Graphs and Combinatorics (2001) 17:73–78

Graphs and Combinatorics

Maximal τ-Critical Linear Hypergraphs*

Zoltán Füredi

Department of Mathematics, University of Illinois, 1409 W. Green St., Urbana, IL 61801, USA. e-mail: z-furedi@math.uiuc.edu

Abstract. A construction using finite affine geometries is given to show that the maximum number of edges in a τ-critical linear hypergraph is $(1 - o(1))\tau^2$. This asymptotically answers a question of Roudneff [14], Aharoni and Ziv [1].

Key words. Linear hypergraphs, Finite affine geometries, τ-critical hypergraphs

1. Linear Hypergraphs

A *linear hypergraph* $\mathscr{H}$ is an ordered pair $\mathscr{H} = (V, \mathscr{E})$, where $V = V(\mathscr{H})$ is a finite set of vertices and $\mathscr{E} = \mathscr{E}(\mathscr{H})$ is a collection of subsets of V, called *edges*, such that any two edges have at most one common vertex. The *size* of the hypergraph means the number of its edges. A set T is a *cover* (or *transversal*) of $\mathscr{H}$ if it intersects every edge. The minimum cardinality of a cover is the *covering number*, it is denoted by $\tau(\mathscr{H})$.

Let us denote the Desarguesian finite projective plane of order q by $\mathscr{P} = \mathscr{P}_q$. It is obtained from a finite field of size q (cf. [13]). It is a linear hypergraph of vertex set of size $q^2 + q + 1$ (called points) and the same number of hyperedges (lines). It is also intersecting (i.e., $L \cap L' \neq \emptyset$ for every two lines $L, L' \in \mathscr{E}(\mathscr{P})$) thus every line is a cover. It is well-known (and easy) that $\tau(\mathscr{P}_q) = q + 1$.

Call a cover $B \subset V(\mathscr{P}_q)$ *non-trivial or a blocking set* if it contains no line, i.e., $L \cap B \neq \emptyset$ and $L \not\subset B$ hold for every $L \in \mathscr{E}(\mathscr{P})$. We are going to use the following result of Blokhuis [4]. The size of a blocking set B of a Desarguesian $\mathscr{P}_q$ with q an odd prime is at least

$$|B| \geq \frac{1}{2}(3q + 3). \tag{1}$$

---

* AMS classification number: 05D05, 05D15, 51E21

This research was supported in part by the Hungarian National Science Foundation grant OTKA 016389, and by a National Security Agency grant MDA904-98-I-0022

The Desarguesian affine plane, $\mathscr{A}_q$, of order q is again a linear hypergraph on q^2 vertices and with q^2+q q-element hyperedges (for details see [13]). Its edge-set can be partitioned into $q+1$ *parallel classes*, $\mathscr{E}(\mathscr{A}_q) = \mathscr{L}_1 \cup \cdots \cup L_{q+1}$ such that $\mathscr{L}_i = \{L_i^1, L_i^2, \dots, L_i^q\}$ and these form a q-partition of the underlying set $V(\mathscr{A}_q)$ into q-element parts. One has $L_i^\alpha \cap L_j^\beta \neq \emptyset$ for $1 \leq i < j \leq q+1$ and for arbitrary $1 \leq \alpha,\ \beta \leq q$. Take a line L_i^α and choose an element from each parallel line $x^\beta \in L_i^\beta$ $(\beta \neq \alpha)$, then the obtained $(2q-1)$-element set forms a cover of the affine plane. Jamison [12] and (independently) Brouwer and Schrijver [7] proved that for the Desarguesian affine plane $\mathscr{A}_q$ with a prime order q these are among the smallest covers, i.e.,

$$\tau(\mathscr{A}_q) = 2q - 1. \tag{2}$$

The affine plane can be obtained from the projective plane $\mathscr{P}_q$ by deleting the vertices of a line L_0 from its vertex set, and by restricting the remaining q^2+q lines into $V(\mathscr{P}_q)\backslash L_0$, i.e., deleting one vertex (the one in $L \cap L_0$) from each of the remaining lines $L \in \mathscr{E}(\mathscr{P}_q)\backslash\{L_0\}$. In this case L_0 is called the *line of infinity* of $\mathscr{A}_q$.

Proofs and further results about covers and especially about blocking sets in projective geometries can be found in the excellent surveys by Blokhuis [5] and Szőnyi [16].

2. τ-Critical Hypergraphs

A hypergraph is τ-*critical* if omitting one edge always reduces its covering number. Erdős, Hajnal and Moon [9] proved that any τ-critical *graph* has at most $\binom{\tau+1}{2}$ edges, with equality holding only for the complete graph on $\tau+1$ vertices. As a generalization Roudneff [14] as well as Aharoni and Ziv [1] conjectured the same upper estimate for every τ-critical linear hypergraph. Let $\ell(\tau)$ denote the maximum size of a τ-critical linear hypergraph. It is easy to see that the maximum degree of such a hypergraph is at most τ and there are sets of size $\tau-1$ that cover all but one of the edges, we obtain $\ell(\tau) \leq \tau^2 - \tau + 1$. Sudakov [15] proved the conjecture for $\tau \leq 5$ and obtained an upper bound $\ell(\tau) \leq \tau^2 - 3\tau + 5$ for $\tau \geq 5$.

The aim of this note is give an example showing that τ-critical linear hypergraphs are much closer to finite geometries than to graphs. The construction given in the next Section yields

Theorem 1. $\ell(\tau) \geq \tau^2 - O(\tau^{8/5})$.

For example, (starting with the affine geometry of order 7) we have $\ell(10) \geq 56 > \binom{11}{2}$.

3. Construction

Using affine geometries we are going to define a linear, τ-critical hypergraph $\mathscr{H}$. Our construction is inspired by an example of Blokhuis [3] (given for a different problem) which was inspired by a result of Drake [8], etc.

Let q be an odd prime, $q \geq 7$ and let r be the largest integer with $\binom{r}{2} < q$, and let $m := \binom{r}{2}$. We have $r := \left\lfloor \frac{1}{2}\left(1 + \sqrt{8q+1}\right) \right\rfloor = \sqrt{2q} + O(1)$, and $4 \leq r < m < q$. Let $\mathscr{A}_q$ be the Desarguasian affine geometry of order q with the q^2-element vertex set V_0 and with parallel classes $\mathscr{L}_1, \ldots, \mathscr{L}_{q+1}$, $\mathscr{L}_i = \{L_i^1, \ldots, L_i^q\}$. Consider $q+1$ complete graphs of order r with pairwise disjoint vertex sets $V_1, \ldots, V_{q+1}$ also disjoint to V_0. Thus $V := \bigcup_{0 \leq i \leq q+1} V_i$ has $q^2 + r(q+1)$ elements. Label the edges of the i'th complete graph from 1 to m, we obtain $\{E_i^1, E_i^2, \ldots, E_i^m\}$. Finally, define the edge-set of $\mathscr{H}$ as

$$\mathscr{E}(\mathscr{H}) := \{L_i^j \cup E_i^j \text{ for } 1 \leq i \leq q+1 \text{ and } 1 \leq j \leq m\} \\ \cup \{L_i^j \text{ for } 1 \leq i \leq q+1 \text{ and } m < j \leq q\}. \tag{3}$$

We claim that $\mathscr{H}$ is a linear, τ-critical hypergraph and the size of the minimum cover is t where this defined as

$$t := q + (r-2) + (q-m) = q + O(\sqrt{q}). \tag{4}$$

This immediately implies for this value of t that

$$\ell(t) \geq |\mathscr{E}(\mathscr{H})| = q^2 + q = t^2 - O(t^{3/2}). \tag{5}$$

To see the linearity of $\mathscr{H}$ observe that its restriction to V_0 is the affine plane, hence $|E \cap E' \cap V_0| \leq 1$ holds for every pair of edges $E, E' \in \mathscr{E}(\mathscr{H})$. Similarly, $|E \cap E' \cap (V_1 \cup \cdots \cup V_{q+1})| \leq 1$. Thus, linearity of $\mathscr{H}$ follows from the fact that if two of its edges E, E' meet in V_0, $E \cup E' \cup V_0 \neq \emptyset$, then $E \cap V_0$ and $E' \cap V_0$ belong to distinct parallel classes hence E and E' are disjoint in $V_1 \cup \cdots \cup V_{q+1}$. Similarly, if E and E' meet in $V_1 \cup \cdots \cup V_{q+1}$ then they are disjoint in V_0.

It is obvious that $\tau(\mathscr{H}) \leq t$, just consider the cover $T := L_1^1 \cup (V_1 \backslash E_1^1) \cup \{x_1^j : m < j \leq q\}$ where x_1^j is an arbitrary point of L_1^j.

Next we show that $\tau(\mathscr{H}) \geq t$. Consider a cover T and suppose that $|T| \leq t$ thus $|T| < \frac{3}{2}(q+1)$. We are going to show that $|T| \geq t$. The restriction of $\mathscr{H}$ to V_0 is just the affine plane, $\mathscr{H}|V_0 \cong \mathscr{A}_q$, thus it is not possible that $T \cap (V_1 \cap \cdots \cup V_{q+1}) = \emptyset$. Indeed, otherwise T is a cover for $\mathscr{A}_q$, too, and then (2) implies that $|T| \geq 2q - 1$, a contradiction. On the other hand, there must be a V_i with $1 \leq i \leq q+1$ such that $V_i \cap T = \emptyset$. Indeed, $|T| < 2q$ implies that $\min_{1 \leq i \leq q+1} |V_i \cap T| \leq 1$. If it is exactly 1, say $|V_1 \cap T| = 1$ and $|V_i \cap T| \geq 1$ for $i > 1$, then $|(V_1 \cup \cdots$

$\cup V_{q+1}) \cap T| \geq q+1$ hence $|V_0 \cap T| \leq t-(q+1) < m-(r-1)$. Thus in this case $V_0 \cap T$ could not cover the (at least $m-(r-1)$) edges of $\mathscr{H}$ obtained from $\mathscr{L}_1$ and uncovered by $V_1 \cap T$.

Form a set T^* by replacing each subset $V_i \cap T$ by an element y_i for $1 \leq i \leq q+1$ whenever this set is non-empty, $V_i \cap T \neq \emptyset$. Then T^* is a cover of size at most $|T|$ of the projective plane corresponding to $\mathscr{A}_q$. Blokhuis' result (1) implies that T^* contains a line L of the projective plane. Our argument in the previous paragraph gives that L is not the line of infinity, thus it is of the form $L_i^j \cup \{y_i\}$. We may suppose that $i=1$, i.e., $L_1^j \subset T$ for some $1 \leq j \leq q$. As L_1^j meets all the lines L_i^k of the affine plane for $i>1$, we may also suppose that $(V_2 \cup \cdots V_{q+1}) \cap T = \emptyset$.

In the case of $1 \leq j \leq m$ one needs at least $(q-m)$ vertices (inside $\bigcup_{k>m} L_1^k$) to cover the hyperedges $\{L_1^k : k>m\}$ and at least $r-2$ more (inside $\bigcup_{1\leq k\leq m} L_1^k \cup V_1$) to cover the hyperedges of the form $L_1^k \cup E_1^k$ for $1 \leq k \leq m$, $k \neq j$. In the case of $j>m$ one needs another $(q-m-1)$ vertices of T inside $\bigcup_{k>m} L_1^k$ and at least $r-1$ more inside $\bigcup_{1\leq k\leq m} L_1^k \cup V_1$. In both cases we obtain $|T| \geq t$.

Finally, we show that $\mathscr{H}$ is critical, i.e., removing any edge E one has $\tau(\mathscr{H}\setminus\{E\}) \leq t-1$. Indeed, removing $L_i^j \cup E_i^j$ (for $1 \leq j \leq m$) one can define the set T_i^j as follows. $T_i^j := (V_i \setminus E_i^j) \cup L_i^{m+1} \cup \{x_i^k : k > m+1\}$, where x_i^k is an arbitrary element of L_i^k. This T_i^j has $t-1$ vertices and meets all edges of $\mathscr{H}$ but $L_i^j \cup E_i^j$. Removing L_i^j (for $j>m$) from $\mathscr{H}$ one can construct again a set T_i^j covering all the other edges as follows. $T_i^j := L_i^1 \cup (V_i \setminus E_i^1) \cup \{x_i^k : k>m, k \neq j\}$.

Proof of Theorem 1. It is well-known that primes are relatively dense among integers, e.g., it follows from [11] that for every integer $\tau > \tau_0$ one can find a prime q in the interval $\tau - \tau^{3/5} < q < \tau - 2\sqrt{\tau}$. Then $\ell(\tau) \geq q^2+q = \tau^2 - O(\tau^{8/5})$ follows from (4), (5) and from the strict monotonicity of ℓ, $\ell(\tau-1) < \ell(\tau)$. (More is true, considering vertex disjoint examples we even get $\ell(a+b) \geq \ell(a)+\ell(b)$). Q.E.D.

4. Generalized Construction from Double Critical Hypergraphs

In the same way as above one can prove the following statement. Let q be an odd prime, $q \geq 7$, and let $\mathscr{A}_q$ be the Desarguesian affine plane of order q with parallel classes $\mathscr{L}_i$ and vertex set V_0. Let $\mathscr{S} = \{E^1, \ldots, E^m\}$ be a linear, τ-critical hypergraph with covering number $t \geq 1$ such that $m < q$ and $m - t \geq \frac{1}{2}(q-5)$. Consider $q+1$ copies of $\mathscr{S}$ with pairwise disjoint vertex sets also disjoint to V_0. We obtain $\{E_i^1, E_i^2, \ldots, E_i^m\}$ for $1 \leq i \leq q+1$. Define the edge-set of $\mathscr{H}$ in the same way as in (3), then $\mathscr{H}$ is a linear, τ-critical hypergraph with

$$\tau(\mathscr{H}) = q + (t-1) + (q-m). \tag{6}$$

The construction can be extended even to the case $m = q$ if the hypergraph $\mathscr{S}$ is *double critical*. We call a τ-critical hypergraph $\mathscr{S}$ double critical if for every $E \in \mathscr{E}(\mathscr{S})$ there exists another edge E' such that removing both of them the covering number decreases by 2, $\tau(\mathscr{S}\backslash\{E,E'\}) = \tau(\mathscr{S}) - 2$. The Desarguesian affine plane $\mathscr{A}_q$ is double critical ($q \geq 3$, a prime number), and any vertex disjoint union of τ-critical hypergraphs yields double critical systems.

Replace the complete graph in the construction of the previous Section by an (almost) optimal double critical linear hypergraph. Induction implies that for $\tau = q + o(\sqrt{q})$ (where q is a prime) one has

$$\ell(\tau) \geq \tau^2 - (2 + o(1))\tau^{3/2}. \tag{7}$$

This is the natural limit of our method in improving the error term for $\ell(\tau)$. However, our constructions are not necessary optimal (in some cases they are not even maximal) so it might be that Sudakov's upper bound is closer to the truth.

5. A Remark on Set-Pair Systems

The usual approach for dealing τ-critical hypergraphs is by the so-called set-pair method. Call a collection of pair of sets $(A_i, B_i)_{i=1,2,\ldots,m}$ *cross-intersecting* of size m if it consists of disjoint pairs $A_i \cap B_i = \emptyset$ for every i but otherwise $A_i \cap B_j \neq \emptyset$ holds for every $i \neq j$. It is called a cross-intersecting (a,b)-system if in addition $|A_i| \leq a$ and $|B_i| \leq b$ hold for all $1 \leq i \leq m$. The set-pair method was started by Bollobás [6] who proved that the maximum size of a cross-intersecting (a,b)-system, is exactly $\binom{a+b}{a}$. Since then (1965) several generalizations (e.g., Alon [2]) and applications were proved, see the surveys [10] and [17] for more details.

Call a cross-intersecting set-pair system $(A_i, B_i)_{i=1,2,\ldots,m}$ an $(a,b)^*$-system if $|A_i| \leq a$ and $|B_i \cap B_j| < b$ hold for all $1 \leq i \neq j \leq m$. Denote by $f(a,b)$ the maximum possible size of an $(a,b)^*$-system. From every linear τ-critical hypergraph $\mathscr{H}$, with $\mathscr{E}(\mathscr{H}) = \{B_1, \ldots, B_m\}$ one can construct a $(\tau - 1, 2)^*$-system of size m in the following natural way. By definition, for any edge B_i there exists a subset A_i of size at most $\tau - 1$ meeting all edges of $\mathscr{H}$ except B_i.

The conjecture in [1] was originaly formulated in the stronger form $f(a,b) \leq \binom{a+b}{b}$. The Construction in Section 3 shows that it does not hold for $b = 2$ and $a > a_0$. In general, the best upper bound is due to Sudakov [15], $f(a,b) \leq a^b$. It follows from the recursion $f(a,b) \leq af(a,b-1) + 1$. It might give the right order of magnitude of f.

Acknowledgments. The author is indepted to Tao Jiang (Urbana) for fruitful conversations, and for the referees for helpful comments.

References

1. Aharoni, R., Ziv, R.: Private communication (1992). Also see in [17] as Problem (9) on p. 508

2. Alon, N.: An extremal problem for sets with applications to graph theory. J. Comb. Theory, Ser. A **40**, 82–89 (1985)
3. Blokhuis, A.: More on maximal intersecting families of finite sets. J. Comb. Theory, Ser. A **44**, 299–303 (1987)
4. Blokhuis, A.: On the size of a blocking set in $PG(2, p)$. Combinatorica **14**, 111–114 (1994)
5. Blokhuis, A.: Blocking sets in Desarguesian planes. In: D. Miklós et al.: Combinatorics, Paul Erdős is eighty, vol. 2 (Keszthely, 1993), Bolyai Soc. Math. Studies, vol. 2, pp. 133–155. (János Bolyai Math. Soc.) Budapest 1996
6. Bollobás, B.: On generalized graphs. Acta Math. Acad. Sci. Hungar. **16**, 447–452 (1965)
7. Brouwer, A.E., Schrijver, A.: The blocking number of an affine space. J. Comb. Theory, Ser. A **24**, 251–253 (1978)
8. Drake, D.A.: Embedding maximal cliques of sets in maximal cliques of bigger sets. Discrete Math. **58**, 229–242 (1986)
9. Erdős, P., Hajnal, A., Moon, J.W.: A problem in graph theory. Am. Math. Mon. **71**, 1107–1110 (1964)
10. Füredi, Z.: Matchings and covers in hypergraphs. Graphs Comb. **4**, 115–206 (1988)
11. Ivaniec, H., Pintz, J.: Primes in short intervals. Monatsh. Math. **98**, 115–143 (1984)
12. Jamison, R.E.: Covering finite fields with cosets of subspaces. J. Comb. Theory, Ser. A **22**, 253–266 (1977)
13. van Lint, J.H., Wilson, R.M.: A course in combinatorics. Cambridge University Press 1992
14. Roudneff, J.-P.: Private communication (1986). Also see in [17] as Problem (9) on p. 508, and in [10] as Problem 2.24 on p. 130
15. Sudakov, B.: A note on τ-critical linear hypergraphs. Graphs Comb. **13**, 281–285 (1997)
16. Szőnyi, T.: Blocking sets in Desarguesian affine and projective planes. Finite Fields and Appl. **3**, 187–202 (1997)
17. Tuza, Z.: Applications of the set pair method in extremal hypergarph theory. In: P. Frankl et al.: Extremal problems for finite sets. Bolyai Soc. Math. Studies 1994 (János Bolyai Math. Soc., vol. 3, pp 479–514) Budapest 1994

Received: June 28, 1999
Final version received: October 22, 1999