ON THE PRAGUE DIMENSION OF KNESER GRAPHS

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ABSTRACT. In this note we point out another connection between the Prague dimension of graphs and the dimension theory of partially ordered sets by giving a very short proof of a theorem of Poljak, Pultr and Rödl [10]. We show that the dimension of the Kneser graph is bounded as $\dim_{\mathrm{P}}(K(n,k)) < C_k \log\log n$, where C_k is depending only on k.

1. DIMENSION OF GRAPHS

The Kneser graph $K(n,k)$ is the graph whose vertices are the k-subsets of the n-element set $[n] := \{1, 2, ..., n\}$, with vertices are adjacent when the corresponding k-sets are disjoint.

The *product* of the graphs (V_1, E_1) and (V_2, E_2) is a graph with vertex set $V_1 \times V_2$; two vertices (v_1, v_2) and (w_1, w_2) are adjacent in the product graph if (v_1, w_1) is adjacent in G_1 and (v_2, w_2) is adjacent in G_2. In particular, v_i and w_i must be distinct. The *Prague dimension* (or product dimension) of the graph G, $\dim_{\mathrm{P}}(G)$, is the minimum number d such that G is an induced subgraph of the product of d complete graphs. In other words, it is the minimum d such that the vertices x of G can be represented by vectors $\mathbf{v}(x) = (v_1(x), \dots, v_d(x))$ such that (x, y) forms an edge if and only if $v_i(x) \neq v_i(y)$ for all $1 \leq i \leq d$. Again, another form, it is the minimum number of good colorings of the vertices of G, $\varphi_1, \dots, \varphi_d$, (not necessarily with minimum number of colors), such that for every non-edge (a, b) one has at least one i with $\varphi_i(a) = \varphi_i(b)$.

The Prague dimension was introduced and investigated in a series of papers by Nešetřil, Pultr [9], and other Czech mathematicians. Poljak, Pultr and Rödl [10] proved that

$$\log_2 \log_2 (n/(k-1)) \leq \dim_{\mathrm{P}}(K(n,k)) \leq C_k \lceil \log_2 \lceil \log_2 n \rceil \rceil \ , \tag{1}$$

with $C_k \leq (k-1)k^2$. Later (for n sufficiently large) they [11] improved this to $C_k \leq (81/64)k^2/(\ln k)$. Very recently Körner [4] showed $C_k \leq (k/2) + o(1)$ (again for $n \to \infty$), which is conjectured to be tight in [7]. The case $n = 2k$ was discussed by Lovász, Nešetřil and Pultr [8], they proved that the dimension of the product of d (nontrivial) complete graphs is d. This implies $\dim_{\mathrm{P}}(K(2k,k)) = \lceil \log_2 \binom{2k}{k} \rceil = 2k - O(\log k)$.

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The aim of this note is to point out another connection between the Prague dimension of graphs and the dimension theory of partially ordered sets by giving a very short proof of the upper bound in (1).

2. Scrambling permutations and dimension of posets

The dimension of a partially ordered set P is the minimum d such that P can be embedded into $\mathbf{R}^d$ in an order preserving way. In other words, it is the minimum number of linear extensions $\pi_1, \dots, \pi_d$ such that for all $x, y \in P$ there exists a π_i with $x <_i y$ (x precedes y in π_i) except, of course, if $y <_P x$. In the latter case y precedes x in all linear extensions. Additional background material on dimension theory can be found in the monograph [13].

Let 2^S be denote the collection of subsets of S, and let $\mathbf{B}_n = (2^{[n]}, \subseteq)$ denote the *Boolean lattice*, the subsets of $[n]$ ordered by inclusion. For a set S, let $\binom{S}{k}$ denote the collection of k-element subsets of S. For $0 \le s < t \le n$ let $\mathbf{B}_n(s,t)$ denote the restriction of $\mathbf{B}_n$ to $\binom{[n]}{s} \cup \binom{[n]}{t}$. Finally, let $\dim(n; s, t)$ denote the (order) dimension of $\mathbf{B}_n(s,t)$. The function $\dim(n; s, t)$ was first studied by Dushnik [1] in 1950, he determined the exact value for $\dim(n; 1, t)$ when $2\sqrt{n} - 2 \le t < n - 1$.

Call the set of permutations of $[n]$, Π, *t-scrambling* if for every (now unordered) t-subset $\{p_1, \dots, p_t\} \subset [n]$ and for every distinguished element of the set, say p_j, there is a permutation $\pi \in \Pi$ such that $\pi(p_j)$ precedes all the other $(t-1)$ p_i's. The cardinality of the smallest t-scrambling family is denoted by $N(n,t)$. It is easy to see that determination of $N(n,t)$ is equivalent to the question of the dimension of the partially ordered set formed by the $(t-1)$ and 1-element subsets of $[n]$ and ordered by inclusion, i.e., $N(n,t) = \dim(n; 1, t-1)$. For t is fixed and $n \to \infty$ an argument due to Hajnal and Spencer [12] gives that

$$\log_2 \log_2 n \le N(n,t) \le \frac{t}{\log_2(2^t/(2^t-1))} \log_2 \log_2 n. \tag{2}$$

In [3] the asymptotic $N(n,3) = \log_2 \log_2 n + (\frac{1}{2} + o(1)) \log_2 \log_2 \log_2 n$ was proved.

Theorem 2.1. $\dim_{\mathrm{P}}(K(n,k)) \le N(n, 2k-1)$.

Proof. Let $\pi_1. \dots, \pi_d$ be a $(2k-1)$-scrambling set of permutations of $[n]$. We define $\varphi_1, \dots, \varphi_d$ good colorings of the Kneser graph $K(n,k)$, $\varphi_i : \binom{[n]}{k} \to [n]$, as follows. Let $\varphi_i(K) = x$ where $x \in K$ is the smallest element of K in the linear order π_i.

As $\varphi_i(K) \in K$, for disjoint k-sets, $K, L \in \binom{[n]}{k}$, we have that $\varphi_i(K) \neq \varphi_i(L)$ for all i. However, for a non-edge, i.e, for an intersecting pair (K, L), for $x \in K \cap L$, one can find a permutation π_i which puts x to the first place among the elements in $K \cup L$. □

Remark 2.2. The constructions in [10, 11, 4] use qualitatively independent partitions and k-independent families of sets. Let us note that the upper bound in (2) also uses k-independent families of sets so it cannot give a better bound for C_k as 2^k. However, together with the upper bound from [3] for $N(n,3)$, it gives the asymptotic for the case $k=2$, which was also showed in [10]. Finally, Theorem 2.1 also gives a number of new upper bounds for $\dim_P(K(n,k))$, when n is "not too large" with respect to k, e.g., $k \sim \log n$, where Kierstead's bound [5] gives $O(\log^3 n/\log\log n)$.

Remark 2.3. One can easily see, that, similarly to the examples in [10, 11, 4], our construction is *faithful*, i.e., $\varphi(K) \cap \varphi(L) = K \cap L$ holds for every two k-sets, where $\varphi(K) := \{\varphi_i(K) : 1 \le i \le d\}$.

Remark 2.4. (*Binary intersection representations.*) Körner and Monti [6] defined the *Bohemian representation* of the Kneser graph $K(n,k)$ as a set of colorings of its vertex set, $\varphi_1, \dots, \varphi_t$, where now $\varphi_i : \binom{[n]}{k} \to N$ is not necessarily a good coloring of the graph, and a function $\varphi : 2^{[t]} \to 2^{[n]}$ with the following property. For a pair of distinct sets $A, B \in \binom{[n]}{k}$ let $\delta(A,B)$ denote a sequence from $\{0,1\}^t$ with $\delta_i = 1$ for $\varphi_i(A) = \varphi_i(B)$ and 0 otherwise. In a Bohemian representation $(\varphi_1, \dots, \varphi_t, \varphi)$ we want to be able to read out the intersection structure of the complete hypergraph knowing only the binary vectors, $\delta(A,B)$, i.e., we have $\varphi(\delta(A,B)) = A \cap B$. The minimum of such t is called the *Bohemian dimension*, and is denoted by $T(n,k)$. Körner and Monti [6] proved that

$$k - 1 \le \liminf_{n\to\infty} \frac{T(n,k)}{\log_2 n} \le \limsup_{n\to\infty} \frac{T(n,k)}{\log_2 n} \le k(k-1).$$

Using a different kind of set of scrambling permutations, one can see that $T(n,k) = O(\log n)$ as k is fixed and $n \to \infty$ as follows. Call a family of permutations $\pi_1, \dots, \pi_t$ of $[n]$ *completely k-scrambling* if for every ordered k-subset $\{p_1, \dots, p_k\}$ of k distinct elements of $[n]$ there is a permutation π_i with $\pi_i(p_1) < \cdots < \pi_i(p_k)$. This means that all k-subsets appear in all $k!$ possible orderings. The cardinality of the smallest completely k-scrambling family is denoted by $N^\star(n,k)$. It is known (for $k \ge 3$) that $\frac{1}{2}(k-1)!\log_2 n < N^\star(n,k) \le (1+o(1))\frac{k}{\log_2(k!/(k!-1))}\log_2 n$. Here the lower bound is from [2] and the upper bound is due to Spencer [12].

Now, one can easily see, that a completely $(4k-2)$-scrambling set of permutations in the same way as in Theorem 2.1 provides a Bohemian representation of $K(n,k)$ thus proving $T(n,k) \le N^\star(n, 4k-2)$. Even more, again, the obtained φ_i's are proper colorings of the Kneser graph.

Further problems and connections between permutations and order dimensions can be found in [2].

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