thereby determine the cut-cover size of a wide range of random graphs.

Proof of Theorem 5. Note that (7) immediately implies that $n \geqslant 100$, so that this theorem only applies to 'big' graphs. By (7) and (9),

$$\mathrm{Cut}'(G) \leqslant \alpha(G)(n-\alpha(G)) \leqslant \frac{d^2 n}{100}(n-1) < \frac{10}{43} dn^2 \leqslant \mathrm{Cut}(G),$$

so that G cannot be type AB.

Next, we show that

$$\text{If } U \subset V(G) \quad \text{and} \quad |U| \geqslant dn/5+1, \text{ then } |E(U)| \geqslant \frac{5}{6} d \binom{|U|}{2}. \tag{13}$$

Indeed, let $t = \lfloor |U|/2 \rfloor \geqslant dn/10$. Then (13) is implied by (8)

$$2\binom{|U|-2}{t-1} |E(U)| = \sum_{T \subset U, |T|=t} |[T, U-T]| \geqslant \binom{|U|}{t} \frac{5}{6} dt(|U|-t).$$

To see that G is type B let $\{[S_i, \bar{S}_i] : 1 \leqslant i \leqslant k\}$ be an optimal cover. We can assume without loss of generality that $|S_i| \leqslant n/2$ for all $1 \leqslant i \leqslant k$. We let $s = \sum_{i=1}^{k} |S_i|$ and claim that

$$s < 2.15n. \tag{14}$$

Indeed, $dn^2 > 2e(G) > \mathrm{cs}(G) = \sum_{i=1}^{k} |[S_i, \bar{S}_i]| \geqslant \sum_{i=1}^{k} \frac{20}{43} dn|S_i| = \frac{20}{43} dns$.

Furthermore we define, for $0 \leqslant j \leqslant k$,

$$E_j := \{e \in E(G) : e \text{ is covered in exactly } j \text{ cuts}\}, \quad E_{\geqslant j} := \bigcup_{j \leqslant \ell \leqslant k} E_\ell,$$

$$W_j := \{v \in V(G) : v \text{ is in exactly } j \text{ of the } S_i\}, \quad W_{\geqslant j} := \bigcup_{j \leqslant \ell \leqslant k} W_\ell.$$

In the labeling equivalent to the covering, the vertices in W_j are exactly those of weight j. Therefore, an edge between a vertex in W_j and $W_{j'}$ with $j \leqslant j'$ is covered in exactly $j' - j + 2m$ cuts for some $0 \leqslant m \leqslant j$. The edges in $E_1 \cap [S_i, \overline{S}_i]$ form a graph with the property that the neighborhood of any vertex is a stable set in G. Indeed, if the edges uv and uw are only established in $[S_i, \overline{S}_i]$, then v and w must be in the same partite set for every $[S_\ell, \overline{S}_\ell]$ so that vw would not be covered, and thus there can be no such edge in $E(G)$. So we can use (7) to conclude that the maximum degree in this graph is at most $\alpha \leqslant d^2 n/100$. Thus, $|E_1 \cap [S_i, \bar{S}_i]| \leqslant |S_i| d^2 n/100$ and

$$|E_1| = \sum_{i=1}^{k} |E_1 \cap [S_i, \bar{S}_i]| \leqslant \sum_{i=1}^{k} |S_i| \frac{d^2 n}{100} = s\frac{d^2 n}{100} < 0.0215 d^2 n^2. \tag{15}$$

Also $|E_0|=0$, so that $2e(G)>\mathrm{cs}(G)=\sum_{j=1}^{k} j|E_j|$ implies that

$$|E_{\geqslant 3}|<|E_1|<0.0215d^2n^2 \quad \text{and} \quad |E_4|<|E_1|/2<0.011d^2n^2. \tag{16}$$

It is also important that in our labeling, the set of vertices with a fixed label is a stable set, so that for example $|W_0|\leqslant d^2n/100$.

To show that G is type B we give a two-step proof that W_1 is large and in a third step we show that in an optimal cover $W_{\geqslant 2}=\emptyset$. This will finish our proof, since W_0 is a stable set, the edges in $[W_0,W_1]$ are covered once, and all other edges are covered twice.

Step 1: $|W_1|>dn/10$. Note that $n=\sum_{j=0}^{k}|W_j|$ so that

$$\begin{aligned}2.15\,n>s=\sum_{i=1}^{k}|S_i|=\sum_{j=0}^{k} j|W_j|=2n+\sum_{j=0}^{k}(j-2)|W_j|\\ \geqslant 2n-2|W_0|-|W_1|+|W_{\geqslant 3}|.\end{aligned}$$

But then

$$\begin{aligned}|W_2|&=n-|W_0|-|W_1|-|W_{\geqslant 3}|\\ &>n-|W_0|-|W_1|-(0.15n+2|W_0|+|W_1|)\\ &=0.85n-3|W_0|-2|W_1|\geqslant 0.85n-3\frac{d^2n}{100}-2|W_1|,\end{aligned}$$

so that if we assume that $|W_1|\leqslant dn/10$, then $|W_2|>0.62n$.

Next, we define $A_i=S_i\cap W_2$ and $B_i=\bar{S}_i\cap W_2$. Notice that $|B_i|\geqslant|W_2|-|S_i|>0.12n$. The edges in $[A_i,B_i]$ are covered either two times or four times, since they are between vertices in W_2. If $v\in B_1$, then without loss of generality $v\in A_2$, A_3 and $v\in B_i$, for $i>3$, so the label of v is $011000\ldots$. If the edge $uv\in[A_1,B_1]$ is covered twice, then $u\in A_1$, so that the label of u is $110000\ldots$ or $101000\ldots$. However, each of these labels induces an independent set, so that at most $2d^2n/100$ edges at v in $[A_1,B_1]$ are covered twice.

But now we can conclude that $|A_1|\leqslant dn/6$, since otherwise

$$\begin{aligned}|E_4|&\geqslant|[A_1,B_1]|-|B_1|2\frac{d^2n}{100}\geqslant\frac{5}{6}d|A_1||B_1|-|B_1|\frac{d^2n}{50}\\ &>|B_1|\left(\frac{5}{6}d\frac{1}{6}dn-\frac{d^2n}{50}\right)>0.12n\frac{107}{900}d^2n>0.011d^2n^2,\end{aligned}$$

which is a contradiction to (16). Since we can argue in the same fashion for $i>1$, we can now assume that $|A_i|\leqslant dn/6$ for all i.

Again, we observe that if an edge uv in $[A_i,B_i]$ is covered twice, then its vertices u and v have a 1 in the same position somewhere, that is $u,v\in A_\ell$ for some ℓ. Also $\sum_{i=1}^{k}|A_i|=2|W_2|$, since each vertex in W_2 is in exactly $2A_i$, so that the number of edges covered twice within W_2 is at most

$$\sum_{i=1}^{k} \binom{|A_i|}{2} \leqslant \frac{2|W_2|}{dn/6} \binom{dn/6}{2} < \frac{dn}{6}|W_2|.$$

All other edges in W_2 are covered 4 times, so that, using (13) and the fact that $n \geqslant 100$, we get a contradiction to (16) again

$$|E_4| \geqslant |E(W_2)| - \frac{dn}{6}|W_2| \geqslant \frac{5}{6}d\binom{|W_2|}{2} - \frac{dn}{6}|W_2|$$

$$= \frac{d|W_2|}{6}\left(\frac{5}{2}(|W_2|-1) - n\right) > 0.1dn(1.55n - 2.5 - n) > 0.05dn^2.$$

Step 2: $|\bar{W}_1| < 0.1dn$. Suppose to the contrary that $|\overline{W}_1| \geqslant dn/10$. From an argument similar to the argument above we see that only few edges from W_1 to $\bar{W}_1$ can be covered twice: These edges are in $[W_1, W_3]$ and for every vertex $v \in W_3$, say with label 1110000..., there are only three labels possible for a vertex $u \in W_1$ such that the edge uv is covered exactly twice: 1000000..., 0100000... and 0010000..., so that

$$|\bar{E}_2| \geqslant |[W_1, \bar{W}_1]| - 3\frac{d^2n}{100}|W_3| \geqslant \frac{5}{6}d|W_1||\bar{W}_1| - \frac{3d^2n}{100}|\bar{W}_1|$$

$$= (n - |W_1|)d\left(\frac{5}{6}|W_1| - \frac{3dn}{100}\right).$$

As a function in $|W_1|$ this represents a parabola opening downwards, so that it is minimized at the endpoints $|W_1| = dn/10$ or $|W_1| = n - dn/10$. However, both of the values obtained are still greater than $0.043d^2n^2$, which is by (16) an upper bound on $|\bar{E}_2| = |E_1| + |E_{\geqslant 3}|$.

Step 3: $|W_{\geqslant 2}| = 0$. If $|W_{\geqslant 2}| > 0$, then we will be able to obtain a better cover by moving the vertices in $W_{\geqslant 2}$ to W_1. We leave the vertices in $W_{\leqslant 1}$ unchanged, but change the labels of the vertices in $W_{\geqslant 2}$ so as to obtain a cover by stars. That is for every vertex in $W_{\geqslant 2}$ we introduce a new coordinate, and make its label in that coordinate 1, all other coordinates will be zero. Vertices in $W_{\leqslant 1}$ will receive zeros in the new coordinates, so that the number of times the edges in $E(W_{\leqslant 1})$ are covered does not change. All other edges are now covered at most twice.

Before this change, the only edges not in $E(W_{\leqslant 1})$ that were covered at most twice were contained in $E(W_{\geqslant 2})$, $[W_0, W_2]$, $[W_1, W_2]$ or $[W_1, W_3]$ which, using the fact that $|\bar{W}_1| \leqslant 0.1dn$, adds up to at most

$$\binom{|W_{\geqslant 2}|}{2} + |W_0||W_2| + 2\frac{d^2n}{100}|W_2| + 3\frac{d^2n}{100}|W_3| \leqslant |W_{\geqslant 2}|\left(\frac{1}{2}|W_{\geqslant 2}| + \frac{3d^2n}{100}\right)$$

$$\leqslant 0.08dn|W_{\geqslant 2}|.$$

However, from (9) we conclude $|[W_{\geqslant 2}, W_{\leqslant 1}]| \geqslant \frac{20}{43} dn|W_{\geqslant 2}| > 2(0.08dn|W_{\geqslant 2}|)$, so that more of the edges that we changed were covered three or more times, than once or twice. □

Proof of Theorem 6. The proof requires only a few basic results about random graphs, most of which can be found in the classical book of Bollobás [4].

A graph is *unicyclic* if it contains exactly one cycle. Trees are bipartite and thus type AB. Unicyclic graphs are either bipartite or a bipartite graph plus an edge — in either case $\mathrm{Cut} = \mathrm{Cut}'$, so that unicyclic graphs are also type AB. Thus, the second statement of the theorem follows immediately from the following facts:

Theorem 12 (cf. Bollobas [4, V.7(i)]). *If* $p = \mathrm{o}(1/n)$, *then almost every graph in* $\mathscr{G}(n, p)$ *is a forest.*

Corollary 13 (cf. Bollobas [4, V.8]). *Suppose* $p = c/n$ *with* $0 < c < 1$. *Then almost every graph in* $\mathscr{G}(n, p)$ *is such that every component is a tree or a unicyclic graph.*

To prove that graphs with big probability are type $A'B$ it suffices to check (7)–(9). Theorem II.8 of [4] implies that if $252n^{-1}\log n < p \leqslant \frac{1}{2}$, then almost every graph in $\mathscr{G}(n, p)$ has $|\mathrm{e}(G) - p\binom{n}{2}| < \sqrt{7pn^{-1}\log n}\binom{n}{2}$. Thus, almost every graph with edge probability p has $|d-p| < (7p\log n/n)^{1/2}$ as long as $1-(252\log n)/n > p > (252\log n)/n$. Therefore, since $p \geqslant 6(\log n/n)^{1/3}$,

$$\left|\frac{d}{p} - 1\right| = \frac{|d-p|}{p} < \sqrt{\frac{7\log n}{np}} \leqslant \sqrt{7/6}\left(\frac{\log n}{n}\right)^{1/3} = \mathrm{o}(1).$$

To establish (7) we will apply

Theorem 14 (cf. Bollobas [4, XI.22(ii)]). *If* $2.27/n < p \leqslant \frac{1}{2}$, *then almost every graph in* $\mathscr{G}(n, p)$ *has independence number at most* $2\log pn/p$.

For $p \leqslant \frac{1}{2}$ it suffices to show that $2(\log pn)/p \leqslant p^2 n(1-\mathrm{o}(1))/100 (\leqslant d^2 n/100)$, or equivalently $200\log pn \leqslant p^3 n(1-\mathrm{o}(1))$. Because $p \geqslant 6(\log n/n)^{1/3}$ the right-hand side eventually exceeds $200\log n$, and thus the left-hand side. For $p > \frac{1}{2}$ we simply use the values for $p = \frac{1}{2}$, since then $d \geqslant \frac{1}{2} - \mathrm{o}(1)$, so that $d^2 n/100 > n/500$ for almost every graph. Furthermore, for almost every graph with $p \geqslant \frac{1}{2}$ we get $\alpha(G) \leqslant 4\log(n/2)$, which grows slower.

To prove (8) we need

Theorem 15 (cf. Bollobas [4, II.11]). *Let* $0 < p \leqslant \frac{1}{2}$. *Then almost every graph in* $\mathscr{G}(n, p)$ *is such that*

$$||[S_1, S_2]| - p|S_1||S_2|| < (7p|S_1|^{-1}\log n)^{1/2}|S_1||S_2|,$$

whenever S_1, S_2 *are disjoint vertex-sets satisfying* $(252/p)\log n < |S_1| \leqslant |S_2|$.

Note that for n sufficiently large, $d \geqslant \frac{5}{6} p \geqslant 5(\log n/n)^{1/3}$. Therefore,

$$dn/10 \geqslant n^{2/3} > (252/6)(n \log^2 n)^{1/3} \geqslant (252 \log n)/p$$

for almost every graph. So for $p \leqslant \frac{1}{2}$ we can apply Theorem 15 to prove (8):

$$\begin{aligned} |[S_1, S_2]| &> p|S_1||S_2| - (7p \log N/N^{2/3})^{1/2}|S_1||S_2| \\ &\geqslant p|S_1||S_2|(1 - o(1)) \geqslant d|S_1||S_2|(1 - o(1)). \end{aligned}$$

For $1/2 < p \leqslant 1 - 6(\log n/n)^{1/3}$ we can argue similarly with $\bar{G}$ and for other $p \geqslant 1 - o(1)$ we can interpolate again.

If $|S| \geqslant n^{2/3}$, then the statement that we just proved implies (9). Erdős and Rényi [8] proved that if $\varepsilon > 0$ and $\log n/n = o(p)$ then for almost every graph in $\mathscr{G}(n, p)$, $\delta > (1 - \varepsilon)pn$. Hence (9) also follows for $|S| < n^{2/3}$:

$$\begin{aligned} |[S, \bar{S}]| &\geqslant \delta(G)|S| - 2\binom{|S|}{2} \\ &> pn(1 - \varepsilon)|S| - |S|n^{2/3} = pn|S|(1 - \varepsilon - o(1)). \qquad \square \end{aligned}$$

6. Complementary graphs

In this section, we determine the maximum and minimum value $\mathrm{cs}(G) + \mathrm{cs}(\overline{G})$ can take when G is an n-vertex graph. Results of this kind are frequently referred to as 'Nordhaus–Gaddum-type results'.

Proof of Theorem 7. The upper bound is immediate, since every cut-cover for K_n yields a simultaneous cover for G and $\overline{G}$, and this is sharp for $G = K_n$. For the lower bound observe that, for some maximum cuts $[S_1, \overline{S}_1]$ in G and $[S_2, \overline{S}_2]$ in $\bar{G}$

$$\begin{aligned} \mathrm{cs}(G) + \mathrm{cs}(\overline{G}) &\geqslant (2\mathrm{e}(G) - |[S_1, \overline{S}_1]|) + (2\mathrm{e}(\overline{G}) - |[S_2, \overline{S}_2]|) \\ &= 2\mathrm{e}(K_n) - (|[S_1, \overline{S}_1]| + |[S_2, \overline{S}_2]|) \geqslant 2\binom{n}{2} - t(n, 4), \end{aligned}$$

since the graph with edge-set $[S_1, \overline{S}_1] \cup [S_2, \overline{S}_2]$ is 4-colorable. To show that the lower bound in Theorem 7 is optimal we construct the following example. Take blow-ups of the self-complementary path on 4 vertices as follows: Partition the n vertices into 4 parts V_i, $1 \leqslant i \leqslant 4$, that are in size as equal as possible. Now let G be the subgraph of K_n formed by taking all edges in $[V_1, V_2]$, $[V_2, V_3]$, $[V_3, V_4]$ and all edges induced by V_1 and V_4 as indicated in Fig. 2.

For G we can take the cover that consists of stars centered at the vertices in $V_1 \cup V_4$ and the single cut $[V_1 \cup V_2, V_3 \cup V_4]$. Every edge in $[V_1 \cup V_3, V_2 \cup V_4]$ is covered once

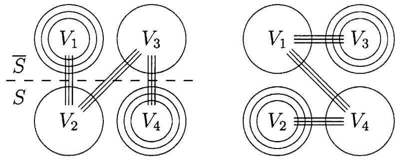

Fig. 2. G and $\bar{G}$.

and every other edge twice. Since $\overline{G}$ has the same structure as G, we obtain

$$\begin{aligned}
&\mathrm{cs}(G) + \mathrm{cs}(\overline{G}) \\
&\quad \leqslant (2\mathrm{e}(G) - |[V_1 \cup V_3, V_2 \cup V_4]|) + (2\mathrm{e}(\overline{G}) - |[V_1 \cup V_2, V_3 \cup V_4]|) \\
&\quad = 2\binom{n}{2} - t(n,4). \qquad \square
\end{aligned}$$

Remark 16. While the lower bound is achieved for a number of graphs and the upper bound gives the exact answer only for $G = K_n$, the latter is optimal in the following sense: for almost every graph in $\mathscr{G}(n, p)$, with fixed probability p and $q = 1 - p$, it can be seen from Theorem 6 and the facts that $\alpha(G) = (2\log n/\log(1/p))(1 + \mathrm{o}(1))$ and that G is almost pn-regular, that

$$\mathrm{cs}(G) + \mathrm{cs}(\bar{G}) = n^2 - \left(\frac{2p}{\log(1/q)} + \frac{2q}{\log(1/p)}\right) n \log n(1 + \mathrm{o}(1)).$$

7. More bounds and exact values

Proof of Proposition 8. For $k = 1$, $G = \overline{K}_n$ and all statements are true, so that we can assume $k > 1$. Let G be a complete k-partite graph with part sizes $n_1 \leqslant n_2 \leqslant \cdots \leqslant n_k$. By Remark 10.2, we can assume that the cuts in an optimal cut-cover will not cut through any of the partite sets. Thus for every cut $[S_i, \overline{S}_i]$ in the cover we get that

$$|S_i| = n_{i_1} + n_{i_2} + \cdots + n_{i_j} \quad |[S_i, \overline{S}_i]| = |S_i|(n - |S_i|),$$

so that the size of the cover can be viewed as a function in the k variables n_i:

$$|\mathscr{C}| = \sum |[S_i, \overline{S}_i]| = f(n_1, n_2, \ldots, n_k).$$

If we keep all but two of the coordinates $i < j$ fixed, then by combining like terms we obtain

$$f(n_1, n_2, \ldots, n_k) = a n_i n_j + b n_i + c n_j + d,$$

for some $a, b, c, d \geqslant 0$. Furthermore, $a > 0$ (since the edges between the two parts need to be covered at least once) and $b \geqslant c$ (since otherwise we could swap the roles of n_i and n_j in the cover and not increase its total size). When $n_i > 1$, we can decrease n_i

by one and increase n_j by one, thus defining another complete k-partite graph G' on n vertices. The same cover as the one defined by f now shows that

$$\begin{aligned}\mathrm{cs}(G') &\leqslant f(n_1,n_2,\ldots,n_i-1,\ldots,n_j+1,\ldots,n_k)\\ &= a(n_i-1)(n_j+1)+b(n_i-1)+c(n_j+1)+d\\ &= [an_in_j+bn_i+cn_j+d]+a(n_i-n_j-1)+(c-b)<\mathrm{cs}(G).\end{aligned}$$

With this observation, (10) follows and we observe that the inequalities are strict, except when $G=T(n,k)$ or $G=K_{k-1}\vee\overline{K}_{n-k+1}$.

To compute the exact values, note that for $k=n$ the values were given in (1) so we can assume that $n>k>1$. Clearly, for $G=K_{k-1}\vee\overline{K}_{n-k+1}$ we get that $\mathrm{Cut}'(G)=\max\{(k-1)(n-k+1),n-1\}=(k-1)(n-k+1)$, so that $\mathrm{cs}(G)\leqslant(k-1)(n-1)$. Furthermore, K_k is a subgraph of G so that for $k\neq 4,8$,

$$\mathrm{cs}(G)\geqslant\mathrm{cs}(K_k)+\mathrm{e}(G)-\mathrm{e}(K_k)=(k-1)(n-1). \tag{17}$$

For $k=4$ we get that $K_3\vee\overline{K}_{n-3}$ is type B from Proposition 3, since $n-3\geqslant 2=\Delta(K_3)$. For $k=8$ the same proof as in (17) yields a lower bound that is only off by one:

$$\mathrm{cs}(K_7\vee\overline{K}_{n-7})\geqslant\mathrm{cs}(K_8)+\mathrm{e}(K_7\vee\overline{K}_{n-7})-\mathrm{e}(K_8)=7(n-1)-1. \tag{18}$$

If this inequality were sharp, then (in a given optimal cover) every induced K_8 must be covered optimally, that is by 3 $K_{4,4}$'s. In this covering of K_8 the edges involving a fixed vertex are covered a total of 12 times, so that it follows that

$$\mathrm{cs}(K_7\vee\overline{K}_{n-7})=\mathrm{cs}(K_8)+12(n-8)=12n-48. \tag{19}$$

Combining (18) and (19) we get $12n-48=7n-8$, or $n=8$.

The upper bound for (12) is given by (6), since $\mathrm{Cut}'(T(n,k))=\lceil n/k\rceil(n-\lceil n/k\rceil)$. For $k\leqslant 4$, $T(n,k)$ is 4-colorable, so that $\mathrm{cs}(T(n,k))$ can be easily computed from Theorem 4. If $k>200$, then Theorem 5 applies. For the case that $k>8$, observe again that we can assume that the cuts in an optimal cut-cover will not cut through any of the partite sets — thus we essentially have a cut-cover of a weighted K_k where edge ij has weight n_in_j. The cover achieving (12) corresponds to the cover of K_k with stars, which is the only cover achieving $\mathrm{cs}(K_k)$. Thus, every other cover of K_k has total size at least $(k-1)^2+1$, so to show (12) it suffices to check that

$$2t(n,k)-\left\lceil\frac{n}{k}\right\rceil\left(n-\left\lceil\frac{n}{k}\right\rceil\right)\leqslant((k-1)^2+1)\lfloor n/k\rfloor^2. \tag{20}$$

If k divides n, or $n>2(k-1)^3$ then this is indeed the case. More detailed computations are possible to provide more cases of equality. □

Observe that it suffices to compute all relevant parameters on the blocks of G, since if $G=G_1\cup G_2$ with $|V(G_1)\cap V(G_2)|\leqslant 1$, then $\mathrm{cs}(G)=\mathrm{cs}(G_1)+\mathrm{cs}(G_2)$ and the same holds for $\mathrm{e}(G)$, $\mathrm{Cut}(G)$ and $\mathrm{Cut}'(G)$. Furthermore, we will define $G_1\leftrightarrow G_2$ to be any graph obtained by identifying G_1 and G_2 at any one of their vertices. By the previous remark $G_1\leftrightarrow G_2$ is type A (B) exactly when G_1 and G_2 are both of type A (B).

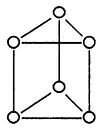

Fig. 3. $K_3 \Box K_2$.

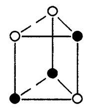

Fig. 4. $\mathrm{Cut}(K_3 \Box K_2) = 7$.

To be able to construct many graphs whose cut-cover size is easily computed we define the *box product* $G \Box H$ of the graphs G and H to be the graph with vertex set

$$V(G \Box H) = V(G) \times V(H) = \{(v, w): v \in V(G), w \in V(H)\},$$

and (v_1, w_1) adjacent to (v_2, w_2) if $v_1 = v_2$ and w_1 adjacent to w_2, or if $w_1 = w_2$ and v_1 adjacent to v_2 (see Fig. 3).

The following facts summarize all relevant properties of the box product:

(1) $n(G \Box H) = n(G)n(H)$,
(2) $\mathrm{e}(G \Box H) = \mathrm{e}(G)n(H) + n(G)\mathrm{e}(H)$,
(3) $\chi(G \Box H) = \max\{\chi(G), \chi(H)\}$,
(4) $\mathrm{Cut}(G \Box H) = \mathrm{Cut}(G)n(H) + n(G)\mathrm{Cut}(H)$,
(5) $\mathrm{Cut}'(G \Box H) \leqslant \mathrm{Cut}'(G)n(H) + n(G)\mathrm{Cut}'(H)$,
(6) $\mathrm{cs}(G \Box H) = \mathrm{cs}(G)n(H) + n(G)\mathrm{cs}(H)$,
(7) $G \Box H$ is type A exactly when G and H are type A,
(8) If $G \Box H$ is type B then G and H are type B.

For example, to prove (6) we observe that $G \Box H$ contains $n(H)$ copies of G and each one of them contributes at least $\mathrm{cs}(G)$ towards the total sum in an optimal cut-cover. Combining this with a similar argument for the copies of H, we get that the left-hand side is no smaller than the right-hand side. A cover achieving this can be obtained from optimal covers of G and H, by putting all copies of a given vertex in $V(G)$ in the same partition as in the original cut of G (similarly for the cuts from H). The other proofs are similar and will be omitted.

Note that K_2 and K_3 are both type AB, but $K_3 \Box K_2$ is type AB', since the cut illustrated in Fig. 4 is, up to isomorphism, the only cut of size 7.

To see that Theorem 4 is indeed best possible we define, for a given type T,

$$\chi(T) = \min\{\chi(G): G \text{ is type } T\}.$$

Hence $\chi(AB) = 1$, $\chi(AB') = 3$ (achieved by $K_3 \Box K_2$), $\chi(A'B) = 5$ (achieved by K_5) and $\chi(A'B') = 5$ (achieved by $K_5 \Box K_2$). We will show that for all types T there are infinitely many graphs G with $\chi(G) = k \geqslant \chi(T)$ in a strong sense:

Proposition 17. *For every type T and graph G with $\chi(G) \geqslant \chi(T) - 1$, there exists a graph G' of type T containing G as an induced subgraph so that $\chi(G') = \chi(G) + 1$ and $n(G') \leqslant n(G) + \Delta(G) + \chi(T)$.*

Proof. For $T=AB$, we can choose $G'=G\vee\overline{K}_{\Delta+1}$. For $T=A'B$ we let $G'=(G\vee\overline{K}_{\Delta})\leftrightarrow K_5$. If $T=AB'$ and $\chi(G)\geqslant 3$, then we let $G'=(G\vee\overline{K}_{\Delta})\leftrightarrow K_4$. If $T=AB'$ and $\chi(G)=2$ then we let $G'=G\leftrightarrow(K_3\Box K_2)$, which works if $\Delta(G)\geqslant 2$ — otherwise, $\Delta(G)=1$ and G is a matching plus isolated vertices so that we can extend one of the matching edges to a $K_3\Box K_2$. Finally, in the case $T=A'B'$ we note that $\Delta(G)\geqslant\chi(G)$ unless G contains a $K_{\Delta+1}$. When $\chi(G)=4$ we can let $G'=G\leftrightarrow(K_5\Box K_2)$, except if $\Delta(G)\leqslant 3$. In this case, G must have a component that is a K_4 and we just extend this component to a $K_5\Box K_2$. For $\chi(G)>4$ we can let $G'=(G\vee K_1)\leftrightarrow(K_5\Box K_2)$, except if $\Delta(G)=4$. In this case G must have a component that is a K_5 and we just extend this component by a vertex v to a K_6 and let $G'=(G+v)\leftrightarrow(K_3\Box K_2)$. □

8. Cut-covers and L_∞-representations

For $x\in\mathbb{R}^d$ we define $||x||_\infty=\max\{|x_i|: 1\leqslant i\leqslant d\}$ and $||x||_1=\sum_{1\leqslant i\leqslant d}|x_i|$. An L_∞-representation (in $\mathbb{R}^d$) of a graph G is an assignment

$$f: V(G)\to\mathbb{R}^d, \tag{21}$$

such that $||f(u)-f(v)||_\infty\geqslant 1$ whenever $uv\in E(G)$.

For a given L_∞-representation (G,f) we define

$$L_1(G,f):=\sum_{uv\in E(G)}||f(u)-f(v)||_1. \tag{22}$$

So the average L_1-distance between adjacent vertices in the L_∞-representation is $L_1(G,f)/\mathrm{e}(G)$.

Theorem 18.

$$\mathrm{cs}(G)=\inf\{L_1(G,f): f \textit{ is an } L_\infty\textit{-representation of } G\}$$

Proof. The $\{0,1\}$-labeling f associated with a given cut-cover is an L_∞-representation of G and $L_1(G,f)$ counts the total size of the cut-cover. Thus, $\mathrm{cs}(G)\geqslant\inf\{L_1(G,f)$: f is an L_∞-representation of $G\}$.

For the reverse inequality it suffices to show that for every L_∞-representation f we can find a $\{0,1\}$-representation f^*, maybe higher dimensional, such that $L_1(G,f^*)\leqslant L_1(G,f)$.

We denote the value of the ith coordinate in $f(v)$ by $f(v)_i$. Among L_∞-representations let f' be one with a maximum number of integer coordinates, subject to $L_1(G,f')\leqslant L_1(G,f)$. We claim that all $f'(v)_i\in\mathbb{Z}$. Indeed, let $U_c(i)=\{v\in V: f'(v)_i=\lfloor f'(v)_i\rfloor+c\}$ and suppose that $U_c(i)\neq\emptyset$ for some $1\leqslant i\leqslant d$ and $c>0$. We can partition $V=\bigcup U_{c_j}(i)$ with $0\leqslant c_1<c_2\cdots<c_k<1$, and define $x^+:=1-c_k$ and $x^-:=c_{k-1}-c_k$ where we set $c_0=0$. Now f^x defined by

$$f^x(u)_j=\begin{cases} f'(u)_j+x, & j=i, u\in U_{c_k}(i),\\ f'(u)_j, & \text{otherwise,}\end{cases}$$

is an L_∞-representation as long as $x^- \leqslant x \leqslant x^+$. Furthermore, if we let

$$e_1 = |\{uv \in E(G)\colon u \in U_{c_k}, v \notin U_{c_k}, f(u)_i > f(v)_i\}|,$$

$$e_2 = |\{uv \in E(G)\colon u \in U_{c_k}, v \notin U_{c_k}, f(u)_i < f(v)_i\}|,$$

then $L_1(G, f^x) = L_1(G, f') + e_1x - e_2x$. Hence, $L_1(G, f^x) \leqslant L_1(G, f')$ for either $x = x^+$ or $x = x^-$. But if $x = x^+$ or $k = 1$ then we increased the number of integer coordinates. However if $x = x^-$ and $k > 1$ then we decreased the number of sets in our partition, and by iterating this process we must eventually be in the previous case.

Now, we are in the position to obtain f^*. We can assume that, for all $1 \leqslant i \leqslant d$, $\min\{f'(v)_i\colon v \in V\} = 0$ since we can shift the coordinates appropriately. Set $k = \max\{f'(v)_i\colon v \in V, 1 \leqslant i \leqslant d\}$ and for $0 \leqslant j \leqslant k$ let $s(j)$ be the k-dimensional vector whose first j coordinates are 1, all other coordinates 0. Define

$$f^*\colon V \to \{0,1\}^{kd}, v \mapsto (s(f'(v)_1), s(f'(v)_2), \ldots, s(f'(v)_d)).$$

If $||f^*(v) - f^*(u)||_\infty < 1$, then $f^*(v) = f^*(u)$, so that already $f'(u) = f'(v)$ and thus f^* is an L_∞-representation. Finally, $||f^*(v) - f^*(u)||_1 = \sum |f'(v)_i - f'(u)_i| = ||f'(v) - f'(u)||_1$, so that $L_1(G, f^*) = L_1(G, f') \leqslant L_1(G, f)$. □

Remark 19. We can define L_q-representations for G and $L_p(G, f)$ for a given graph G by replacing the L_∞-norm in (21) and L_1-norm in (22) with the L_q and L_p-norms, respectively. So we can define

$$L_{p,q}(G) := \inf\{L_p(G, f)\colon f \text{ is an } L_q\text{-representation of } G\}$$

for all parameters $1 \leqslant p, q \leqslant \infty$. For $p < p'$ we have $L_{p,q}(G) \geqslant L_{p',q}(G)$ and $L_{q,p}(G) \leqslant L_{q,p'}(G)$ for all q, since $||x||_p \geqslant ||x||_{p'}$ for all $x \in \mathbb{R}^d$. Furthermore, $L_{p,q}(G) = 0$ when $p > q$ and $L_{p,p}(G) = e(G)$. In the case $p < q$ it is not obvious what values $L_{p,q}(G)$ takes for non-bipartite graphs and this might be related to other graph parameters.

Remark 20. The bandwidth-sum is defined by

$$BS(G) := \min \left\{ \sum_{uv \in E(G)} |f(u) - f(v)|\colon \text{ bijections } f : V(G) \to [n] \right\}.$$

It is an immediate consequence of Theorem 18 that $cs(G) \leqslant BS(G)$, since the labelings for $BS(G)$ are L_∞-representations in $\mathbb{R}^1$. However, $BS(G)$ is typically much larger than $cs(G)$. Recent results on the bandwidth-sum problem can be found in [19,25,27,28].

9. Open questions

Since the investigation of $cs(G)$ has just started, there are still many basic questions that need to be answered — a few of which we will mention.

Problem 21. *Can the bounds of Theorem* 1 *be improved*? *Find improvements for special classes of graphs*, *like triangle-free graphs.*

Problem 22. *For any given probability function* $p(n)$: *Is almost every graph type* B?

Problem 23. *Is there a threshold* $f(n)$ *such that almost every graph is type* A *if* $p(n) < f(n)$ *and type* A' *if* $p(n) > f(n)$? *Determine* $f(n)$.

Problem 24. *Determine the value of* $\mathrm{cs}(T(n,k))$.

Problem 25. *Is there a constant* $c \in \mathbb{N}$ *such that for every graph there exists an optimal covering so that no edge is covered more than* c *times*?

For Problem 25, $c \geqslant 3$ since every optimal cover of K_8 contains an edge that is covered 3 times. We are not aware of any graph for $c = 4$ — obviously, such a graph would have to be type $A'B'$. In this direction we have

Proposition 26. *For every* $e \in E(G)$

$$1 \leqslant \mathrm{cs}(G) - \mathrm{cs}(G - e) \leqslant n - 1.$$

Proof. $\mathrm{cs}(G - e) \leqslant \mathrm{cs}(G) - 1$ since every cover for G also covers $G - e$, and this is best possible since, for example, cut-edges are covered only once in every optimal covering.

To show $\mathrm{cs}(G) \leqslant \mathrm{cs}(G - e) + n - 1$ we note that every optimal cover has at most $n - 1$ cuts. Indeed, in every cut there is at least one edge covered only in this cut, and taking one such edge from every cut must result in an acyclic graph (since no cycle crosses a cut just once) — but acyclic graphs have at most $n - 1$ edges. Thus, we can take any optimal cover for $\mathrm{cs}(G - e)$ and use it as a cover for G — if it does cover e it can do so at most $n - 1$ times, and if it does not than we add one more cut: a star centered at an endpoint of e. Note that this upper bound is also best possible, since for $G = K_n$ we have $G - e = K_{n-2} \vee \overline{K}_2$, so that for $n \geqslant 9$

$$\mathrm{cs}(G - e) = (n - 2)(n - 1) = (n - 1)^2 - (n - 1) = \mathrm{cs}(G) - (n - 1). \qquad \square$$

Remark 27. This does not settle Problem 25, since the sharp drop in $\mathrm{cs}(G)$ can result from having a different cover (i.e. one fewer star). Probably Proposition 26 can be sharpened:

Problem 28. *Is* $\mathrm{cs}(G) \leqslant \mathrm{cs}(G - e) + \Delta(G)$?

Problem 29. *Is* $\mathrm{cs}(G) \leqslant \mathrm{cs}(G - uv) + \min\{d(u), d(v)\}$?

Both questions can be answered in the affirmative if $G - e$ is type B.

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