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The Maximum Size of 3-Uniform Hypergraphs Not Containing a Fano Plane

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A conjecture of V. Sós [3] is proved that any set of $\frac{3}{4}\binom{n}{3}+cn^2$ triples from an n-set, where c is a suitable absolute constant, must contain a copy of the Fano configuration (the projective plane of order two). This is an asymptotically sharp estimate. © 2000 Academic Press

Given a 3-uniform hypergraph $\mathscr{F}$, let $\mathrm{ex}_3(n, \mathscr{F})$ denote the maximum possible size of a 3-uniform hypergraph of order n that does not contain any subhypergraph isomorphic to $\mathscr{F}$. Our terminology follows that of [1], which is a comprehensive survey of Turán-type extremal problems. An elementary and well known averaging argument shows that the ratio $\mathrm{ex}_3(n, \mathscr{F})/\binom{n}{3}$ is a non-increasing sequence, so that $\pi(\mathscr{F}) := \lim_{n\to\infty} \mathrm{ex}_3(n, \mathscr{F})/\binom{n}{3}$ exists.

THEOREM. *If $\mathscr{F} = PG(2, 2)$ is the Fano plane then $\pi(\mathscr{F}) = \frac{3}{4}$.*

Proof. First we present a construction due to V. Sós [3]. She conjectures that this gives the exact value of $\mathrm{ex}_3(n, \mathscr{F})$. For each n let $\mathscr{H}^n$ be the hypergraph obtained by splitting a ground set of cardinality n into two sets, say A and B, of nearly equal size; the hyperedges of $\mathscr{H}^n$ consist of all triples that meet both A and B. Since, as is well known and easy to check, the Fano plane is not two-colourable, we see that $\mathscr{H}^n$ does not contain $\mathscr{F}$. The number of hyperedges of $\mathscr{H}^n$ equals $\frac{3}{4}\binom{n}{3} - O(n^2)$, which establishes the lower bound $\pi(\mathscr{F}) \geqslant \frac{3}{4}$.

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In the other direction, let $\mathscr{H}$ be any 3-uniform hypergraph of order n that does not contain the Fano plane. We will prove that, for some suitable absolute constant c, there exists a point of $\mathscr{H}$ that lies in at most $\frac{3}{4}\binom{n}{2}+cn$ hyperedges. A straightforward inductive argument then gives the upper bound $\pi(\mathscr{F})\leqslant\frac{3}{4}$; we leave the details to the reader. Given any point x, the link graph $\mathscr{H}[x]$ is defined as the set of pairs $\{y, z\}$ such that $\{z, y, z\}$ is a hyperedge of $\mathscr{H}$. Fix any hyperedge $\{1, 2, 3\}$ of $\mathscr{H}$. We claim that, given any four-element set $S=\{a, b, c, d\}$ of points disjoint from $\{1, 2, 3\}$, the three links $\mathscr{H}[1]$, $\mathscr{H}[2]$ and $\mathscr{H}[3]$ have altogether at most fifteen edges (counting multiplicities) contained in S, from a maximum possible of $3\cdot\binom{4}{2}=18$. In fact, it is obvious that the maximum $3\times 6=18$ minus any two edges always guarantee the 3×2 appropriate edges which are needed to get a Fano configuration.

We may assume, without loss of generality, that $\mathscr{H}$ contains a tetrahedron, i.e. a complete hypergraph on four points $K_3(4)=\{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}$. Indeed, if every 4-point set spans at most 3 triples, out of the maximum possible $\binom{4}{3}=4$, then a standard averaging argument implies that $\mathscr{H}$ can have already at most $\frac{3}{4}\binom{n}{3}$ hyperedges, in which case there is nothing to worry about, or $\mathscr{H}$ is so dense that it must contain a $K_3(4)$.

If S is any 4-set disjoint from the point-set $\{1, 2, 3, 4\}$ of a $K_3(4)$ in $\mathscr{H}$, then the four links $\mathscr{H}[i]$, $i=1$ to 4, have altogether at most 20 edges contained in S; this follows from our earlier upper bound of 15 edges per three links and simple averaging. Now we will invoke the following result, which is a very special case of a general theorem of Füredi and Kündgen [2]. To make this paper self-contained in the Appendix we give a short summary of the proof of the case we are going to use. Let $m(n, k, r)$ denote the maximum possible number of edges that a multigraph of order n can have, given that every k-set of points contains a total of at most r edges.

Lemma. $m(n, 4, 20)=3\binom{n}{2}+O(n)$.

Thus we may conclude that given any tetrahedron $K_3(4)$ in $\mathscr{H}$, its four links have altogether at most $3\binom{n}{2}+O(n)$ edges. (Note that in applying the Lemma one should, strictly speaking, disregard hyperedges involved in two of the links of the points of the $K_3(4)$; but this involves only $O(n)$ additional hyperedges.) Hence at least one of the four points of the $K_3(4)$ is contained in at most $\frac{3}{4}\binom{n}{2}+O(n)$ hyperedges of $\mathscr{H}$. As explained at the outset, this is sufficient for a proof that $\pi(PG(2, 2))\leqslant\frac{3}{4}$. This completes the proof of our theorem.

Turán type problems for hypergraphs are notoriously difficult; even the determination of $\pi(K_3(4))$ remains open. Thus the above result $\pi(PG(2, 2)) =\frac{3}{4}$ is gratifying, especially since the Fano plane is such a nice and famous configuration. Unfortunately, the method of this paper does not seem to generalize to a broad class of configurations $\mathscr{F}$. We refer once more to [1], especially Section 6, for a review of most of the known exact results.

APPENDIX

Here we prove $m(n, 4, 20) \leqslant 3\binom{n}{2} + n - 2$.

First, we use induction to show $m(n, 3, 10) \leqslant 3\binom{n}{2} + n - 2 (n \geqslant 3)$. Let G be a (3, 10) multigraph on n vertices, i.e., every 3 vertices span at most 10 edges. If every pair of vertices in G has multiplicity at most 3 then $e(G) \leqslant 3\binom{n}{2}$ and we are done. If one can find a pair $\{x, y\}$ with multiplicity at least 4, then for every z the sum of multiplicities of the edges from z to x and y is at most 6. Hence the total degrees of x and y is at most $8 + 6(n-2)$. One of them has degree at most $3n - 2$, and we can finish by induction.

Finally, consider a multigraph G on n vertices such that every 4 vertices span at most 20 edges. If it is a (3, 10)-graph then we have $e(G) \leqslant m(n, 3, 10)$. Otherwise, if there exists a 3 subset $\{x, y, z\}$ spanning at least 11 edges, then for every w the sum of multiplicities of the edges from w to $\{x, y, z\}$ is at most 9. As before, one can conclude that the total degrees of x, y and z is at most $22 + 9(n-3)$, one of them has degree at most $3n - 2$, and use induction.

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