Embedding of Graphs in Two-Irregular Graphs

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Abstract: A graph G on n vertices is called *two-irregular* if there are at most two vertices having the same degree for all possible degrees. We show that every graph with maximal degree at most $n/8 - O(n^{3/4})$ can be embedded into a two-irregular graph. We obtain it as a corollary of an algorithmic proof of a result about packing the graphs. This improves the bound of $O(n^{1/4})$ given by Faudree et al. J Graph Theory 36: 75–83, 2001

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1. INTRODUCTION

One of the most fascinating problems in graph theory is the problem of packing of graphs, i.e., the problem of determining what graphs could be placed edge-disjointly on the same set of vertices. An equivalent version is the following:

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when could a graph be embedded into another graph on the same set of vertices, or, in other words, when is a graph a spanning subgraph of another? Sauer and Spencer [4] showed that if a product of maximal degrees of two graphs is at most half the order of the graphs, then these two graphs can be packed. Wozniak [6] showed that every graph of order n and size at most $n-2$ is 3-placeable in K_n unless it is isomorphic to two special graphs, where 3-placeable means that three copies of a graph could be placed edge-disjointly. There are many other related results on packing of several graphs or packing special classes of graphs, most of them are formulated in terms of the number of edges or the maximal degree. We obtain a new type of criterion for graphs to be packed (or embedded).

We claim that if the maximal degree of a graph is not too large then it could be embedded in a graph with a certain prescribed degree sequence.

This question is motivated by a paper of Faudree et al. [2]. They show that every graph with maximal degree at most $O(n^{1/4})$ is embeddable into a two-irregular graph of the same order. We improve their bound to $n/8 - O(n^{3/4})$. Moreover, we show that if a graph G has maximal degree at most $n/12$ and a graph H satisfies some special conditions, then for any permutation of vertices of G there is a graph F with the same degree sequence as H, such that G and F are packable with corresponding vertex order.

2. DEFINITIONS

For any two graphs $G = (V, E_1)$ and $H = (V, E_2)$, let $G \backslash H = (V, E_1 \backslash E_2)$ and $G \cup H = (V, E_1 \cup E_2)$. The complement of a graph G is denoted as $\bar{G}$, the subgraph of G induced by vertex set A as $G[A]$. For $A, B \subset V, A \cap B = \emptyset$ we denote the induced bipartite graph of G as $G[A, B]$. Two graphs G and H are said to be *packed* if there is a permutation of vertices σ, such that for all $u, v \in V$ when $uv \in E(G), \sigma(u)\sigma(v) \notin E(H)$. We say that G is embedded in $H, G \subseteq H$, if there is a permutation of vertices σ, such that for all $u, v \in V$ when $uv \in E(G), \sigma(u)\sigma(v) \in E(H)$. A graph G is called *two-irregular* if it has no more than two vertices having the same degree for each possible degree. We define the graph $B_n = (V, E)$, the complement of *half graph*, as follows: $V = A \cup B$, $A = \{a_1, \ldots, a_{\lceil n/2 \rceil}\}, B = \{b_1, \ldots, b_{\lfloor n/2 \rfloor}\}, E = \{\{a_i, a_j\} : i, j \in \{1, \ldots, \lceil n/2 \rceil\}\} \cup \{\{b_i, b_j\} : i, j \in \{1, \ldots, \lfloor n/2 \rfloor\}\} \cup \{\{a_i, b_j\} : i < j, i \in \{1, \ldots, \lceil n/2 \rceil\}, j \in \{1, \ldots, \lfloor n/2 \rfloor\}\}$. This graph has degrees $d(a_i) = n - i - 1, i \in \{1, \ldots, \lceil n/2 \rceil\}, d(b_j) = \lfloor n/2 \rfloor - 2 + j, j \in \{1, \ldots, \lfloor n/2 \rfloor\}$. This B_n is a two-irregular graph.

Given an underlying graph H and another graph H' on the same vertex set let the *excessive degree* of $x \in V$ with respect to $H, d_{\text{ex}}(H', x)$, be $d(H', x) - d(H, x)$. Let $\text{EX}(H') = \{x \in V : d_{\text{ex}}(H', x) > 0\}$ be the set of vertices with positive excessive degrees. The graphs G and H have the *same degrees* if $V(G) = V(H)$ and $d(G, x) = d(H, x)$ for every $x \in V$. They have the same *degree sequence* if there is a bijection $f : V(G) \to V(H)$ such that $d(G, x) = d(H, f(x))$.

3. RESULTS

Let H be a complement of a bipartite graph on n vertices with almost equal parts A and B such that $A = \{a_1, \ldots, a_{\lceil n/2 \rceil}\}, B = \{b_1, \ldots, b_{\lfloor n/2 \rfloor}\}$ and for any $I \subset \{1, \ldots, \lceil n/2 \rceil\}$ of size at least $n/8$, the set of vertices $\{a_i, b_i : i \in I\}$ induces at most $|I|^2/2$ edges in $H[A, B]$,

$$|E(H[A_I, B_I])| \leq |I|^2/2. \tag{1}$$

Theorem 1. *Let G be a simple graph on n vertices with maximal degree $d \leq n/8 - O(n^{3/4})$. Then G can be embedded into a graph F with the same degree sequence as H.*

Corollary 2. *If G is a simple graph on n vertices with maximal degree $d \leq n/8 - O(n^{3/4})$, then G is embeddable into a two-irregular graph having the same degree sequence as B_n.*

Theorem 3. *Let G be a graph on the same vertex set as H. Suppose that maximal degree of G is $d \leq n/12$. Then there exists a graph F with every vertex having the same degree as H and $G \subseteq F$.*

Moreover, for Theorem 3 we give the algorithm for constructing a graph F with the required properties.

4. PROOFS OF THE RESULTS

Proof of Theorem 1. Let $d_0 = \frac{1}{2}d + 6\sqrt{n}$. By a result of Spencer [5] there exists a partition of $V(G) = A' \cup B', A' = \{a'_1, \ldots, a'_{\lceil n/2 \rceil}\}, B' = \{b'_1, \ldots, b'_{\lfloor n/2 \rfloor}\}$ such that

$$|N(G, x) \cap A'| \leq d_0 \quad \text{and} \quad |N(G, x) \cap B'| \leq d_0 \quad \text{for all}\, x \in V. \tag{2}$$

Place G into the vertex set of H by identifying a_i with a'_i and b_i with b'_i. We obtain $G \cup H$.

Define a class of graphs, $\mathbf{H}_1$, such that every member $Q \in \mathbf{H}_1$ has the same vertex set $V(Q) = V(H)$ and satisfies the following conditions:

- **C1.** Q contains G and $H[A, B]$.
- **C2.** Q is a subgraph of $G \cup H$.
- **C3.** Q is obtained by deleting the same number of edges of $H \backslash G$ in A and in B, so it has the same excess with respect to H on A and B, i.e., $\sum_{x \in A} d_{\text{ex}}(Q, x) = \sum_{x \in B} d_{\text{ex}}(Q, x)$.
- **C4.** Q majorizes the degrees of H, $d_{\text{ex}}(Q, x) \geq 0$ for every $x \in V$.

Notice that $\mathbf{H}_1$ is not empty since $G \cup H$ satisfies C1–C4. Hence

$$\sum_{x \in V} d_{\text{ex}}(Q, x) \leq \sum_{x \in v} d_{\text{ex}}(G \cup H, x)$$

for every member Q of $\mathbf{H}_1$. Let $H_1 \in \mathbf{H}_1$ minimize the total excess

$$\sum_{x\in V} d_{\mathrm{ex}}(H_1,x) \leq \sum_{x\in V} d_{\mathrm{ex}}(Q,x) \quad \text{for all } Q \in \mathbf{H}_1. \tag{3}$$

We note that a minimal element of $\mathbf{H}_1$ (with respect to simultaneous edge deletion) is sufficient for our purposes. Finding such a minimal member of $\mathbf{H}_1$ is easy from an algorithmic point of view, as we shall see later in Section 5. ■

Claim 1. The edges of G together with the deleted edges (i.e., the pairs from $\mathbf{H}\backslash\mathbf{H}_1$) form a complete graph on the vertices of positive excessive degrees either in A or in B. That is, one of the following holds:
(a) $xy \in E(G) \cup E(\bar{H}_1)$ for all $x,y \in \mathrm{EX}(H_1) \cap A$,
(b) $xy \in E(G) \cup E(\bar{H}_1)$ for all $x,y \in \mathrm{EX}(H_1) \cap B$.

Proof of Claim 1. Suppose that there are $a,a' \in \mathrm{EX}(H_1) \cap A$ and $b,b' \in \mathrm{EX}(H_1) \cap B$ such that $aa',bb' \notin E(G)$ and $aa',bb' \in E(H_1)$. Then, by deleting the edges aa' and bb' from H_1 we obtain another member of $\mathbf{H}_1$ with smaller total excess (3), a contradiction.

From now on, we assume that (a) holds. Case (b) can be handled in the same way.

Claim 2. The number of vertices with positive excessive degree in A, $|\mathrm{EX}(H_1) \cap A|$ is at most d. The total excess of H_1 in A is at most $(d+1)^2/4$, that is

$$\sum_{x\in A} d_{\mathrm{ex}}(H_1,x) \leq (d+1)^2/4.$$

Proof of Claim 2. It follows from C1 and C2 that H_1 is obtained from $G \cup H$ by deleting some edges of H in parts A and B which are not edges of G. In particular, $H_1[A,B] = (G\cup H)[A,B]$. Thus, we could express the excessive degree of $x \in A \cap \mathrm{EX}(H_1)$ in H_1 as the number of $G\backslash H$ edges joining x to B minus the number of deleted edges starting from x. We have

$$\begin{aligned} d_{\mathrm{ex}}(H_1,x) &\equiv d(H_1,x) - d(H,x) \\ &= d(H_1[A],x) + d(H_1[A,B],x) - d(H[A],x) - d(H[A,B],x) \\ &= -d(\bar{H}_1[A],x) + d((G\backslash H)[A,B],x) \\ &\leq -d(\bar{H}_1[A],x) + d(G[A,B],x). \end{aligned}$$

From this we conclude that

$$d(\bar{H}_1[A],x) \leq d(G[A,B],x) - d_{\mathrm{ex}}(H_1,x).$$

Adding $d(G[A], x)$ to both sides gives

$$d((G \cup \bar{H}_1)[A], x) \le d(G, x) - d_{\text{ex}}(H_1, x). \tag{4}$$

Let us note that one can prove in the same way that for all $x \in B$

$$d((G \cup \bar{H}_1)[B], x) \le d(G, x) - d_{\text{ex}}(H_1, x). \tag{5}$$

Let q be the maximal excessive degree in A,

$$q = \max\{d_{\text{ex}}(H_1, x) : x \in \text{EX}(H_1) \cap A\}. \tag{6}$$

Claim 1(a) implies that x is joined to all vertices of $\text{EX}(H_1) \cap A$ by edges of $G \cup \bar{H}_1$, thus we have

$$|\text{EX}(H_1) \cap A| \le 1 + d((G \cup \bar{H}_1)[A], x) \le 1 + d - q. \tag{7}$$

Therefore,

$$\sum_{x \in A} d_{\text{ex}}(H_1, x) = \sum_{x \in \text{EX}(H_1) \cap A} d_{\text{ex}}(H_1, x) \le (d - q + 1)q \le (d+1)^2/4.$$

If $q = 0$, then there is nothing left to prove anymore ($\text{EX}(H_2) = \emptyset$), and for $q \ge 1$ the first half of the claim follows from (7).

Now lets define the class of graphs $\mathbf{H}_2$. Each member of $Q \in \mathbf{H}_2$ is constructed from H_1 by adding some edges between A and B and deleting the same number of edges in each of the parts A and B such that Q satisfies the conditions D1–D6below.

D1. $Q[A]$ and $Q[B]$ are subgraphs of $H_1[A]$ and $H_1[B]$ respectively and Q contains G and $H[A, B]$. (Recall that $(G \cup H)[A, B] = H_1[A, B]$.)

D2. $d(H, x) \le d(Q, x) \le d(H_1, x)$, so $0 \le d_{\text{ex}}(Q, x) \le d_{\text{ex}}(H_1, x)$ for every $x \in V$.

D3. $|E((H_1 \backslash Q)[A])| = |E((H_1 \backslash Q)[B])| = |E((Q \backslash H_1)[A, B])|$, so $\sum_{x \in A} d_{\text{ex}}(Q, x) = \sum_{x \in B} d_{\text{ex}}(Q, x)$.

D4. There are no new edges in Q incident to the vertices of positive excessive degree in A, $Q[(\text{EX}(H_1) \cap A), B] = H_1[(\text{EX}(H_1) \cap A), B]$.

D5. Define $d_1 = \lfloor d/2 \rfloor$. There are at most d_1 new edges at every vertex of B, $d(Q \backslash H_1, x) \le d_1$ for every $x \in B$.

D6. Total excess of $H_2[A]$ is the total excess of $H_1[A]$ minus the number of added (i.e., $Q \backslash H_1$) edges, hence $\sum_{x \in A} d(H_2 \backslash H_1, x) = \sum_{x \in A} (d_{\text{ex}}(H_1, x) - d_{\text{ex}}(H_2, x))$.

Let $H_2 \in \mathbf{H}_2$ minimize the total excessive degree

$$\sum_{x \in V} d_{\text{ex}}(Q, x) \ge \sum_{x \in V} d_{\text{ex}}(H_2, x) \quad \text{for all } Q \in \mathbf{H}_2.$$

Claim 3. Total excessive degree of H_2 is zero, $\sum_{x\in V} d_{\mathrm{ex}}(H_2,x)=0$.

Proof of Claim 3. Suppose that there are two vertices $a\in A$ and $b\in B$ with positive excessive degrees. We are going to find $a'\in A$ and $b'\in B$ such that deleting aa' and bb' (from $H_1\backslash G$) and adding $a'b'$ (from $\overline{G\cup H}$) produces another member of $\mathbf{H}_2$ with smaller total excess, a contradiction. We have $\mathrm{EX}(H_2)\subseteq \mathrm{EX}(H_1)$, so $a\in \mathrm{EX}(H_1)\cap A$.

Define A' as a set of vertices $a'\in A$ with zero excessive degree such that aa' could be deleted,

$$A'=\{a' : aa'\in (E(H_2)\backslash G), a'\in A\ a'\notin \mathrm{EX}(H_2)\}.$$

The last condition for $A'(a'\notin \mathrm{EX}(H_2))$ is implied by the first, since Claim 1 gives $\mathrm{EX}(H_1)\cap A\subseteq \{a\}\cup N(G\cup \bar{H}_1,a)$. We are going to show that

$$|A'|\geq \lceil n/2\rceil - d - 1. \tag{8}$$

We have

$$A\backslash A' = \{a\}\cup N((G\cup \bar{H}_1)[A],a)\cup N((\bar{H}_2\backslash \bar{H}_1)[A],a).$$

Obviously, $d((\bar{H}_2\backslash\bar{H}_1)[A],a)=d(H_1\backslash H_2,a)$. Then D4 implies that this is equal to $d_{\mathrm{ex}}(H_1,a)-d_{\mathrm{ex}}(H_2,a)$. Now by D2 we have

$$d((\bar{H}_2\backslash\bar{H}_1)[A],a)\leq d_{\mathrm{ex}}(H_1,a). \tag{9}$$

Adding (4) (with $x=a$) to (9) we have $d((G\cup\bar{H}_2)[A],a)\leq d$, implying $|A\backslash A'|\leq d+1$ and (8) follows.
Define B' as a set of vertices $b'\in B$ satisfying *strict* inequality in D5 such that bb' can be deleted,

$$B'=\{b' : bb'\in (E(H_2)\backslash G),\ b'\in B,\ d(H_2\backslash H_1,b')<d_1\}.$$

Let t be the number of vertices for which the equality of D5 is achieved,

$$t=|\{y'\in B : d(H_2\setminus H_1,y')=d_1\}|. \tag{10}$$

We are going to show that

$$|B'|\geq \lfloor n/2\rfloor - d - d_1 - t - 1. \tag{11}$$

We have

$$\begin{aligned} B\backslash B' = &\{b\}\cup N((G\cup\bar{H}_1)[B],b)\cup N((\bar{H}_2\backslash\bar{H}_1)[B],b)\\ &\cup\{b'\in B : d(H_2\backslash H_1,b')=d_1\}.\end{aligned}$$

As before we have $d((\bar{H}_2\backslash\bar{H}_1)[B],b) = d(H_1\backslash H_2,b)$. This is equal to $d_{\text{ex}}(H_1,b) - d_{\text{ex}}(H_2,b) + d((H_2\backslash H_1)[B,A],b)$. Thus, D2 and D5 imply that this is at most

$$d((\bar{H}_2\backslash\bar{H}_1)[B],b) \leq d_{\text{ex}}(H_1,b) + d_1.$$

Adding (5) to this inequality (with $x = b$), we obtain

$$d((G \cup \bar{H}_2)[B],b) \leq d + d_1.$$

Together with (10) we have

$$d((G \cup \bar{H}_2)[B],b) + |\{b' \in B : d(H_2\backslash H_1,b') = d_1\}| \leq d + d_1 + t,$$

and (11) follows.

Define $I = \{i : a_i \in A', b_i \in B'\}, A_I = \{a_i : i \in I\}, B_I = \{b_i : i \in I\}$. The inequalities (8) and (11) give

$$|I| \geq |A'| + |B'| - \lceil n/2 \rceil \geq \lfloor n/2 \rfloor - 2d - d_1 - t - 2. \tag{12}$$

Now, let us estimate the number of edges of H_2 between A_I and B_I. There are three types of edges—edges of H, edges of G and edges of $H_2\backslash(G \cup H)$.

Using the density property (1) of H we have

$$|E(H[A_I,B_I])| \leq |I|^2/2. \tag{13}$$

Using (2) we have

$$|E(G[A_I,B_I])| \leq |I|d_0. \tag{14}$$

Concerning the $H_2\backslash(G \cup H)$ edges between A and B, their number is at most $(d+1)^2/4$ by D2 and Claim 2. By definition, at least $d_1 t$ of these join A to $B\backslash B'$, so we have

$$|E((H_2\backslash(G \cup H))[A_I,B_I])| \leq (d+1)^2/4 - d_1 t. \tag{15}$$

Now, summing up (13), (14) and (15) and applying (12), an easy calculation leads to

$$|E(H_2[A_I,B_I])| < |I|^2.$$

Thus, for $d \leq n/8 - O(n^{3/4})$ there exists a nonedge $a'b' \notin E(H_2), a' \in A_I \subseteq A'$, $b' \in B_I \subseteq B'$. This concludes the proof of Claim 3. ■

Now we can take H_2 as the graph F in Theorem 1.

For the proof of Theorem 3 we mimic the proof of Theorem 1 starting with an arbitrary placement of the graph G and letting the constant d_0 be equal to d. The details are left to the reader.

5. ALGORITHM

To construct a graph F for the Theorem 3, we apply a two-step augmentation algorithm.

Step 1. We build graphs $\{H_i^*\}$ on the same set of vertices as H as follows. Let $H_0^* = G \cup H$. Suppose that H_i^* has been built. If there is an edge aa' in A and edge bb' in B such that a, a', b, b' have positive excessive degrees in H_i^* and aa', bb' are not the edges of G, then delete them and set $E(H_{i+1}^*) = E(H_i^*)\backslash(\{aa'\} \cup \{bb'\})$. Repeat this procedure as long as possible. Assume that the last graph obtained is H_k^*. It is easy to see that this graph H_k^* satisfies all the properties of graph H_1 in the Theorem 3. Therefore Claim 1 (say(a)) and Claim 2 hold.

Step 2. We build graphs $\{F_i^*\}$ on the same set of vertices as follows. Let $F_0^* = H_k^*$. Suppose that F_i^* is built. If there is $x \in A$ and $y \in B$ with positive excessive degrees and there are $x' \in A, y' \in B$, such that xx' and yy' are the edges of F_i^* but not the edges of G, and $x'y'$ is not the edge of F_i^*, then delete xx' and yy' and add $x'y'$, i.e., set $E(F_{i+1}^*) = E(F_i^*)\backslash(\{xx'\} \cup \{yy'\}) \cup x'y'$. Repeat this procedure as long as possible. Denote the last graph obtained by F_l^*. It is easy to see that its satisfies D1–D6, and we are done.

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