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On Generalized Ramsey Theory: The Bipartite Case

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Given graphs G and H, a coloring of $E(G)$ is called an (H, q)-coloring if the edges of every copy of $H \subseteq G$ together receive at least q colors. Let $r(G, H, q)$ denote the minimum number of colors in an (H, q)-coloring of G. We determine, for fixed p, the smallest q for which $r(K_{n,n}, K_{p,p}, q)$ is linear in n, the smallest q for which it is quadratic in n. We also determine the smallest q for which $r(K_{n,n}, K_{p,p}, q) = n^2 - O(n)$, and the smallest q for which $r(K_{n,n}, K_{p,p}, q) = n^2 - O(1)$. Our results include showing that $r(K_{n,n}, K_{2,t+1}, 2)$ and $r(K_n, K_{2,t+1}, 2)$ are both $(1+o(1))\sqrt{n/t}$ as $n \to \infty$, thereby proving a special case of a conjecture of Chung and Graham. Finally, we determine the exact value of $r(K_{n,n}, K_{3,3}, 8)$, and prove that $2n/3 \leqslant r(K_{n,n}, C_4, 3) \leqslant n+1$. Several problems remain open.

Key Words: Ramsey theory; edge-coloring of bipartite graphs; projective planes; algebraic constructions.

1. GENERALIZING THE CLASSICAL PROBLEM FOR MULTICOLORINGS

The classical Ramsey problem asks for the minimum n such that every k-coloring of the edges of K_n yields a monochromatic K_p. For each n below this threshold, there is a k-coloring such that every K_p receives at least 2

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colors. We can study the same problem by fixing n and asking for the minimum k such that $E(K_n)$ can be k-colored with each p-clique receiving at least 2 colors. For integers n, p, q, a (p, q)-coloring of K_n is a coloring of $E(K_n)$ in which the edges of every K_p together receive at least q colors. Let $f(n, p, q)$ denote the minimum number of colors in a (p, q)-coloring of K_n.

This function was first studied in this form by Elekes, Erdős, and Füredi (as described in Section 9 of [14]). Erdős and Gyárfás [15] improved the results about 15 years later, using the Local Lemma to prove an upper bound of $O(n^{c_{p,q}})$, where $c_{p,q} = (p-2)/(\binom{p}{2} - q + 1)$. They also determined, for each p, the smallest q such that $f(n, p, q)$ is linear in n and the smallest q such that $f(n, p, q)$ is quadratic in n. Many cases remain unresolved, most notably the growth rate of $f(n, 4, 3)$ and $f(n, 5, 9)$. In [23] it is shown that $f(n, 4, 3) < e^{O(\sqrt{\log n})}$, thereby proving that $f(n, 4, 3)$ grows more slowly than any power of n, but it remains open whether $f(n, 4, 3)/\log n \to \infty$. In [4] it is shown that

$$\frac{1+\sqrt{5}}{2} n - 3 \leqslant f(n, 5, 9) \leqslant 2n^{1 + c/\sqrt{\log n}},$$

which still leaves open the problem of determining the growth rate exactly.

In this paper we generalize this problem beyond cliques.

Definition. Given graphs G and H, and an integer $q \leqslant |E(H)|$, an (H, q)-*coloring* of G is a coloring of $E(G)$ in which the edges of every copy of $H \subseteq G$ together receive at least q colors. Let $r(G, H, q)$ denote the minimum number of colors in an (H, q)-coloring of G.

To recall some examples, we have $r(K_6, K_3, 2) > 2$ and $r(K_5, K_3, 2) = 2$.

Note that determining $r(K_n, K_p, 2)$ is hopeless, since it is equivalent to determining the classical Ramsey numbers for multicolorings. Let $r_k(H)$ be the minimum n such that every k-coloring of $E(K_n)$ yields a monochromatic copy of the subgraph H. Then $r_k(K_p) = n$ is equivalent to the statements $r(K_n, K_p, 2) > k$ and $f(K_{n-1}, K_p, 2) = k$.

Although the function $r(K_n, H, q)$ was studied (in the form $r_k(H)$) by Erdős and Rado [16] as early as 1956, and the case $r(K_{n,n}, K_{p,p}, q)$ was considered by Chvátal [11] in relation to Zarankiewicz's problem, our results and techniques have a different flavor. In our investigation of $r(K_{n,n}, K_{p,p}, q)$, we always assume that p and q are fixed and $n \to \infty$.

In Section 2 we reprove a result of Chung and Graham [8] about $r(K_n, C_4, 2)$ and extend it to $r(K_n, K_{2,t+1}, 2)$ and $r(K_{n,n}, K_{2,t+1}, 2)$, thereby proving a special case of a conjecture of theirs. Both of these Ramsey numbers are asymptotic to $\sqrt{n/t}$ as $n \to \infty$. We also observe that a recent result of Alon *et al.* [3] implies $r(K_{n,n}, K_{3,3}, 2) = n^{1/3}(1 + o(1))$. Here the corresponding colorings are obtained by algebraic constructions.

Using the Local Lemma a very general upper bound is given in Section 3. Following the Erdős–Gyárfás results on cliques, for fixed p, in Section 4 we determine the smallest q for which $r(K_{n,n}, K_{p,p}, q)$ is linear in n and the smallest q for which $r(K_{n,n}, K_{p,p}, q)$ is quadratic in n; these values are $q = p^2 - 2p + 3$ and $q = p^2 - p + 2$, respectively. In Section 5 we prove that the smallest q for which $r(K_{n,n}, K_{p,p}, q) = n^2 - O(1)$ is $q = p^2 - \lfloor p/2 \rfloor + 1$. In Section 6 we prove that the smallest q for which $r(K_{n,n}, K_{p,p}, q) = n^2 - O(n)$ is $q = p^2 - \lfloor (2p-1)/3 \rfloor + 1$. Our main tool is a density result of Brown *et al.* [7] that gives estimates on the maximum number of edges in a k-uniform hypergraph such that every u vertices span at most v edges.

In Section 7 we determine the exact value of $r(K_{n,n}, K_{3,3}, 8)$ by relating the allowable colorings to four-cycle packings of $K_{n,n}$; this value is $(3/4)\,n^2$ if n is even, and $\lceil (3/4)\,n^2 + n/4 \rceil$ if n is odd. Finally, in Section 8 we investigate $r(K_{n,n}, C_4, 3)$. We prove that $2n/3 \leqslant r(K_{n,n}, C_4, 3) \leqslant n+1$, and study a related function defined by relaxing the requirements of a $(C_4, 3)$-coloring. In a *weak* $(C_4, 3)$-coloring every copy of C_4 either has at least 3 colors or has its edges alternately 2-colored. Using Steiner Triple systems and a theorem of Pippenger and Spencer about edge-coloring k-uniform almost regular hypergraphs with small codegree, we obtain bounds for the minimum number of colors needed in a week $(C_4, 3)$-coloring of the edges of $K_{n,n}$.

2. MULTICOLOR RAMSEY NUMBERS ($q = 2$)

Let $\mathrm{ex}(G, H)$ be the maximal t such that there is a (not necessarily induced) subgraph of G with t edges not having H as a subgraph, i.e., the size of the largest H-free subgraph. Usually, $\mathrm{ex}(K_n, H)$ is called the *Turán number* of H, and $\mathrm{ex}(K_{n,n}, K_{a,b})$ is a symmetric version of the *Zarankiewicz number*. The classical upper bound for the Zarankiewicz number, due to Kővári *et al.* [22], has recently been improved in [19], where it is shown that for $1 \leqslant a \leqslant b$

$$2\,\mathrm{ex}(K_n, K_{a,b}) \leqslant \mathrm{ex}(K_{n,n}, K_{a,b}) \leqslant (b-a+1)^{1/a}\, n^{2-(1/a)} + an^{2-(2/a)} + an. \quad (1)$$

These are believed to be asymptotically optimal as $n \to \infty$. Chung and Graham [8] noticed that knowledge of the Turán number $\mathrm{ex}(K_n, G)$ can be used to deduce a lower bound on the multicolored Ramsey number $r(G, H, 2)$ through the obvious inequality

$$r(G, H, 2) \geqslant \frac{|E(G)|}{\mathrm{ex}(G, H)} \geqslant \frac{|E(G)|}{\mathrm{ex}(K_n, H)}, \quad (2)$$

where $n = |V(G)|$. Summarizing (1) and (2) we obtain the following lower bound on $r(K_{n,n}, K_{p,p}, 2)$:

$$n^{1/p}(1+o(1)) \leqslant \frac{n^2}{\mathrm{ex}(K_{n,n}, K_{p,p})} \leqslant r(K_{n,n}, K_{p,p}, 2). \tag{3}$$

It was pointed out by Spencer [8] that a standard probabilistic argument shows

$$r(G, H, 2) \leqslant r(K_n, H, 2) \leqslant \frac{n^2}{\mathrm{ex}(K_n, H)} \log n. \tag{4}$$

The following lemma connects the Ramsey numbers $r(K_{n,n}, K_{a,b}, 2)$ and $r(K_n, K_{a,b}, 2)$ in the same way as the Turán numbers $\mathrm{ex}(K_n, K_{a,b})$ and $\mathrm{ex}(K_{n,n}, K_{a,b})$ are related in (1) by Bollobás (cf. [5, p. 310]). Note that $r(K_{n,n}, K_{a,b}, 2) \leqslant r(K_{2n}, K_{a,b}, 2)$ is obvious, but this lemma enables us to determine the asymptotic values of some $r(K_{n,n}, K_{a,b}, 2)$.

LEMMA 2.1. *Suppose that* $b \geqslant 2$. *Then* $r(K_{n,n}, K_{a,b}, 2) \leqslant r(K_n, K_{a,b}, 2) + 1$.

Proof. Let $c: E(K_n) \to [m]$ be an edge-coloring of K_n without a monochromatic $K_{a,b}$. Let $V(K_n) = \{v_1, ..., v_n\}$ and $V(K_{n,n}) = A \cup B$, with $A = \{a_1, ..., a_n\}$, $B = \{b_1, ..., b_n\}$. Then the following edge-coloring c': $E(K_{n,n}) \to [m+1]$ is a $(K_{a,b}, 2)$-coloring. Let $c'(a_i, b_j) = c(v_i, v_j)$ if $i \neq j$ and $m+1$ if $i = j$. ▮

COROLLARY 2.2. $r(K_{n,n}, K_{2,2}, 2) = n^{1/2}(1+o(1))$ *and* $r(K_{n,n}, K_{3,3}, 2) = n^{1/3}(1+o(1))$.

Proof. The lower bounds for these Ramsey numbers follow from (3) while the upper bounds are implied by Lemma 2.1 and the following two asymptotics. Chung and Graham [8] proved $r(K_n, K_{2,2}, 2) = n^{1/2}(1+o(1))$ by constructing a k-coloring of the edges of K_{k^2-k+1} if $k-1$ is a power of a prime, such that no monochromatic C_4 occurs. A $(K_{3,3}, 2)$-coloring implying $r(K_n, K_{3,3}, 2) = n^{1/3}(1+o(1))$ was recently given by Alon *et al.* [3]. ▮

Chung and Graham [8] conjectured that $r(K_{n,n}, K_{s,t+1}, 2)$ is asymptotic to $(n/t)^{1/s}$ for fixed $t+1 \geqslant s \geqslant 2$ and proved it for $K_{s,t+1} = C_4$. Chung [9] proved the case $s = 2$ for some special values of t using a complicated argument based on high-dimensional projective geometries over finite fields; however, the values of t grew with n and were not fixed. The proof in [8] for C_4 used Singer's theorem on the existence of difference sets. Below we prove the conjecture for $s = 2$ and fixed $t \geqslant 1$ using a simple self-contained argument.

THEOREM 2.3. *Let t be a positive integer. Then the Ramsey numbers $r(K_{n,n}, K_{2,t+1}, 2)$ and $r(K_n, K_{2,t+1}, 2)$ are both asymptotic to $\sqrt{n/t}$ as $n \to \infty$.*

Proof. A lower bound $r(K_{n,n}, K_{2,t+1}, 2) > \sqrt{n/t} - O(n^{1/4})$ follows from (2) using (1) (or using the original bound by Kővári *et al.* [22] which was extended to multicolored graphs by Chung and Graham [8].)

To prove $r(K_n, K_{2,t+1}, 2) \leqslant (1+o(1)) \sqrt{n/t}$ we give a coloring based on the construction from [18], where it was proved that $\text{ex}(n, K_{2,t+1}) = \frac{1}{2}\sqrt{t}\, n^{3/2} + O(n^{4/3})$ for any fixed $t \geqslant 1$. Then the asymptotics for the Ramsey numbers $r(K_{n,n}, K_{2,t+1}, 2)$ follow from Lemma 2.1.

Let q be a prime power such that $(q-1)/t$ is an integer, and let $n = (q-1)^2/t$. We define a coloring c of the edges of K_n by $(q-1)/t + O(\sqrt{q} \log q)$ colors such that no monochromatic copy of $K_{2,t+1}$ occurs. Then the upper bound for the Ramsey number for all n follows from the fact that for every sufficiently large n there exists a prime q satisfying $q \equiv 1 \pmod{t}$ and $\sqrt{nt} - n^{1/3} < q < \sqrt{nt}$ (see [21]).

Let $\mathbf{F}$ be the q-element finite field, $h \in \mathbf{F}$ an element of order t, $H = \{1, h, ..., h^{t-1}\}$. H is a t-element subgroup of $\mathbf{F} \setminus \{0\}$. Let $H_1, ..., H_{(q-1)/t}$ be the cosets of H. These cosets give the decomposition $\mathbf{F} \setminus \{0\} = H_1 \cup \cdots \cup H_{(q-1)/t}$. The vertices of K_n are labeled by the t-element orbits of $(\mathbf{F} \setminus \{0\}) \times (\mathbf{F} \setminus \{0\})$ under the action of multiplication by powers of h. Thus the vertex set consists of equivalence classes in $(\mathbf{F} \setminus \{0\}) \times (\mathbf{F} \setminus \{0\})$, $n = (q-1)^2/t$, where $(a, b) \sim (x, y)$ if there is an $\alpha \in H$ such that $a = \alpha x$ and $b = \alpha y$. The class represented by (a, b) is denoted by $\langle a, b \rangle$. Color the edge joining two classes $\langle a, b \rangle$ and $\langle x, y \rangle$ with color i if $ax + by \in H_i$. This relation is symmetric, and compatible with the equivalence classes, i.e., $ax + by \in H_i$, $(a, b) \sim (a', b')$, and $(x, y) \sim (x', y')$ imply $a'x' + b'y' \in H_i$. Note that the edges $(\langle a, b \rangle, \langle x, y \rangle)$ with $ax + by = 0$ are still uncolored.

Let G_i denote the graph consisting of the edges colored i. We claim that G_i contains no copy of $K_{2,t+1}$. The proof follows [18]. We show that for $(a, b), (a', b') \in (\mathbf{F} \setminus \{0\}) \times (\mathbf{F} \setminus \{0\})$, $(a, b) \not\sim (a', b')$ these two vertices have at most t common neighbors in G_i. Consider the equation system

$$\begin{aligned} ax + by &= u \\ a'x + b'x &= v. \end{aligned} \qquad (5)$$

We claim it has at most one solution (x, y) for every $u, v \in H_i$. Indeed, the solution is unique if the determinant of the system, $\det\begin{pmatrix} a & b \\ a' & b' \end{pmatrix}$, is not 0. Otherwise, there exists an α such that $a = \alpha a'$, $b = \alpha b'$. If there exists a solution of (5) at all, then multiplying the second equation by α and subtracting it from the first one we get on the right hand side $u - \alpha v = 0$. We know that $(u/v) \in H$ hence $\alpha \in H$, contradicting the fact that (a, b) and (a', b') are not equivalent. Finally, there are t^2 possibilities for $u, v \in H_i$ in (5). The set of solutions form t-element equivalent classes, so there are at most t classes $\langle x, y \rangle$ joined simultaneously to $\langle a, b \rangle$ and $\langle a', b' \rangle$.

Now turn to the still uncolored edges $(\langle a, b \rangle, \langle x, y \rangle)$ with $ax + by = 0$. Let G_0 be the graph formed by them. We are going to finish the proof of the Theorem by coloring the edges of G_0 by an additional $O(\sqrt{q} \log q)$ colors. Partition the underlying set of K_n into equivalence classes, $V_1, ..., V_{q-1}$, of size $(q-1)/t$ as follows: $\langle a, b \rangle$ and $\langle x, y \rangle$ are in the same class if $a/b = x/y$. If $\langle a, b \rangle \in V_i$ and $\langle x, y \rangle \in V_j\ (i \neq j)$ and the edge $(\langle a, b \rangle, \langle x, y \rangle)$ is in G_0, then clearly every edge between V_i and V_j is also in G_0, and no edge in G_0 has only one endpoint in $V_i \cup V_j$. If some edge with both endpoints in V_i is in G_0, then all edges with both endpoints in either V_i or V_j are in G_0. Hence the graph G_0 consists of vertex disjoint unions of complete bipartite graphs $K_{(q-1)/t, (q-1)/t}$ joining a V_i to a V_j completely, and perhaps also some complete graphs. For these graphs we can use (4) together with the lower bound for $\mathrm{ex}(K_n, K_{2, t+1})$ from [18] to color the edges of each of them simultaneously using the same set of at most $O(\sqrt{q} \log q)$ new colors such that each color class is $K_{2, t+1}$-free. ∎

The applications of symmetric block designs to construct $K_{2, t+1}$-free graphs is not new. To cite one example, Parsons [24] extended the "Friendship Theorem" of Erdős *et al.* [17] and used symmetric (v, k, λ)-block designs admitting a polarity to obtain certain Ramsey numbers.

3. A GENERAL UPPER BOUND

Erdős and Gyárfás obtained an upper bound for $f(n, p, q)$ from the Local Lemma. Using the same method, we obtain an upper bound for $r(G, H, q)$. We always assume that G has n vertices and that H has v vertices and e edges. Below we present the symmetric version of the Lemma. For a proof, see [2].

THEOREM 3.1 (Lovász Local Lemma). *Let $A_1, A_2, ..., A_n$ be events in an arbitrary probability space. Suppose that each event A_i is mutually independent of a set of at least $n - D$ other events, and suppose that* $\Pr(A_i) \leqslant p$ *for all* $1 \leqslant i \leqslant n$. *If* $3pD \leqslant 1$, *then* $\Pr(\bigcap_i \overline{A_i}) > 0$.

THEOREM 3.2. *For graphs G, H with $n = |V(G)|$, $v = |V(H)|$, $e = |E(H)|$, and $1 \leqslant q \leqslant e$, there is a constant $c = c(H, q)$ such that*

$$r(G, H, q) < cn^{(v-2)/(e-q+1)}.$$

Proof. If $q=1$, the result is trivial, so assume that $q\geqslant 2$. Color the edges of G independently with t colors, where the colors are assigned with equal probability. The probability that a given copy of H receives at most $q-1$ colors is bounded by

$$P=\binom{t}{q-1}\left(\frac{q-1}{t}\right)^{e}<t^{q-1}\left(\frac{q-1}{t}\right)^{e}.$$

Furthermore, the coloring of a fixed H is independent of the colorings of all other H's except those that intersect it in at least one edge. The number of these is at most

$$D=e\binom{n}{v-2}<en^{v-2}.$$

Choosing t sufficiently large we obtain $3PD<1$, e.g., let

$$t\geqslant(3(q-1)^{e}\,e)^{1/(e+1-q)}\,n^{(v-2)/(e-q+1)}. \tag{6}$$

The Local Lemma therefore implies that if t is at least this large, then an (H,q)-coloring with t colors exists. We also obtained that $c(H,q)\leqslant(3(q-1)^{e}\,e)^{1/(e+1-q)}$. ∎

COROLLARY 3.3. $r(K_{n,n},K_{p,p},q)\leqslant c(K_{p,p},q)\,n^{(2p-2)/(p^2-q+1)}$.

4. THRESHOLDS FOR LINEAR AND QUADRATIC $R(K_{n,n},K_{p,p},q)$

For fixed p, we find the smallest q for which $r(K_{n,n},K_{p,p},q)$ is linear in n, and the smallest q for which $r(K_{n,n},K_{p,p},q)$ is quadratic in n. It turns out that these values are fairly close.

THEOREM 4.1. *Suppose that $q=e-v+3$ and that H is connected. Then*

$$\frac{n-1}{2v-4}\leqslant r(K_n,H,q)<cn,$$

and $r(K_n,H,q-1)\leqslant c'n^{1-1/(v-1)}$ for some constants $c=c(H,q)$ and $c'=c(H,q-1)$.

Proof. The upper bounds follow from Theorem 3.2. For the lower bound, it is sufficient to show that in an (H,q)-coloring of K_n each color class contains at most $(v-2)\,n$ edges. Let S be a spanning tree of H. A monochromatic copy of S can be completed to a copy of H with a total of at most $e-(v-1)+1=q-1$ colors. Thus each color class of $E(K_n)$ contains at most $\text{ex}(K_n, S)$ edges. It is well-known (and easy, see [5]) that $\text{ex}(K_n, S) \leqslant (v-2)\,n$. ∎

Note that the Erdős–Sós Conjecture (i.e., $\text{ex}(K_n, S) \leqslant (v-2)\,n/2$; for latest developments, see Ajtai *et al.* [1]) would yield a twice larger lower bound.

The above proof can easily be modified to give the following. Let S be a spanning tree of the bipartite graph H. Then $n^2/\text{ex}(K_{n,n}, S) \leqslant r(K_{n,n}, H, e-v+3)$. Let $S_{a,b}$ denote the double star, a spanning tree of $K_{a,b}$ with an adjacent pair of degrees a and b. By considering edges with an endpoint at a vertex of small degree, it is easy to see that $\text{ex}(K_{n,n}, S_{a,b}) < 2n(b-1)$ for $b \geqslant a$. Thus we have the following.

Corollary 4.2. *Fix $p \geqslant 2$. If $q=p^2-2p+3$, then $r(K_{n,n}, K_{p,p}, q)$ is linear in n, in particular, $\frac{n}{2p-2} < r(K_{n,n}, K_{p,p}, q) < c(K_{p,p}, q)\,n$. On the other hand, $r(K_{n,n}, K_{p,p}, q-1) \leqslant c(K_{p,p}, q-1)\,n^{1-1/(2p-1)}$, where the values of c come from Theorem* 3.2.

Remark. It can easily be shown from (6) in the proof of Theorem 3.2 that $c(K_{p,p}, q-1) < 3p^{p+2}$.

Next we compute the threshold for quadratic $r(K_{n,n}, K_{p,p}, q)$.

Theorem 4.3. *Let $q=p^2-p+2$, $p \geqslant 3$. Then $r(K_{n,n}, K_{p,p}, q) \geqslant C(n^2-n)$, where $C=(\lfloor p/2\rfloor^2+\lfloor p/2\rfloor+1)/(\lfloor p/2\rfloor^3+\lfloor p/2\rfloor^2+\lfloor p/2\rfloor+1)$. If $p \geqslant 6$ and $n > p^{3/2}$, then $n^{4/3}-2n^{2/3}+1 \leqslant r(K_{n,n}, K_{p,p}, q-1) \leqslant c'n^{2-2)/p}$ for $c' \leqslant 2c(K_{p,p}, q-1)$.*

Proof. Let $E(K_{n,n}) = C_1 \cup \cdots \cup C_r$ be a $(K_{p,p}, q)$-coloring. Then every color class has at most $p-1$ edges; hence $r(K_{n,n}, K_{p,p}, q) \geqslant n^2/(p-1)$ is immediate. Next we improve the coefficient $1/(p-1)$ to C. Here C is slightly less then $1/\lfloor p/2\rfloor$ and (as it will be shown in Theorem 7.1) gives the right coefficient of n^2 for $p=3$.

Denote the partite sets of $K_{n,n}$ by X and Y. Let e_i denote the size of C_i, let V_i be the set of vertices incident to an edge of C_i, and let $E_i = K_{n,n} \mid V_i$ be the edges contained in V_i. Call a color class C_i *large* if $e_i \geqslant \lfloor p/2\rfloor+1$. Let $\ell+m$ be the number of large color classes. Suppose that ℓ of these, $C_1, ..., C_\ell$ are matchings, but for each C_i with $\ell < i \leqslant \ell+m$ one can find a vertex v_i incident to at least two edges of color i. For the rest of the colors $C_{\ell+m+1}, ..., C_r$ we have $e_i \leqslant \lfloor p/2\rfloor$.

We claim that $\sum_{\ell < i \leqslant \ell+m} e_i \leqslant n$. Indeed, if $\ell < i < j \leqslant \ell+m$, then the vertices of degree at least 2, v_i and v_j belong to the same partite set X or Y. Assume that $v_i, v_j \in Y$. Then $V_i \cap V_j \cap X = \varnothing$, implying that large color classes which are not matchings altogether span at most n edges. Considering the three types of colors we obtain

$$n^2 = \sum e_i = \sum (e_i - \lfloor p/2 \rfloor) + r \lfloor p/2 \rfloor \leqslant \sum_{1 \leqslant i \leqslant \ell} (e_i - \lfloor p/2 \rfloor) + n + r \lfloor p/2 \rfloor. \quad (7)$$

We may suppose that $\ell \geqslant 2$, otherwise a slightly sharper version of (7) gives a better lower bound than $C(n^2 - n)$. For each large color class C_i which is a matching observe that $|V_i| = 2e_i$, and $|E_i| = e_i^2$. It follows that for $1 \leqslant i < j \leqslant \ell$ we have $E_i \cap E_j = \emptyset$. Even more, if $e, e' \in E_i \cup E_j$, then e and e' have different colors unless they both belong to one of C_i or C_j. Thus, letting t denote the number of distinct colors in $\bigcup_{1 \leqslant i \leqslant \ell} E_i$, we have

$$\sum_{1 \leqslant i \leqslant \ell} (e_i^2 - e_i + 1) = t \leqslant r. \quad (8)$$

Let $\alpha = 1/(\lfloor p/2 \rfloor^2 + \lfloor p/2 \rfloor + 1)$. Multiplying (8) by α, adding the result to (7), and rearranging yields

$$n^2 - n \leqslant r(\lfloor p/2 \rfloor + \alpha) + \sum_{1 \leqslant i \leqslant \ell} (e_i - \lfloor p/2 \rfloor - \alpha(e_i^2 - e_i + 1)).$$

Since $x - \lfloor p/2 \rfloor - \alpha(x^2 - x + 1) \leqslant 0$ for $x \geqslant \lfloor p/2 \rfloor + 1$, the number of colors r is at least $(n^2 - n)/(\lfloor p/2 \rfloor + \alpha)$, as claimed.

Now we are going to prove the polynomial bounds for $r(K_{n,n}, K_{p,p}, q-1)$. The upper bound follows from Theorem 3.2. For the lower bound, consider a $(K_{p,p}, q-1)$-coloring of $K_{n,n}$. If every color class has at most $n^{2/3}$ edges, then the total number of color classes is at least $n^{4/3}$. We may therefore suppose that there is a color class $C \subseteq E(K_{n,n})$ of size at least $n^{2/3} > p$. Let V_C be the set of vertices incident to an edge from C, let $V_X = V_C \cap X$, and let $V_Y = V_C \cap Y$. Let G be the graph formed by the edges in C, i.e., $V(G) = V_C$ and $E(G) = C$. If there exist $x \in V_X$ and $y \in V_Y$ with $\min\{d_G(x), d_G(y)\} \geqslant 2$, then there is a $(q-2)$-colored $K_{p,p}$, (containing $p+1$ edges from C) so we may assume by symmetry that $d_G(x) \leqslant 1$ for all $x \in V_X$. Thus $|V_X| \geqslant \max\{n^{2/3}, |V_Y|\}$. Let $H \subseteq K_{n,n}$ be the complete bipartite graph spanned by V_C. Observe that all edges other than the edges from C have distinct colors in H. If $|V_Y| \leqslant |V_X| \leqslant |V_Y| + 1$ then the number of colors on $E(H)$ is at least

$$|V_X|\,|V_Y| - |V_X| + 1 \geqslant |V_X|(|V_Y| - 1) + 1 \geqslant n^{4/3} - 2n^{2/3} + 1.$$

If $|V_X| > |V_Y| + 1$, then either there are $u, v \in V_Y$ with $d_G(u) \geqslant 2$ and $d_G(v) \geqslant 2$, or there is a $w \in V_Y$ with $d_G(w) \geqslant 3$. In the first case, let $Y' = Y - \{u, v\}$, and in the second case, let $Y' = Y - \{w\}$. Since $p \geqslant 6$, there are no repeated colors on the edges between Y' and V_X except the color on edges of C; otherwise there would be a $K_{p,p}$ using $q-2$ colors including either u and v or w. Thus the total number of colors is at least $(n-3)|V_X|+1 \geqslant n^{4/3}-2n^{2/3}+1$. ▮

5. WHEN IS $r(K_{n,n}, K_{p,p}, q)=n^2-O(1)$?

In this section we determine, for fixed p, the threshold for q beyond which all edges but a constant number must be colored with distinct colors. We also determine an infinite family of Ramsey numbers.

THEOREM 5.1. *If* $q \geqslant p^2-\lfloor p/2 \rfloor+1$, *then* $r(K_{n,n}, K_{p,p}, q)=n^2-(p^2-q)$. *However,* $f(K_{n,n}, K_{p,p}, p^2-\lfloor p/2 \rfloor) \leqslant n^2-\lfloor n/2 \rfloor$, *with equality for* $p \geqslant 7$ *and* p *odd, and* $r(K_{n,n}, K_{5,5}, 23)=n^2-2\lfloor n/2 \rfloor+2$. *Moreover,* $r(K_{n,n}, K_{p,p}, p^2-\lfloor p/2 \rfloor)=n^2-\lceil n/2 \rceil$ *for* $p \geqslant 14$ *and* p *even.*

Proof. The upper bounds for $r(K_{n,n}, K_{p,p}, q)$ are provided by the following constructions. Suppose that the partite sets of $K_{n,n}$ are $X=\{x_1, ..., x_n\}$ and $Y=\{y_1, ..., y_n\}$.

When $q \geqslant p^2-\lfloor p/2 \rfloor+1$, color the edges $x_{2i-1}y_{2i-1}$ and $x_{2i}y_{2i}$ with color i, for $1 \leqslant i \leqslant p^2-q$. When $q=p^2-\lfloor p/2 \rfloor$, color in the same way, except let $1 \leqslant i \leqslant \lfloor n/2 \rfloor$. In both cases, color all the other edges with new distinct colors. The total number of colors used is $n^2-(p^2-q)$ in the first case, and $n^2-\lfloor n/2 \rfloor$ in the second case. When n is odd and p is even, we can also color the pair x_1y_n and y_3x_n with the same color; this saves one color, giving only $n^2-\lceil n/2 \rceil$ colors.

Our construction for $r(K_{n,n}, K_{5,5}, 23)$ is slightly different. For $2 \leqslant i \leqslant \lfloor n/2 \rfloor$, let x_1y_{2i-1} and x_2y_{2i} have the same color, with different pairs getting distinct colors. Similarly, let y_1x_{2i-1} and y_2x_{2i} have the same color (but distinct from the previous color set), with different pairs getting distinct colors. Give all other edges new distinct colors. This is a $(K_{5,5}, 23)$-coloring, since no $K_{5,5}$ contains three monochromatic matchings. The number of colors is $n^2-2\lfloor n/2 \rfloor+2$.

To prove the lower bounds consider a $(K_{p,p}, q)$-coloring with r colors. Let $e_1^i, e_2^i, ..., e_s^i$ be the edges of color i with $s \geqslant 2$. Form the 4-element sets F_j^i $(1 \leqslant j < s)$ by taking the union $e_1^i \cup e_{j+1}^i$ and adding an arbitrary additional vertex of $X \cup Y$ if needed in such a way that $|F_j^i \cap X|=|F_j^i \cap Y|=2$. Finally, let $\mathbf{F}$ be the edge-set of the (multi)hypergraph of the four-tuples obtained in this way. Since every color class with t edges gives rise to precisely $t-1$ four-tuples, $|\mathbf{F}|=n^2-r$. For p-element sets $X' \subset X$, $Y' \subset Y$, we have

$$X' \cup Y' \text{ contains at most } p^2-q \text{ members of } \mathbf{F}. \tag{9}$$

If $2(p^2-q+1)\leqslant p$ and $|\mathbf{F}|>p^2-q$, then we can take p^2-q+1 four-tuples from $\mathbf{F}$ that are contained in a copy H of $K_{p,p}$; $E(H)$ will have fewer than q colors. Hence if $2(p^2-q+1)\leqslant p$, then $|\mathbf{F}|\leqslant p^2-q$ and we are done.

Call a member $F\in\mathbf{F}$ of type X (type Y) if no other member of $\mathbf{F}$ contains any vertex from $F\cap X$ ($F\cap Y$, resp.). If each edge is of type X then $|\mathbf{F}|\leqslant|X|/2$ and we are done. The same is true for type Y.

Suppose that F_1 is not of type X and F_2 is not of type Y, for example $F_1\cap F_3\cap X\neq\varnothing$ and $F_2\cap F_4\cap Y\neq\varnothing$. Suppose first, that these four sets are distinct members of $\mathbf{F}$. In the case p odd, $p\geqslant 7$ adding $(p-7)/2$ arbitrary additional members to F_1, F_2, F_3, F_4 we get a contradiction to (9) and we are done.

Suppose that $\mathbf{F}$ contains another 4 members $F_5, ..., F_8$ such that $F_5\cap F_7\cap X\neq\varnothing$ and $F_6\cap F_8\cap Y\neq\varnothing$. In the case p even, $p\geqslant 14$, adding $(p-14)/2$ arbitrary additional members of $\mathbf{F}$ to $F_1, ..., F_8$ we get a contradiction to (9) and we are done.

In case of coincidences among $F_1, ..., F_4$ one needs to add more members. The details are omitted.

It remains to consider the case $p=5$. Suppose that F_1 is of neither type, i.e., $F_1\cap F_2\cap X\neq\varnothing$, and $F_1\cap F_3\cap Y\neq\varnothing$. Then $F_1\cup F_2\cup F_3$ can be covered by the vertex set of a $K_{5,5}$ (by a $K_{3,3}$ when F_2 coincides with F_3) a contradiction to (9). Thus every member is of type X or of type Y. If each member is of both types we obtain $|\mathbf{F}|\leqslant(2n)/4$; otherwise we have $|\mathbf{F}|\leqslant 2\lfloor(n-2)/2\rfloor$. ▮

6. DENSITIES OF HYPERGRAPHS

In this section we determine, for fixed p, the threshold for q beyond which all edges but $\Theta(n)$ must be colored with distinct colors. Our main tool is an estimate of the size of a hypergraph with bounded densities of small subhypergraphs.

A k-uniform hypergraph with edge-set $\mathbf{F}$ is called $(u, v+1)$-free if every u vertices span at most v members of $\mathbf{F}$. Let $g_k(n, u, v)$ be the maximum number of edges of a $(u, v+1)$-free k-uniform hypergraph with n vertices. Turán's classical theorem determines $g_2(n, u, \binom{u}{2}-1)$; for example, $g_2(n, 3, 2)=\lfloor n^2/4\rfloor$. For a recent account on graph-density questions see Griggs *et al.* [20]. Brown *et al.* [7] proved that

$$g_k(n, u, v)>cn^{k-(u-k)/v}$$

by constructing a $(u, v+1)$-free k-uniform hypergraph on n vertices with $cn^{k-(u-k)/v}$ edges (here $c=c(k, u, v)>0$ is independent of n). Consider the hypergraph $\mathbf{H}$ on $2n$ vertices obtained from their construction for $k=4$ and

$u = 2p$. Their proof also implies that for the case $v \leqslant p-2$ one can also suppose that $|H \cap H'| \leqslant 1$ for all $H, H' \in \mathbf{H}$. Randomly partition the vertices of $\mathbf{H}$ into two equal sets X and Y. As the probability that a 4-element set is partitioned into two equal parts is 6/16, this yields a family of 4-subsets $\mathbf{F} = \{F_1, ..., F_m\}$ of an underlying set $X \cup Y$ such that every $2p$-element subset contains at most v of the F_i's and

(1) $|F_i \cap X| = |F_i \cap Y| = 2$ for every F_i,

(2) $|F_i \cap F_j| \leqslant 1$ for $i \neq j$ (assuming that $v \leqslant p-2$), and

(3) $m > c_p n^{4-(2p-4)/v}$, where $c_p > 0$ depends only on p.

(Here c_p is smaller than the constant in the result of Brown *et al.*) Now replace each 4-element set F_i by two disjoint pairs contained in it connecting X to Y, color these two edges with color i, and color the rest of the pairs between X and Y with distinct new colors. Since the total number of colors used is $n^2 - m$, we obtain

$$r(K_{n,n}, K_{p,p}, p^2 - v) < n^2 - c_p n^{4-(2p-4)/v} \tag{10}$$

for $1 \leqslant v \leqslant p-2$ and some constant $c_p > 0$.

THEOREM 6.1. *If* $p^2 - \lfloor (2p-1)/3 \rfloor + 1 \leqslant q \leqslant p^2 - \lfloor p/2 \rfloor$, *then* $n^2 - 2\lfloor (p-2)/3 \rfloor (n-1) < r(K_{n,n}, K_{p,p}, q) \leqslant n^2 - \lfloor p/2 \rfloor$. *However*, $r(K_{n,n}, K_{p,p}, p^2 - \lfloor (2p-1)/3 \rfloor) < n^2 - c_p n^{1+\varepsilon_p}$. *Here* c_p *and* ε_p *are positive constants depending only on* p.

Proof. The upper bound in the last statement follows from (10) by letting $v = \lfloor (2p-1)/3 \rfloor$.

For the case $\lfloor p/2 \rfloor \leqslant p^2 - q < \lfloor (2p-1)/3 \rfloor$, a $(K_{p,p}, q)$-coloring with $n^2 - \lfloor n/2 \rfloor$ colors was given in Section 5. We have to prove that all such colorings use more than $n^2 - 2\lfloor (p-2)/3 \rfloor (n-1)$ colors. Let E be the set of edges whose color is used also on at least one other edge, and let $G \subseteq K_{n,n}$ be the subgraph spanned by E. Set $t = \lfloor (p+1)/3 \rfloor$. We claim that if $u, v \in V(G)$ with $d_G(u), d_G(v) \geqslant t$, then $uv \notin E$, i.e., high degree vertices in G are nonadjacent in G. To prove this claim, suppose that $uv \in E$.

Case 1. $p \not\equiv 1 \pmod 3$. Then there is a $K_{p,p}$ containing a pair of edges of each color that appears on the edges incident with uv (and perhaps some more pairs e_i, f_i, with color i if there are colors adjacent to both u and v). The number of colors on this copy is at most $p^2 - (2t-1) < q$, a contradiction.

Case 2. $p \equiv 1 \pmod 3$. Then $3t - 1 = p - 2$, so in addition to the edges in the previous case, our copy of $K_{p,p}$ can be chosen to contain another 2 edges with the same color. The number of colors on this copy is $p^2-(2t-1)-1<q$, a contradiction.

Counting the edges in E by their endpoint of lower degree gives $|E| \leqslant 2(n-1)(t-1)$, which yields the required lower bound on the number of colors. ∎

The coefficient $2\lfloor (p-2)/3 \rfloor$ in Theorem 6.1 can be improved by choosing t more carefully, noting its dependence on q. We could also include the colors from the nontrivial color classes. We do not attempt to find the optimal bound.

Note that by substituting $v=p-2$ into (10) we obtain a matching upper bound for $q=p^2-p+2$ (cf. Theorem 4.3):

$$r(K_{n,n}, K_{p,p}, p^2-p+2) < (1-c_p)\, n^2. \tag{11}$$

7. THE EXACT VALUE OF $r(K_{n,n}, K_{3,3}, 8)$

When $p=3$ and $2 \leqslant q \leqslant 8$, our upper bounds are those in Theorem 3.2. (See the chart in Section 9.) We have nontrivial lower bounds only for $q \in \{6, 8\}$. Corollary 4.2 states that $n/4 < r(K_{n,n}, K_{3,3}, 6) \leqslant cn$ for some constant c. Theorem 4.3 states that $(3/4)(n^2-n) < r(K_{n,n}, K_{3,3}, 8)$. In this section we give the exact value of this Ramsey number.

THEOREM 7.1. *$r(K_{n,n}, K_{3,3}, 8)$ is $(3/4)\, n^2$ if n is even, and $\lceil (3/4)\, n^2 + n/4 \rceil$ if n is odd.*

Proof. First, we show the upper bound by constructing the colorings. Let the partite sets of $K_{n,n}$ be $X=\{x_1, ..., x_n\}$ and $Y=\{y_1, ..., y_n\}$. We color $E(K_{n,n})$ with ordered pairs as follows.

Case 1. $n=2k$. For $i, j \in \{1, ..., \lfloor n/2 \rfloor\}$, let $x_{2i}y_{2j}$ and $x_{2i-1}y_{2j-1}$ both have color (i, j). Let all other edges have new distinct colors. Since every $K_{3,3}$ has at most one pair of edges of the form $x_{2i}y_{2j}$, $x_{2i-1}y_{2j-1}$, our construction is a $(K_{3,3}, 8)$-coloring.

Case 2. $n=4k+1$. Let $c(x_{j-(2i-1)}y_j)=c(x_{j+2i}y_{j+1})=(j, i)$ for $1 \leqslant i \leqslant (n-1)/4$, $1 \leqslant j \leqslant n$, where addition is taken modulo n. Color all other edges with distinct colors. Since the union of any two color classes of size 2 spans at least four vertices either in X or in Y, every $K_{3,3}$ has at most one color class of size 2. Thus our construction is a $(K_{3,3}, 8)$-coloring.

Case 3. $n=4k+3$. Let $c(x_{j-(2i-1)}y_j)=c(x_{j+2i}y_{j+1})=(j, i)$ as before for $1 \leqslant i \leqslant (n-3)/4$, $1 \leqslant j \leqslant n$, and let $c(x_iy_i)=c(x_{i+(n-1)/2}y_{i+(n-1)/2})=(i, 0)$ for $1 \leqslant i \leqslant (n-1)/2$. Color all other edges with distinct colors. It is easy to check that the color classes of size 2 induce edge-disjoint copies of C_4, so the obtained coloring is a $(K_{3,3}, 8)$-coloring. The total number of colors in the last case is $n^2 - n(n-3)/4 - (n-1)/2 = (3/4)n^2 + n/4 + 1/2$.

For the lower bound, consider a $(K_{3,3}, 8)$-coloring of $K_{n,n}$. Obviously, each color class contains at most two edges. Let $C_1, ..., C_t$ be the color classes of two edges, $C_i = \{\{x_1^i, y_1^i\}, \{x_2^i, y_2^i\}\}$. We are going to define t edge-disjoint cycles of length 4, $Q_1, ..., Q_t$, $C_i \subset Q_i$. Consider a color class C_i forming $2K_2$, i.e., $x_1^i \neq x_2^i$ and $y_1^i \neq y_2^i$. Then let Q_i be the four edges spanned by the vertices of C_i. Consider a color class C_i forming P_3, for example $x_1^i = x_2^i$ and $y_1^i \neq y_2^i$. Then choose a vertex x_3^i arbitrarily from $X \setminus \{x_1^i\}$, and let Q_i be spanned by $\{x_1^i, x_3^i, y_1^i, y_2^i\}$. It is easy to check that edges of Q_i and Q_j are disjoint for $i \neq j$.

Finally we need an upper bound for the number of edge-disjoint four-cycles. Each $x \in X$ is contained in at most $n/2$ of the Q_i's, thus $2t \leqslant n \lfloor n/2 \rfloor$. ∎

It is easy to see that although our construction is not unique, every optimal $(K_{3,3}, 8)$-coloring contains no two adjacent edges of the same color, and the coloring can be obtained from a four-cycle packing of $K_{n,n}$. Note that in the same way one can show that if $m, n \geqslant 3$, then $r(K_{n,m}, K_{3,3}, 8) = nm - t$, where t is the maximum number of edge-disjoint four-cycles packed into $E(K_{n,m})$. On the other hand (denoting this maximum t by $t(m, n)$) one can easily extend the above constructions, or use a recurrence like $t(m, n) \geqslant t(m, n-2) + \lfloor m/2 \rfloor$ to determine the exact value of t. This yields

$$r(K_{n,m}, K_{3,3}, 8) = nm - \min\left\{\left\lfloor \frac{n}{2} \left\lfloor \frac{m}{2} \right\rfloor \right\rfloor, \left\lfloor \frac{m}{2} \left\lfloor \frac{n}{2} \right\rfloor \right\rfloor\right\}. \tag{12}$$

8. BOUNDS FOR $r(K_{n,n}, C_4, 3)$

The next case we consider is $r(K_{n,n}, C_4, 3)$. Since monochromatic P_4's are forbidden, each color class consists of disjoint stars. Using this observation, it is easy to prove that $r(K_{n,n}, C_4, 2) \geqslant n^2/(2n-2) \sim n/2$. Later we improve this lower bound, but first we provide a simple construction.

THEOREM 8.1. *If n is odd, then $r(K_{n,n}, C_4, 3) \leqslant n$. If n is even, then $r(K_{n,n}, C_4, 3) \leqslant n+1$.*

Proof. First suppose that n is odd. Let the partite sets of $K_{n,n}$ be $X = \{x_1, ..., x_n\}$ and $Y = \{y_1, ..., y_n\}$. Color $E(K_{n,n})$ with n colors by letting the jth color class consist of the edges $x_i y_{i+j}$, $1 \leqslant i \leqslant n$, $0 \leqslant j \leqslant n-1$, where the subscripts are taken modulo n.

Since each color class is a matching, a 2-colored C_4 must consist of 2 monochromatic matchings of size 2. Assume without loss of generality that one of these matchings is in color 0 and that the four vertices of the 4-cycle are x_1, y_1, x_k, y_k. Since n is odd, $n+1-k \neq k-1$. Thus $x_1 y_k$ and $x_k y_1$ have distinct colors, and our construction is a $(C_4, 3)$-coloring.

When n is even we color the edges of $K_{n+1, n+1}$ as before and consider the coloring restricted to $K_{n, n}$. This gives an upper bound of $n+1$. ∎

Improving this upper bound seems to be very hard. Eichhorn [13] improved it by one when $n=4$, 12, 20, 36, and 60 by exhibiting $(C_4, 3)$-colorings of $K_{n, n}$ with n colors. In the matrix below, the (i, j)th entry represents the color of $x_i y_j$, where the partite sets of G are $X=\{x_1, ..., x_n\}$ and $Y=\{y_1, ..., y_n\}$. A construction for $n=12$ is shown:

$$\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 12 & 9 & 10 & 11 \\
4 & 1 & 2 & 3 & 8 & 5 & 6 & 7 & 12 & 11 & 9 & 10 \\
4 & 3 & 1 & 2 & 6 & 8 & 7 & 5 & 9 & 10 & 11 & 12 \\
2 & 4 & 3 & 1 & 8 & 7 & 5 & 6 & 10 & 12 & 11 & 9 \\
5 & 6 & 7 & 8 & 12 & 9 & 10 & 11 & 1 & 2 & 3 & 4 \\
8 & 5 & 6 & 7 & 12 & 11 & 9 & 10 & 4 & 1 & 2 & 3 \\
6 & 8 & 7 & 5 & 9 & 10 & 11 & 12 & 4 & 3 & 1 & 2 \\
8 & 7 & 5 & 6 & 10 & 12 & 11 & 9 & 2 & 4 & 3 & 1 \\
12 & 9 & 10 & 11 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
12 & 11 & 9 & 10 & 4 & 1 & 2 & 3 & 8 & 5 & 6 & 7 \\
9 & 10 & 11 & 12 & 4 & 3 & 1 & 2 & 6 & 8 & 7 & 5 \\
10 & 12 & 11 & 9 & 2 & 4 & 3 & 1 & 8 & 7 & 5 & 6
\end{pmatrix}$$

$$r(K_{12, 12}, C_4, 3) \leqslant 12.$$

We have already observed that $r(K_{n, n}, C_4, 2) \geqslant n/2$. Through a more careful examination of both the structure of each color class, and the interaction between color classes in a $(C_4, 3)$-coloring, we improve the lower bound to $2n/3$.

THEOREM 8.2. $r(K_{n, n}, C_4, 3) > \lfloor \frac{2n}{3} \rfloor$.

Proof. Consider a $(C_4, 3)$-coloring of $K_{n, n}$ with color classes $D_1, D_2, ..., D_g$. Suppose that the ith color class D_i consists of l_i disjoint stars S_{ij}, where $1 \leqslant j \leqslant l_i$. Let $S_{i, j}$ have $d_{i, j}$ edges, and set $L = \sum_i l_i$.

Since every edge is covered once and $\sum_{j=1}^{l_i} d_{i,j} \leqslant 2n - l_i$, we have

$$n^2 = \sum_{i=1}^{g} \sum_{j=1}^{l_i} d_{i,j} \leqslant 2ng - L. \tag{13}$$

Two monochromatic paths of length 2 with common endpoints would yield a 2-colored C_4. Letting t denote the number of monochromatic paths of length 2, we thus obtain

$$\sum_{i=1}^{g} \sum_{j=1}^{l_i} \binom{d_{i,j}}{2} = t \leqslant 2\binom{n}{2}. \tag{14}$$

From (14) we obtain

$$\frac{\sum_{i=1}^{g} \sum_{j=1}^{l_i} (d_{i,j})^2}{\sum_{i=1}^{g} \sum_{j=1}^{l_i} d_{i,j}} = \frac{2\sum_{i=1}^{g} \sum_{j=1}^{l_i} \binom{d_{i,j}}{2} + \sum_{i=1}^{g} \sum_{j=1}^{l_i} d_{i,j}}{\sum_{i=1}^{g} \sum_{j=1}^{l_i} d_{i,j}} \leqslant \frac{3n^2 - 2n}{n^2}. \tag{15}$$

Since the double sum in (14) has $\sum_i l_i = L$ terms, the Cauchy–Schwarz inequality yields

$$\left(\sum_{i=1}^{g} \sum_{j=1}^{l_i} d_{i,j} \right)^2 \leqslant L \left(\sum_{i=1}^{g} \sum_{j=1}^{l_i} (d_{i,j})^2 \right). \tag{16}$$

By rearranging (16) and using (15), we obtain $L \geqslant n^3/(3n-2)$. Substituting back into (13) gives $2ng \geqslant n^2 + L \geqslant n^2 + n^3/(3n-2)$. Solving for g yields

$$g \geqslant \left\lceil \frac{n(2n-1)}{3n-2} \right\rceil > \left\lfloor \frac{2n}{3} \right\rfloor. \quad \blacksquare$$

An *alternating* C_4 is a 2-colored C_4 whose edges alternate between its two colors when viewed cyclically. One might feel there is hope in improving the lower bound above because the proof allows alternating C_4's. Unfortunately, we have been unable to obtain any significant improvement from this observation. It is, however, interesting to define a function similar to $r(K_{n,n}, C_4, 3)$ with the exception that alternating C_4's are permitted.

DEFINITION. A *weak* $(C_4, 3)$-coloring of $K_{n,n}$ is a coloring of the edges of $K_{n,n}$ in which every copy of C_4 has at least three colors or is alternately 2-colored. Let $r'(K_{n,n}, C_4, 3)$ denote the minimum number of colors in a weak $(C_4, 3)$-coloring of $K_{n,n}$.

Since the definition of $r'(K_{n,n}, C_4, 3)$ is a relaxation of that of $r(K_{n,n}, C_4, 3)$, we certainly have $r(K_{n,n}, C_4, 3) \geqslant r'(K_{n,n}, C_4, 3)$. Furthermore, the proof of Theorem 7.2 yields

$$r'(K_{n,n}, C_4, 3) > \left\lfloor \frac{2n}{3} \right\rfloor.$$

In the remaining part of this section we prove an upper bound on $r'(K_{n,n}, C_4, 3)$ that is asymptotic to $3n/4$. The proof requires a deep theorem about edge-coloring of hypergraphs. We describe this first.

Given a hypergraph $H = (V, E)$, the *degree* of a vertex $v \in V$, $d(v)$, is the number of edges containing v. For vertices v, w, the *codegree* of v and w, $\text{cod}(v, w)$, is the number of edges containing both v and w. Let

$$\Delta(H) = \max_{v \in V} d(v),$$

$$\delta(H) = \min_{v \in V} d(v),$$

$$C(G) = \max_{u, v \in V, u \neq v} \text{cod}(u, v).$$

A *matching* in H is a set of pairwise disjoint edges of H. A matching is *perfect* if every vertex of H is in exactly one of its edges. Let $\chi'(H)$, the *chromatic index* of H, denote the minimum number of matchings needed to partition the edges of H. A hypergraph H is k-uniform if each of its edges consists of exactly k elements.

THEOREM 8.3 (Pippenger and Spencer [25]). *For every $k \geqslant 2$ and $\varepsilon > 0$, there exist $\varepsilon' > 0$ and n_0 such that if H is a k-uniform hypergraph on $n(H) \geqslant n_0$ vertices satisfying*

$$\delta(H) \geqslant (1 - \varepsilon')\, \Delta(H) \tag{17}$$

and

$$C(H) \leqslant \varepsilon' \Delta(G), \tag{18}$$

then

$$\chi'(H) \leqslant (1 + \varepsilon)\, \Delta(G). \tag{19}$$

We rephrase Theorem 8.3 in more convenient asymptotic notation.
Let $H_1, H_2, \ldots$ be hypergraphs, with $|V(H_i)| \to \infty$. If

$$\delta(H_n) \sim \Delta(H_n) \tag{20}$$

and

$$C(H_n) = o(\Delta(H_n)), \tag{21}$$

then

$$\chi'(H_n) \sim \Delta(H_n). \tag{22}$$

A *Steiner Triple System* (STS) is a 3-uniform hypergraph in which each pair of vertices has codegree one. It is well known that a STS on n points exists if and only if $n \equiv 1, 3 \pmod 6$.

We use Steiner Triple Systems and the following "large deviation" result in probability theory to prove an upper bound on $r'(K_{n,n}, C_4, 3)$.

THEOREM 8.4 (Chernoff [12]). *Suppose that* $p \in [0, 1]$ *and* $X_1, \ldots, X_n$ *are mutually independent random variables with* $\Pr(X_i = 1) = p = 1 - \Pr(X_i = 0)$. *If* $X = X_1 + \cdots + X_n$ *and* $a > 0$, *then* $\Pr(|X - pn| > a) \leqslant 2e^{-2a^2/n}$.

THEOREM 8.5. *As* $n \to \infty$, $r'(K_{n,n}, C_4, 3) \leqslant \frac{3n}{4}(1 + o(1))$.

Proof. We first prove the result for a sufficiently dense set of positive integers. Later we use standard approximation arguments to obtain the result asymptotically for all n. Suppose that $2n + 1 \equiv 1, 3 \pmod 6$, and let S be a STS of $[2n+1]$. Select a set $A \subseteq [2n+1]$ by picking each point of $[2n+1]$ with probability 1/2, independently. Let $A \subseteq [2n+1]$ be the (random) set of points thus picked, and let H be the 3-uniform hypergraph with vertex set $[2n+1]$ and edges from the STS that intersect both A and $[2n+1] - A = B$.

The calculations in the following paragraphs will show that, with high probability, the sizes of A and B differ by very little. Also, the degree of each vertex in H is close to $3n/4$. Since H is 3-uniform and has codegree bounded by 1, the hypothesis for Theorem 8.3 will be satisfied and we therefore obtain a proper edge-coloring of H with about $3n/4$ colors. This coloring of $E(H)$ will yield a weak $(C_4, 3)$-coloring of the underlying bipartite graph with bipartition A, B.

Set $a = |A|$ and $b = |B|$. Let X be the event that $|a - n| \leqslant 2\sqrt{n}$, and let Y be the event that $|d_H(i) - 3n/4| \leqslant \sqrt{n \log(10n)/2}$ for all $i \in [2n+1]$. Since each edge of S is retained in H with probability 3/4, and every vertex i has degree n in the STS, each vertex in H has expected degree $3n/4$. Since the expected size of A is n (actually $n+1/2$, but this is insignificant in the following calculation) Theorem 8.4 gives

$$\begin{aligned}\Pr(\bar{X}\cup\bar{Y}) &\leqslant \Pr(\bar{X})+\Pr(\bar{Y})\\ &\leqslant 2\exp\left\{-\frac{8n}{2n+1}\right\}+(2n+1)\,2\exp\left\{-\frac{n\log(10n)}{n}\right\}<1.\end{aligned}$$

Thus $\Pr(X\cap Y)>0$, so there is a set A such that both X and Y hold. Choose such a set A. Since X holds, we may assume without loss of generality that $n-2\sqrt{n}\leqslant a\leqslant b\leqslant n+2\sqrt{n}$. Let G be the complete bipartite graph with partite sets A and B.

Using this random process, we obtain a hypergraph H satisfying (20) and (21). Theorem 8.3 implies that $\chi'(H)\sim\Delta(H)\sim 3n/4$; consider a decomposition of $E(H)$ into $\chi'(H)$ matchings. An edge in H contains either two vertices from A and one from B or vice versa. In G, this edge corresponds to the three-vertex path with the same vertices. For each color class of edges in H, color all the edges of the corresponding P_3's in G with the same color.

Since each pair of vertices in a STS belongs to a unique edge, all edges in G are colored. Because a color class of edges in G arose from a matching in H, each color class in G consists of disjoint paths of length 2. Last, since every pair of vertices in a STS has codegree one, no two monochromatic P_3's in G share each of their two ends. These remarks together imply that the coloring of G is a weak $(C_4, 3)$-coloring with $(1+o(1))\,3n/4$ colors.

For each m with $2m+1\equiv 1, 3 \pmod 6$, we have obtained a weak $(C_4, 3)$-coloring of $K_{a,b}$ with $(1+o(1))\,3m/4$ colors, where $m-2\sqrt{m}\leqslant a\leqslant m\leqslant b\leqslant m+2\sqrt{m}$. Since weak $(C_4, 3)$-colorings are preserved under taking subgraphs, we have $r'(K_{a,a}, C_4, 3)\leqslant(1+o(1))\,3m/4$ for some a with $m-2\sqrt{m}\leqslant a\leqslant m$, by considering a copy of $K_{a,a}\subseteq K_{a,b}$. It remains to extend this to all n.

Given any n, choose m such that $m-3\sqrt{m}\leqslant n\leqslant m-2\sqrt{m}$ and $m\equiv 1, 3 \pmod 6$. Then certainly $n/m\sim 1$ as $n\to\infty$. Let a correspond to m as in the preceding paragraph. Since $r'(K_{n,n}, C_4, 3)$ is a nondecreasing function of n,

$$r'(K_{n,n}, C_4, 3)\leqslant r'(K_{a,a}, C_4, 3)\leqslant\frac{3m}{4}(1+o(1))\sim\frac{3n}{4}(1+o(1)),$$

completing the proof. ∎

9. CHART OF BOUNDS ON $r(K_{n,n}, K_{p,p}, q)$

In the charts below, "$\ll f(n)$" means "$O(f(n))$," and "$\gg g(n)$" means "$\Omega(g(n))$":

q	$r(K_{n,n}, C_4, q)$	$r(K_{n,n}, K_{3,3}, q)$
2	$\sqrt{n}\,(1+o(1))$ Thm. 2.3	$n^{1/3}(1+o(1))$
3	$>\lfloor (2/3)\,n \rfloor$ Thm. 8.1, $\leqslant n+1$ Thm. 8.2	$\ll n^{4/7}$ Thm. 3.2
4	n^2	$\ll n^{2/3}$ Thm. 3.2
5	—	$\ll n^{4/5}$ Thm. 3.2
6	—	$>n/4$, $<cn$ Cor. 4.2
7	—	$\gg n$, $\ll n^{4/3}$ Thm. 3.2
8	—	$\lceil \frac{n}{2} \lceil \frac{3n}{2} \rceil \rceil$ Thm. 7.1
9	—	n^2

q	$r(K_{n,n}, K_{p,p}, q)$
2	$\gg n^{1/p}$ (3)
p^2-2p+2	$\ll n^{1-1/(2P-1)}$ Cor. 4.2
p^2-2p+3	$\Theta(n)$ Cor. 4.2
p^2-p+1	$\ll n^{2-2/p}$ Thm. 4.3
p^2-p+2	$\geqslant C_p(n^2-n)$, $<(1-c_p)\,n^2$ Thm. 4.3, (11)
$p^2-\lfloor (2p-1)/3 \rfloor$	$<n^2-c_p\,n^{1+\varepsilon}$ Thm. 6.1
$p^2-\lfloor (2p-1)/3 \rfloor+1$	$>n^2-2\lfloor (p-2)/3 \rfloor\,(n-1)$ Thm. 6.1
$p^2-\lfloor p/2 \rfloor$	$\leqslant n^2-\lfloor n/2 \rfloor$, $=n^2-\lfloor n/2 \rfloor$ if p odd and $\geqslant 7$ Thm. 5.1
$p^2-\lfloor p/2 \rfloor+1$	$n^2-\lfloor p/2 \rfloor+1$ Thm. 5.1
p^2	n^2

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