Geometriae Dedicata **81:** 285–293, 2000.

The Smallest Convex Cover for Triangles of Perimeter Two

ZOLTAN FÜREDI[1]⋆ AND JOHN E. WETZEL[2]
[1]*Mathematics Institute, Hungarian Academy of Sciences, POB 127, 1364 Budapest, Hungary*
[2]*Department of Mathematics, University of Illinois at Urbana-Champaign,*
1409 West Green Street, Urbana, IL 61801, U.S.A. e-mail: {z-furedi, j-wetzel}@uiuc.edu

(Received: 7 June 1999)

Communicated by R. Schneider

Abstract. We find the unique smallest convex region in the plane that contains a congruent copy of every triangle of perimeter two. It is the triangle ABC with $AB = 2/3$, $\angle B = 60°$, and $BC \approx 1.00285$.

Mathematics Subject Classification (2000): 52A10.

Key words: worm problems, convex covers, isoperimetric triangles.

1. Introduction

Thirty-five years ago, Leo Moser asked for "the region of smallest area which will accommodate every arc of length 1". This so-called 'worm problem' remains unsolved. A wide variety of similar covering problems can be formulated by restricting the family of arcs or specifying the shape of the containing region. Very few of these problems have been solved. For a glimpse at the literature on such problems, see pp. 129–130 in [2].

A convex region that contains a congruent copy of each curve of a specified family is called a *cover* for the family, and a 'worm problem' for that family is to find the cover, possibly of prescribed shape, whose area is as small as possible. For example, the smallest triangular cover for the family $\mathcal{C}$ of all closed curves of length two is the equilateral triangle of side $2\sqrt{3}/\pi$, a result that follows from an inequality published in 1957 by Eggleston [3] (see [9,11]). The problem of finding the smallest cover for $\mathcal{C}$ is unsolved; it is known that the least area A lies in the range $0.38532 \leqslant A \leqslant 0.49095$ (see [1,6]). We can improve both of these bounds a little and show that $0.38667 \leqslant A \leqslant 0.47016$, but the gap remains wide.

In this same vein, we consider here the smaller family $\mathcal{T}$ of all triangles of perimeter two. The smallest equilateral triangular cover, the smallest rectangular cover, and the smallest rectangular cover of prescribed shape for $\mathcal{T}$ have recently been

⋆Research supported in part by Hungarian National Science Foundation grant OTKA 016389 and by National Security Agency grant MDA904-98-I-0022.

described (see [8,10]). In this article we find the *smallest* cover for $\mathcal{T}$, and we show that this smallest cover is unique up to congruence.

2. The Cover T

In describing the smallest convex cover for the family $\mathcal{T}$ of all triangles of perimeter two we need the maximum s_0 of the trigonometric function

$$f(\theta) = \frac{2\sqrt{3}\sin(\frac{1}{3}\pi + \theta)}{3(1 + \sin\frac{1}{2}\theta)} \tag{1}$$

on the interval $[0, \frac{1}{3}\pi]$. A little numerical work, the details of which we omit, shows that the maximum is

$$s_0 \approx 1.00285\ 14266,$$

and it occurs at the (unique) point

$$\theta_0 \approx 0.07473\ 26242.$$

We shall have occasion to refer to the narrow isosceles triangle Δ_0 of perimeter two whose apex angle is θ_0 (about 4.28 degrees). The two equal sides of Δ_0 have length

$$\ell_0 = \frac{1}{1 + \sin\frac{1}{2}\theta_0} \approx 0.96399;$$

and the length of the apex altitude of Δ_0 exceeds 0.96331.

Let **T** be the triangle ABC with $AB = 2/3$, $\angle B = 60°$, and $BC = s_0$. The altitude of **T** to BC is $\sqrt{3}/3$, and the area of **T** is $s_0\sqrt{3}/6 \approx 0.28950$. In this article we establish that:

(A) The triangle **T** is a cover for $\mathcal{T}$.
(B) If **K** is a cover for $\mathcal{T}$, then area(**K**) $\geqslant$ area(**T**).
(C) If **K** is a cover for $\mathcal{T}$ and area(**K**) = area(**T**), then **K** and **T** are congruent triangles.

3. Triangle T is a Cover for $\mathcal{T}$

The proof that **T** contains a congruent copy of every triangle of perimeter two depends on two preliminary lemmas.

LEMMA 1. *In Figure 1,* $AB = 2/3$, $\angle DBA = 60°$, $BD = 1 < s_0$, *and* $BM = MD = 1/2$. *The locus of the apex* X *of isosceles triangles* XBZ *of perimeter two with longest side* BZ *on* BD *is the arc* $\Gamma = AM$ *of the parabola whose focus is the point* B *and whose directrix is the line* d *perpendicular to* BD *at* D.

Proof. Take the ray BD as the polar axis for a polar coordinate system, and let (r, φ) be the polar coordinates of the point X. The perimeter condition $BX + XZ + ZB = 2$ gives $2r + 2r\cos\varphi = 2$, i.e., $r = 1/(1 + \cos\varphi)$, which (for $0 \leqslant \varphi \leqslant 60°$) is the polar equation of the parabolic arc Γ. □

Note in particular that the parabolic arc $\Gamma = AM$ lies in $\mathbf{T}$.

LEMMA 2. *In Figure 2, $\angle CBA = 60°$, $XZ = YZ \geqslant XY$, and triangle XYZ has perimeter two. Then $BZ \leqslant s_0 = BC$, with equality precisely when $\angle XZY = \theta_0$, i.e., precisely when $XYZ = \Delta_0$.*

Proof. Switching to radians, write $\angle XZY = \theta$, so that $\angle X = \angle Y = \frac{1}{2}(\pi - \theta)$, and let $XZ = YZ = \ell$. Then $XY = 2\ell \sin\theta/2$; and since triangle XYZ has perimeter two it follows that $\ell = 1/(1 + \sin\theta/2)$. Now $\angle BXZ = \frac{2}{3}\pi - \theta$, and from the law of sines applied to triangle BZX we conclude that

$$BZ = \frac{2\ell}{\sqrt{3}}\sin\left(\frac{2}{3}\pi - \theta\right) = \frac{2\sqrt{3}\sin\left(\frac{1}{3}\pi + \theta\right)}{3\left(1 + \sin\frac{1}{2}\theta\right)} = f(\theta).$$

Since $0 < \theta \leqslant \frac{1}{3}\pi$, the claim follows. □

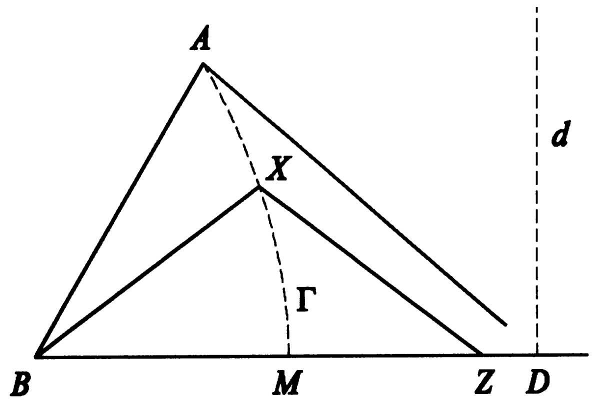

Figure 1. The parabolic locus of X.

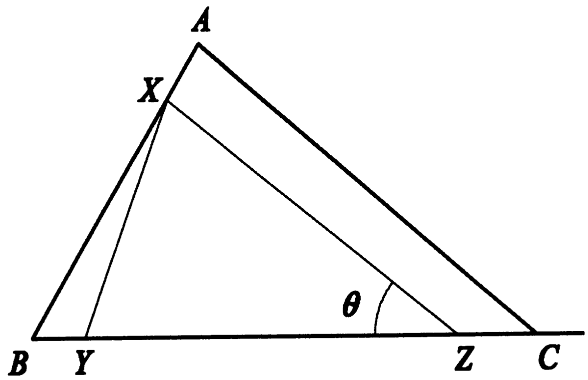

Figure 2. The maximum of BZ.

Our first claim (A) follows easily from these preliminaries:

THEOREM 3. *The triangle* **T** *is a cover for* $\mathcal{T}$.

Proof. Let XYZ be a triangle with perimeter two, and suppose the notation arranged so that $XY \leqslant XZ \leqslant YZ$. Regard XYZ as a triangular tile and place it in $\angle ABC$ with side YZ on the ray BC and with Y as close as possible to B. If $\angle XYZ \leqslant 60°$, then $Y = B$ (Figure 3a) and $YZ < 1 < s_0$. If $BX = XZ$, then Lemma 1 asserts that X lies on the parabolic arc Γ and consequently in **T**. If $BX < XZ$, let Z' be the point on BZ so that $XZ' = XB$. Then triangle XBZ' has perimeter smaller than two, and there are points X_0 on the ray BX and Z_0 on the segment ZZ' so that the isosceles triangle X_0BZ_0 has perimeter two; note that $BX < BX_0$. According to Lemma 1, X_0 lies on the parabolic arc Γ and so in **T**. Thus triangle XYZ fits inside **T**.

If $\angle XYZ > 60°$, then (Figure 3b) X lies on the ray BA, and we must show that X lies on the segment BA and that $BZ \leqslant BC = s_0$. Let h be the altitude from X to YZ, and note that $x = YZ \geqslant \frac{2}{3}$. Thus $\frac{h}{3} \leqslant \frac{1}{2}xh \leqslant \frac{1}{9}\sqrt{3}$, because the equilateral triangle has the greatest area among all triangles of given perimeter [7]. So $h \leqslant \frac{1}{3}\sqrt{3}$; and X lies on the segment BA.

It remains to show that $BZ \leqslant BC = s_0$. According to Lemma 2, this assertion is correct if $YZ = XZ$. Suppose as pictured in Figure 3b that $YZ > XZ$, and let W be the point on YZ so that $WZ = XZ$. Then the points are in the order $B - Y - W - Z$, and the perimeter p of triangle XWZ is strictly less than two. If $X'W'Z'$ is the triangle similar to XWZ with perimeter two having X' on BA and $W'Z'$ along BC, then X' lies between A and X, and the points on the ray BC are in the order $B - Y - W' - Z - Z'$. In particular, $BZ < BZ' \leqslant BC$. It follows that XYZ fits in **T**. □

COROLLARY 4 [11]. *The smallest equilateral triangle that is a cover for the family* $\mathcal{T}$ *of all triangles of perimeter two has side* s_0.

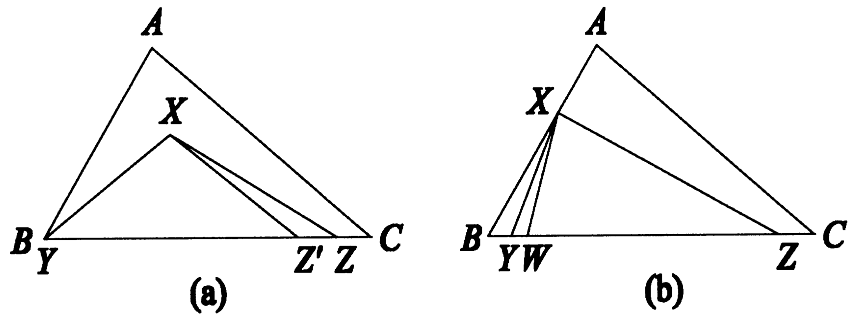

Figure 3. The proof of Theorem 3.

Proof. It is obvious that the (closed) equilateral triangular region Δ with side s_0 is a cover for $\mathcal{T}$, because **T** fits in Δ. That no smaller equilateral triangular region is a cover for $\mathcal{T}$ is an immediate consequence of a recent lemma of K. A. Post [5]: if a triangle ABC contains a triangle PQR, then it also contains a triangle congruent to PQR having two of its vertices on the same side of ABC. If Δ_0 fits in an equilateral triangle Δ' with side $s' < s_0$, then by Post's Lemma it also fits in Δ' with one side along a side of Δ'. Since the apex altitude of Δ_0 is longer than the altitude of Δ, it does not fit with its base on one side of Δ'. Lemma 2 asserts that it does not fit with one equal side along a side of Δ'. Consequently it does not fit at all. (Wetzel [11] gives a direct argument that does not rely on Post's Lemma.) □

4. The Cover T is Minimal and Unique

We conclude by showing that the area of **T** is as small as possible and that the minimal cover **T** is unique up to congruence.

Suppose that a convex set **K** is a cover for $\mathcal{T}$, let PQR be an equilateral triangle of side 2/3 inside **K**, and let XYZ be the equilateral triangle directly similar to PQR that is formed by support lines of K parallel to the sides of PQR. (See Figure 4.) Then

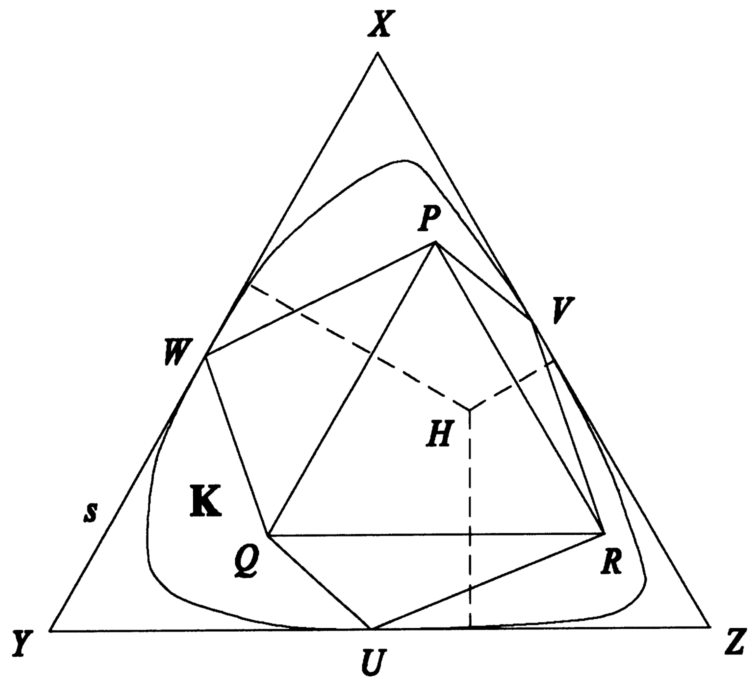

Figure 4. The area inequality.

triangle XYZ is an equilateral triangular cover for $\mathcal{T}$, and it follows from Corollary 4 that $XY = s \geqslant s_0$.

Let U, V, and W be points of **K** that lie on the sides YZ, ZX, and XY, respectively, and let H be a point inside the equilateral triangle PQR. Let x_1, x_2, y_1, y_2, z_1, and z_2 be the distances from H to the lines QR, YZ, RP, ZX, PQ, and XY, respectively. Then the (possibly degenerate) triangles URQ, VPR, and WQP are disjoint and lie in K, and their total area $w =$ area(URQ)+ area(VPR)+area(WQP) is

$$w = \frac{1}{2}\cdot\frac{2}{3}((x_2 - x_1) + (y_2 - y_1) + (z_2 - z_1))$$

$$= \frac{1}{3}(x_2 + y_2 + z_2) - \frac{1}{3}(x_1 + y_1 + z_1)$$

$$= \frac{1}{3}(\text{altitude of } XYZ) - \frac{1}{2}\cdot\frac{2}{3}(\text{altitude of } PQR)$$

$$= \frac{1}{6}s\sqrt{3} - \text{area }(PQR)$$

$$\geqslant \frac{1}{6}s_0\sqrt{3} - \text{area }(PQR), \tag{2}$$

where we have used the familiar fact that the sum of the three distances from a point in an equilateral triangle to the three sides of the triangle is the altitude of the triangle. It follows that

$$\text{area}(\mathbf{K}) \geqslant \text{area(PQR)} + \text{w} \geqslant \frac{1}{6}\text{s}_0\sqrt{3} = \text{area}(\mathbf{T}). \tag{3}$$

We have proved our second claim (**B**):

THEOREM 5. *If* **K** *is a cover for the family $\mathcal{T}$ of all triangles of perimeter two, then* area(**K**) $\geqslant$ area(**T**).

Finally, suppose that area(**K**) = area(**T**). Then the equalities must hold in (3) and consequently in (2), and it follows that the equilateral triangle XYZ must have side $s = s_0$ and that $\mathbf{K} = PQR \cup UQR \cup VRP \cup WPQ$. In particular, this union of triangles must be convex.

The narrow isosceles triangle Δ_0 must fit in **K**. But as noted above, Δ_0 fits in XYZ in only one way, namely, with its apex at a vertex and one of its equal sides along a side of XYZ. We place $\Delta_0 = DEF$ in **K** with its apex D at Y, E on the side YZ, and F on XZ (Figure 5). Then, in particular, D, E, and F lie in **K**. If triangle PQR is disjoint from the segment YZ, then U would be the only point of **K** on YZ, a contradition

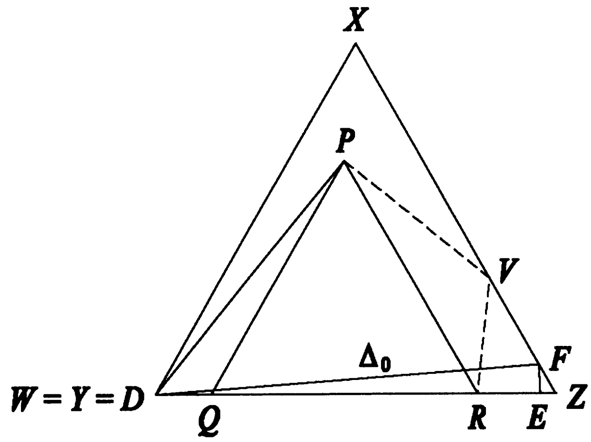

Figure 5. The uniqueness argument.

because the entire side YE of Δ_0 lies in **K**. Consequently the side QR of triangle PQR lies along YZ (as pictured), and the triangle URQ is degenerate. If $Q = Y$, then **K** contains the quadrilateral $PYEF$, whose area is larger than area(**T**). So $Q \neq Y$, and it follows that $W = Y$. We have only to locate V on XZ.

But V depends on R. If R lies between Y and E, then area(**K**) $\geqslant$ area($PYEF$) $>$ area($PYRF$) $=$ area(**T**), contrary to assumption. So R must lie on the segment EZ.

Suppose $R \neq Z$. Then $V = F$, and **K** must be the quadrilateral $PYRF$, which does contain Δ_0. But is the quadrilateral $PYRF$ a cover for $\mathcal{T}$? To see that it is not requires an argument very much like the proof of Lemma 2. Introduce a polar coordinate system (r, θ) having the ray YZ as polar axis, and let YIJ be an isosceles triangle with apex Y, I on YZ, $YI = YJ \geqslant IJ$, and perimeter two. Writing $J = (r, \theta)$, we see that

$$r = \frac{1}{1 + \sin\frac{1}{2}\theta}, \quad 0 < \theta \leqslant \frac{\pi}{3}. \tag{4}$$

The line XZ has polar equation

$$r = \frac{\sqrt{3}s_0}{\sqrt{3}\cos\theta + \sin\theta} = \frac{\sqrt{3}s_0}{2\sin\left(\frac{1}{3}\pi + \theta\right)},$$

and, recalling (1) we see that the condition for the polar curve (4) to lie in triangle XYZ is the inequality

$$\frac{1}{1 + \sin\frac{1}{2}\theta} \leqslant \frac{\sqrt{3}s_0}{2\sin\left(\frac{1}{3}\pi + \theta\right)},$$

i.e., $f(\theta) \leqslant s_0$. It follows that the curve (4) is tangent to XZ at F, and consequently there is a $\theta < \theta_0$ for which the base vertex J of triangle YIJ lies in the triangle FRZ. The quadrilateral $PYRF$ evidently does not contain a congruent copy of this triangle. This contradiction shows that $R = Z$, triangle VPR is degenerate, and $\mathbf{K}$ is the triangle PYZ, which is congruent to $\mathbf{T}$.

This completes the proof of uniqueness, (C):

THEOREM 6. *If* $\mathbf{K}$ *is a cover for* $\mathcal{T}$ *and* $\text{area}(\mathbf{K}) = \text{area}(\mathbf{T})$*, then* $\mathbf{K}$ *is congruent to* $\mathbf{T}$.

5. Triangles of Diameter One

Let $\Phi = ABC$ be the triangle with $AB = 1$, $\angle ABC = 60°$, and $BC = (2\cos 10°)/\sqrt{3} \approx 1.13716$; so the altitude of Φ to the side AB is $\cos 10°$. In 1983 Kovalev [4] showed that Φ is the unique convex cover of least area for the collection $\mathcal{D}$ of all triangles of diameter at most one, i.e., whose longest side has length at most one. Paralleling our (A), (B), and (C) of Section 2, Kovalev showed:

(A_1) Triangle Φ contains a congruent copy of every triangle in $\mathcal{D}$;
(B_1) If $\mathbf{K}$ is a convex region containing both an equilateral triangle T_1 with sides of length one and the particular isosceles triangle T_2 with two equal sides of length one and apex angle 20°, then $\text{area}(\mathbf{K}) \geqslant \text{area}(\Phi)$;
(C_1) If $\mathbf{K}$ is a convex region containing a congruent copy of every triangle in $\mathcal{D}$ and if $\text{area}(\mathbf{K}) = \text{area}(\Phi)$, then $\mathbf{K} \cong \Phi$.

Our method yields a simpler, more transparent proof for the critical steps (B_1) and (C_1). Namely, one shows first that the smallest equilateral triangle containing T_2 has side equal to the length of the side BC of Φ, (cf. Corollary 4, above; see also [10]). Then if $\mathbf{K}'$ is a convex region containing both T_1 and T_2, form an equilateral triangle XYZ with support lines of $\mathbf{K}'$ parallel to the sides of T_1 as in Section 4. Then $XY \geqslant BC$, and the area bounds and uniqueness follow exactly as before.

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