Journal of Combinatorial Theory, Series B **76**, 223–239 (1999)
Article ID jctb.1999.1905, available online at http://www.idealibrary.com on IDEAL®

Signed Domination in Regular Graphs and Set-Systems

Zoltán Füredi

Mathematical Institute of the Hungarian Academy, Pf. 127, H-1364 Budapest, Hungary; and Department of Mathematics, University of Illinois, 1409 West Green Street, Urbana, Illinois 61801-2975
E-mail: furedi@math-inst.hu and z-furedi@math.uiuc.edu

and

Dhruv Mubayi*

Department of Mathematics, University of Illinois, 1409 West Green Street, Urbana, Illinois 61801-2975
E-mail: mubayi@math.gatech.edu

Received January 21, 1998

Suppose G is a graph on n vertices with minimum degree r. Using standard random methods it is shown that there exists a two-coloring of the vertices of G with colors, $+1$ and -1, such that all closed neighborhoods contain more 1's than -1's, and all together the number of 1's does not exceed the number of -1's by more than $(4\sqrt{\log r/r}+1/r)\,n$. For large r this greatly improves earlier results and is almost optimal, since starting with an Hadamard matrix of order r, a bipartite r-regular graph is constructed on $4r$ vertices with signed domination number at least $(1/2)\sqrt{r}-O(1)$. The determination of $\lim_{n\to\infty}\gamma_s(G)/n$ remains open and is conjectured to be $\Theta(1/\sqrt{r})$.

Key Words: discrepancy; domination; random covering of graphs and hypergraphs; Hadamard matrices.

1. DISCREPANCY AND DOMINATION OF HYPERGRAPHS

Discrepancy theory has originated from number theory and in the last few decades it has developed into an elaborate field related also to geometry, probability theory, ergodic theory, computer science, and combinatorics. The combinatorial setting of these problems proved to be a successful approach. See the monograph of Beck and Chen [4], the chapter from the Handbook of Combinatorics [6], or [17].

* Current address: School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160.

0095-8956/99 $30.00

One of the basic problems in combinatorial discrepancy theory is the following: Suppose $\mathscr{H}$ is a hypergraph with vertex set S and edge set $\{A_1, ..., A_m\}$. Our object is to color the elements of S by 2 colors such that all of the edges have almost the same number of elements in each color. A partition of S can be given by a function $f: S \to \{+1, -1\}$. For any set $A \subset S$, we let $f(A) = \sum_{a \in A} f(a)$. The discrepancy of $\mathscr{H}$ with respect to f is defined by

$$\mathscr{D}(\mathscr{H}, f) := \max_{A_i \in \mathscr{H}} |f(A_i)|$$

and the *discrepancy* of the hypergraph $\mathscr{H}$, as it was first defined by Beck [3] is

$$\mathscr{D}(\mathscr{H}) := \min_{f: S \to \{+1, -1\}} \mathscr{D}(\mathscr{H}, f).$$

This measures, in supremum norm, how well the A_i can be partitioned. For a given $\mathscr{H}$ we want to determine or estimate $\mathscr{D}(\mathscr{H})$. A large number of classical theorems in number theory, geometry, and combinatorics can be formulated in this language.

Here we consider a one-sided version of discrepancy. Let α be a real number and suppose that $\mathscr{H}$ is a hypergraph with vertex set S. The function $g: S \to \{+1, -1\}$ defines an α-*dominating* partition of the hypergraph $\mathscr{H}$, if

$$g(A) := \sum_{a \in A} g(a) \geqslant \alpha$$

for every edge A in $\mathscr{H}$. For $\alpha = 1$ we simply want the set $P := g^{-1}(+1)$ to contain strictly more elements of A than the set $N := g^{-1}(-1)$. The α-*domination number* of $\mathscr{H}$ is defined as the minimum of such functions

$$\mathrm{dom}_\alpha(\mathscr{H}) := \min_{\substack{g: S \to \{+1, -1\} \\ g \text{ is } \alpha\text{-dominating}}} g(S).$$

We simply write dom for dom_1. This notion differs from the usual discrepancy in another way, too, namely, we measure our success by minimizing the size of P.

THEOREM 1.1. *Let $\mathscr{H}$ be an n-vertex hypergraph with edge set $\{A_1, A_2, ..., A_m\}$, and suppose that every edge has at least k vertices, where $k \geqslant 100$. Then*

$$\mathrm{dom}(\mathscr{H}) \leqslant 4 \sqrt{\frac{\log k}{k}}\, n + \frac{1}{k}\, m. \tag{1.1}$$

The relatively simple proof is postponed to Section 3. It demonstrates the power of the probabilistic method. In Section 4 we give upper bounds for $3 \leqslant k < 19$.

As we will see in Sections 5 and 6 via an explicit construction, the upper bound in Theorem 1.1 is off from the best possible by at most the $\sqrt{\log k}$ factor. We *conjecture* that, indeed,

$$\operatorname{dom}(\mathscr{H}) \leqslant \frac{c}{\sqrt{k}}(n+m) \qquad (?) \tag{1.2}$$

holds for some constant c (independent of n, m, k and of $\mathscr{H}$). We are able to establish this upper bound in some cases (in Section 7) modulo a discrepancy conjecture of Beck and Fiala [5].

2. SIGNED DOMINATION OF GRAPHS

All graphs we consider are simple. We are especially interested in the class of r-regular, n-vertex graphs, $\mathscr{G}^n(r\text{-reg})$, and the class of n-vertex graphs with minimum degree at least r, $\mathscr{G}^n(\geqslant r)$. In a graph G, the *closed neighborhood* of a vertex $v \in V(G)$, $N_G[v]$ or $N[v]$ for short, consists of v together with its neighbors. A *signed domination function* of a graph G is a function g: $V(G) \to \{+1, -1\}$ such that for every vertex v, the sum of the values of g over the closed neighborhood of v is positive, and the minimum of the sum $\sum_{v \in V} g(v)$ over all such functions is called the signed domination number, $\gamma_s(G)$, i.e.,

$$\gamma_s(G) = \min\{g(V(G)) \colon g \text{ is a signed domination function of } G\}.$$

Using the notations of the previous Section, we have that $\gamma_s(G) = \operatorname{dom}(\mathscr{N}(G))$, where $\mathscr{N}$ is the hypergraph on the vertex set $V(G)$ and its edges are the closed neighborhoods $\{N_G[v] \colon v \in V(G)\}$.

This variant of the usual domination number was introduced by Dunbar *et al.* [9] in the early 1990s. They also observed that $\gamma_s(G) \geqslant n/(r+1)$ for all r-regular n-vertex graphs. This is sharp when r is even and $n/(r+1)$ is an integer, as shown by a vertex disjoint union of complete graphs. However, for r odd, Henning and Slater [13] pointed out that $\gamma_s(G) \geqslant 2n/(r+1)$ for every graph $G \in \mathscr{G}^n(r\text{-reg})$, and that this lower bound is again sharp whenever $n/(r+1)$ is an integer.

Let $\gamma_s(\mathscr{G})$ denote the *maximum* of $\gamma_s(G)$ in the class of graphs $G \in \mathscr{G}$. We consider $\gamma_s(n, r) := \gamma_s(\mathscr{G}^n(r\text{-reg}))$, and $\gamma_s(n, \geqslant r) = \gamma_s(\mathscr{G}^n(\geqslant r))$. Zelinka [20] proved that $\gamma_s(n, 3) \leqslant (4/5)\, n$, and that the fraction 4/5 is best possible. Favaron [10] sharpened this by proving that $\gamma_s(G) \leqslant (3/4)\, n$ for all connected cubic graphs $G \in \mathcal{G}^n(3\text{-reg})$, except for the Petersen graph P^{10}. (It is easy to show that $\gamma_s(P^{10}) = 8$.)

Henning [12] and Favaron [10] independently proved that for r-regular graphs,

$$\gamma_s(n, r) \leqslant \begin{cases} \dfrac{(r+1)^2}{r^2+4r-1}\, n & \text{if } r \text{ is odd} \\ \dfrac{r+1}{r+3}\, n & \text{if } r \text{ is even.} \end{cases}$$

They established these upper bounds for every *locally minimal* signed domination function, in this sense their bounds are sharp. But for the absolute minimum, γ_s, we improve this upper bound for $r = 4$ and $r \geqslant 6$ and extend it to $\mathcal{G}^n(\geqslant r)$ by constructing a signed domination function with value 1 on slightly more than half of the vertices of G. Indeed, applying Theorem 1.1 to the neighborhood hypergraph one gets the following obvious corollary.

THEOREM 2.1 *If G is an n-vertex graph with minimum degree $r \geqslant 99$, then*

$$\gamma_s(G) \leqslant \left(4\sqrt{\frac{\log(r+1)}{r+1}} + \frac{1}{r+1}\right) n.$$

3. THE PROOF OF THE UPPER BOUND BY RANDOM METHOD

We use the following theorem of Chernoff [7] to estimate large deviations. One can refer to [2, p. 238] for a concise proof. If $p \in [0, 1]$, $X_1, ..., X_n$ are mutually independent random variables with

$$\text{Prob}[X_i = 1 - p] = p$$
$$\text{Prob}[X_i = -p] = 1 - p,$$

and $X = X_1 + \cdots + X_n$, $a > 0$, then

$$\text{Prob}[X < -a] < e^{-a^2/2pn}. \tag{3.1}$$

Proof of Theorem 1.1. Set $p = \frac{1}{2} + 2\sqrt{\log k / k}$. Let each vertex receive the label $+1$ with probability p. Let $P \subseteq S$ be the random set of the vertices thus labeled. Let $\mathcal{U} = \mathcal{U}(P)$ be the set of edges $A \in E(\mathcal{H})$ that intersect P in at most $|A|/2$ elements, and call them *uncovered* by P. Let $Q = Q_P$ be the union of uncovered edges. Assign the label $+1$ to every element of $Q - P$;

vertices outside $P \cup Q$ receive -1. The resulting function $g: S \to \{+1, -1\}$ is a signed domination function of $\mathscr{H}$.

Note that $g(S) = 2\,|P \cup Q| - |S|$. We bound the expected value of $g(S)$ by computing $\mathrm{E}(|P|) + \mathrm{E}(|Q|)$. Clearly, $\mathrm{E}(|P|) = pn$. Moreover, $|Q| \leqslant \sum_{A \in \mathscr{U}} |A|$, hence the linearity of expectation gives

$$\mathrm{E}(|Q|) \leqslant \sum_{A \in \mathscr{H}} |A| \;\mathrm{Prob}(|A \cap P| \leqslant |A|/2).$$

To apply (3.1), we decrease the labels of vertices by p and describe P with random variables X_i (for $i \in S$) taking the value $X_i = 1 - p$ with probability p and $X_i = -p$ with probability $1 - p$. Let $X = \sum_{i \in A} X_i$. The edge A intersects P in at most $|A|/2$ elements if and only if $X = \sum_{i \in A} X_i \leqslant (1-p)\,|A|/2 + (-p)\,|A|/2$. This gives $\mathrm{Prob}(|A \cap P| \leqslant |A|/2) = \mathrm{Prob}(X \leqslant (\frac{1}{2} - p)\,|A|)$. Applying (3.1) yields

$$\mathrm{Prob}\left(X \leqslant \left(\frac{1}{2} - p\right) |A|\right) \leqslant \exp\left\{\frac{-((p - \frac{1}{2})\,|A|)^2}{2\,|A|\,p}\right\}$$
$$= \exp\left\{\frac{-(4\,|A| \log k)/k}{1 + 4\sqrt{\log k/k}}\right\}.$$

For $a \geqslant k \geqslant 19$, let

$$f(a, k) = \frac{4a \log k/k}{1 + 4\sqrt{\log k/k}} - \log(2a^2).$$

It is readily observed that $f(a, k)$ is an increasing function of a; thus $f(a, k) \geqslant f(k, k)$. A short calculation shows that $f(k, k) \geqslant 0$ for $k \geqslant 100$. Thus for $|A| \geqslant k \geqslant 100$,

$$\frac{4\,|A| \log k/k}{1 + 4\sqrt{\log k/k}} \geqslant \log(2\,|A|^2).$$

This implies that $\mathrm{Prob}(|A \cap P| \leqslant |A|/2) < 1/(2\,|A|^2)$, and therefore $\mathrm{E}(|Q|) \leqslant \sum_{A \in \mathscr{H}} 1/(2\,|A|) \leqslant m/2k$. Since $\mathrm{E}(|P \cup Q|) \leqslant \mathrm{E}(|P|) + \mathrm{E}(|Q|)$, we have

$$\mathrm{E}(g(S)) = 2\mathrm{E}(|P \cup Q|) - |S| \leqslant (2p - 1)\,n + \frac{m}{k} = 4\sqrt{\frac{\log k}{k}}\,n + \frac{m}{k}.$$

Since there is at least one choice of g such that $g(V) \leqslant \mathrm{E}(G(S))$, this completes the proof of the upper bound for $\mathrm{dom}(\mathscr{H})$. ∎

4. AN IMPROVEMENT OF THE UPPER BOUND FOR SMALL VALUES

In the previous section, the two-step random coloring colored all elements in the uncovered edges with $+1$. In this section we color these edges with the exact number of colors they require. Although this gives a better bound than that in [10, 12] for all $k \leqslant 99$, we only do this for $k \leqslant 19$.

DEFINITION 4.1. Let $a \geqslant 3$ be an integer, and p a real number. Let

$$f(a, p) = \sum_{i=0}^{\lfloor a/2 \rfloor} (i+1) \binom{a}{\lfloor a/2 \rfloor - i} p^{\lfloor a/2 \rfloor - i} (1-p)^{\lceil a/2 \rceil + i}.$$

LEMMA 4.2. *If a is even, then $f(a+1, p) < 2(1-p) f(a, p)$. If a is odd, then $f(a+1, p) < (2p + (1-p)(a+3)/(a+1)) f(a, p)$. In particular, if $2(1-p)(2p + (1-p)(a+3)/(a+1)) < 1$, then*

$$\max_{a \geqslant k} f(a, p) \in \{ f(k, p), f(k+1, p) \}.$$

Proof. Observe that the substitution $j = \lfloor a/2 \rfloor - i$ yields

$$f(a, p) = \sum_{j=0}^{\lfloor a/2 \rfloor} (\lfloor a/2 \rfloor - j + 1) \binom{a}{j} p^j (1-p)^{a-j}.$$

Case 1. a is even. Using $\binom{a+1}{j} = \binom{a}{j}(a+1)/(a+1-j)$ and the fact that $(a+1)/(a+1-j)$ is an increasing function of j we obtain

$$\begin{aligned} f(a+1, p) &= \sum_{j=0}^{a/2} (a/2 - j + 1) \binom{a+1}{j} p^j (1-p)^{a+1-j} \\ &= \sum_{j=0}^{\lfloor a/2 \rfloor} \frac{a+1}{a+1-j} (1-p)(\lfloor a/2 \rfloor - j + 1) \binom{a}{j} p^j (1-p)^{a-j} \\ &< \frac{a+1}{a+1-\lfloor a/2 \rfloor} (1-p) f(a, p) < 2(1-p) f(a, p). \end{aligned}$$

Case 2. a is odd. Letting $S = f(a+1, p) - 2pf(a, p)$ we have

$$\begin{aligned} S = &\sum_{j=0}^{(a+1)/2} \left(\frac{a+1}{2} - j + 1 \right) \binom{a+1}{j} p^j (1-p)^{a+1-j} \\ &- 2p \sum_{j=0}^{(a-1)/2} \left(\frac{a-1}{2} - j + 1 \right) \binom{a}{j} p^j (1-p)^{a-j} \end{aligned}$$

$$= \sum_{j=1}^{(a+1)/2} \left(\frac{a+3}{2} - j\right) p^j (1-p)^{a+1-j} \left(\binom{a+1}{j} - 2\binom{a}{j-1}\right)$$

$$+ \frac{a+3}{2}(1-p)^{a+1}$$

$$= \sum_{j=1}^{(a-1)/2} \left(\frac{\left(\binom{a}{j} - \binom{a}{j-1}\right)\left(\frac{a+3}{2} - j\right)}{\left(\frac{a-1}{2} - j + 1\right)\binom{a}{j}} \right) \left(\frac{a-1}{2} - j + 1\right)\binom{a}{j}$$

$$\times p^j (1-p)^{a+1-j} + \frac{a+3}{2}(1-p)^{a+1}$$

$$= \sum_{j=1}^{(a-1)/2} \frac{a+3-2j}{a+1-j} \left(\frac{a-1}{2} - j + 1\right)\binom{a}{j} p^j (1-p)^{a+1-j} \tag{4.1}$$

$$+ \frac{a+3}{2}(1-p)^{a+1}$$

$$< \frac{a+3}{a+1}(1-p) \sum_{j=0}^{(a-1)/2} \left(\frac{a-1}{2} - j + 1\right)\binom{a}{j} p^j (1-p)^{a-j}$$

$$= \frac{a+3}{a+1}(1-p)\, f(a, p),$$

where the last inequality holds because the leading fraction in (4.1) is a decreasing function of j with value $(a+3)/(a+1)$ at $j=0$. Solving the inequality for $f(a+1, p)$ yields the result. ∎

TABLE I

$\mathrm{dom}(\mathcal{H}) \leqslant (2p-1)\, n + 2qm$, $\gamma_s(G)/n \leqslant (2p+2q-1)$

k	p	$2p-1$	$2q \leqslant$	r	$\gamma_s(G)/n \leqslant$
3, 4	0.9086	0.8172	0.0943	2, 3	0.9115
5, 6	0.8490	0.6980	0.1093	4, 5	0.8073
7, 8	0.8139	0.6278	0.1057	6, 7	0.7335
9, 10	0.7897	0.5794	0.0994	8, 9	0.6788
11, 12	0.7714	0.5428	0.0932	10, 11	0.6360
13, 14	0.7568	0.5136	0.0879	12, 13	0.6015
15, 16	0.7449	0.4898	0.0829	14, 15	0.5727
17, 18	0.7348	0.4696	0.0787	16, 14	0.5483
19, 20	0.7261	0.4522	0.0749	18, 19	0.5271

Theorem 4.2. *If the hypergraph $\mathscr{H}$ has m edges, and every edge contains at least k elements, then*

$$\mathrm{dom}(\mathscr{H}) \leqslant (2p-1)\,n + 2qm,$$

where the values of p and q are given in Table I. The last column of the table applies to graphs in $\mathscr{G}^n(\geqslant r)$.

Proof. The proof follows that of Theorem 1.1. Assign each vertex the label $+1$ with probability p, and let P be the random set of vertices thus labeled. Denote by $\mathscr{U}_i = \mathscr{U}_i(P)$ the set of edges $A_i \in \mathscr{H}$ that intersect P in *exactly* $\lfloor |A|/2 \rfloor - i$ edges, and call these uncovered edges. For $A \in \mathscr{U}_i$ $(i \geqslant 0)$, let Q_A be a set of $i+1$ elements in $A \backslash P$ and set $Q = \bigcup_A Q_A$. We assign the label $+1$ to every element in $P \cup Q$. As before, the expected size of $|P|$ in pm. The expected size of $|Q|$ is

$$\begin{aligned} \mathrm{E}[\,|Q|\,] &\leqslant \sum_{\text{for all } A \in \mathscr{H}} \mathrm{E}[\,|Q_A|\,] \\ &= \sum_{\text{for all } A \in \mathscr{H}} \sum_{i=0}^{\lfloor |A|/2 \rfloor} (i+1)\,\mathrm{Prob}[A \in \mathscr{U}_i] \\ &= \sum_{\text{for all } A \in \mathscr{H}} \sum_{i=0}^{\lfloor |A|/2 \rfloor} (i+1) \binom{|A|}{\lfloor |A|/2 \rfloor - i} p^{\lfloor |A|/2 \rfloor - i} (1-p)^{\lceil |A|/2 \rceil + i} \\ &\leqslant m \times \max_{a \geqslant k} f(a, p). \end{aligned}$$

For each choice of $p = p_k$ given in Table I, it is readily verified that $2(1-p)(2p + (1-p)(a+3)/(a+1)) < 1$ whenever $a \geqslant k$. Lemma 4.2 therefore implies that $\max_{a \geqslant k} f(a, p_k) = f(k, p_k)$ if k is even, and $\max_{a \geqslant k} f(a, p_k) = \max\{f(k, p_k), f(k+1, p_k)\}$ if k is odd. A straightforward calculation

TABLE II

k	p	$2p-1$	$2q \leqslant$	r	$\gamma_s(G)/n \leqslant$
3	0.8165	0.6330	0.1897	2	0.8227
5	0.7877	0.5754	0.1534	4	0.7288
7	0.7675	0.5350	0.1315	6	0.6665
9	0.7520	0.5040	0.1168	8	0.6208
11	0.7396	0.4792	0.1058	10	0.5850
13	0.7293	0.4586	0.0973	12	0.5559
15	0.7206	0.4412	0.0903	14	0.5315
17	0.7130	0.4260	0.0847	16	0.5107
19	0.7063	0.4126	0.0800	18	0.4926

shows that if k is odd and $3 \leqslant k \leqslant 19$, then $\max_{a \geqslant k} f(a, p_k) = f(k+1, p_k)$ whenever $0.5 < p_k < 1$, and thus we group the k values in pairs in the first column of Table I. For each $k \leqslant 19$, we choose p optimally (up to four digits) as shown in the table.

If k is odd and we consider only k-uniform hypergraphs, then a slightly different choice of p_k yields a better bound. This is summarized in Table II; the last column refers to graphs in $\mathscr{G}^n(r\text{-reg})$. ▮

5. A CONSTRUCTION FROM THE HADAMARD MATRIX

In this section we construct a t-regular, bipartite graph $G = G^{4t}$ on $4t$ vertices from an Hadamard matrix H of order t. This G is used in the next section to obtain lower bounds for $\gamma_s(n, r)/n$.

Construction 5.1. An Hadamard matrix H of order t is a $t \times t$ matrix of entries $+1$ and -1 such that $H^T H = I$. After multiplying some columns of H by -1, if necessary we may suppose that the first row of H, $\mathbf{h}_1$, contains only $+1$'s. Then every other row has exactly $t/2$ $+1$'s and -1's.

Let the vertex set $V = V(G)$ consist of four sets, $\{a^+, a^-, d^*\}$, $B = \{b_1, ..., b_t\}$, $C = \{c_2^+, c_2^-, c_3^+, c_3^-, ..., c_t^+, c_t^-\}$, and $D = \{d_2, ..., d_t\}$. The sizes of these sets are 3, t, $2t-2$, and $t-1$, respectively. Let $C^+ = \{c_2^+, ..., c_t^+\}$ and $C^- = \{c_2^-, ..., c_t^-\}$.

To define the edges of G, first, join a^+ to each element of B, $a^+ \leftrightarrow B$, moreover $a^- \leftrightarrow (D \cup \{d^*\})$, and $d^* \leftrightarrow C^+$. The vertices in B correspond to the columns of H, and the vertices in C correspond to the rows of H (and a^+ corresponds to the first row of H). For each i, $(2 \leqslant i \leqslant t)$, let c_i^+ be adjacent to those b_j's for which the (i, j)'th entry of H, $h_{i,j} = 1$; and join c_i^- to b_j if $h_{i,j}$ is -1. Finally, for $2 \leqslant i \leqslant t$, let $c_i^+ \leftrightarrow \{d_i, ..., d_{i+t/2-2}\}$, and $c_i^- \leftrightarrow \{d_i, ..., d_{i+t/2-1}\}$, where the subscripts are taken modulo $t-1$. In Figure 1, some of the edges of G are drawn; an edge to the boundary of a set represents edges to all vertices in that set.

It is easy to check that G is indeed t-regular. The vertex a^+ is adjacent to the t vertices of B; a^- is adjacent to the $t-1$ vertices from D and to d^*; d^* is adjacent to $t-1$ vertices from C and to a^-. Also, each vertex of B is adjacent to $t-1$ vertices from C and to a^+. The neighborhoods of c_i^+ and c_i^- in B are complements to each other, $N(c_i^+, B) = B \backslash N(c_i^-, B)$, so each vertex in C^+ is adjacent to $t/2$ vertices from B, to $t/2-1$ vertices from D, and to d^*; each vertex in C^- is adjacent to $t/2$ vertices from B and to

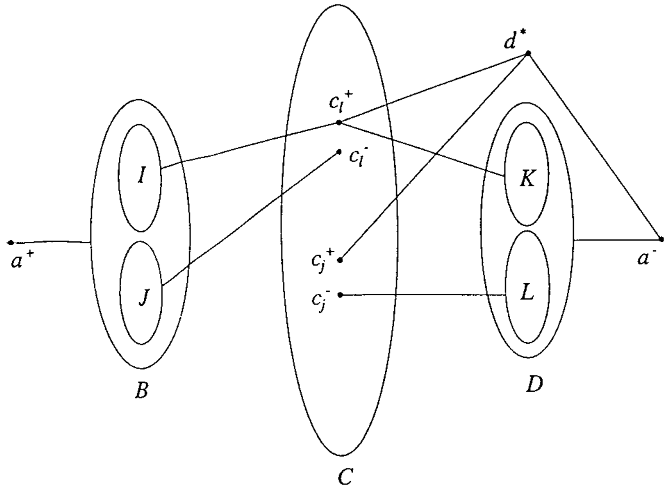

FIG. 1. Construction of G.

$t/2$ vertices from D. Finally, each vertex in D is adjacent to $t-1$ vertices from C and to a^-.

LEMMA 5.2. *Suppose that β is a real number with $-\sqrt{t}<\beta\leqslant\sqrt{t}/2$ and the function f: $V\to\{+1,-1\}$ is such that*

$$f(N(v))\geqslant\beta \tag{5.1}$$

holds for every (open) neighborhood of $v\in V(G)$. Then

$$f(V)\geqslant\sqrt{t}/2+4\beta-5/2.$$

For the proof we are going to use the following theorem of Olson and Spencer [16]. Let $\mathbf{h}_i$ denote the ith row of H and let $\mathbf{f}$: $[t]\to\{+1,-1\}$ be arbitrary. Then there is an ℓ such that for the scalar product we have

$$|\mathbf{f}\cdot\mathbf{h}_\ell|\geqslant\sqrt{t}. \tag{5.2}$$

For completeness, we include their elegant proof (also see in [18]): Since the vectors $\mathbf{h}_1, ..., \mathbf{h}_t$ are pairwise orthogonal they form a basis for $\mathbf{R}^t$. Write $\mathbf{f}$ in the form $\mathbf{f}=\sum_{1\leqslant i\leqslant t}c_i\mathbf{h}_i$. From $\mathbf{h}_i\cdot\mathbf{h}_i=\mathbf{f}\cdot\mathbf{f}=t$ we get $\sum_i c_i^2=1$. So there is an ℓ such that $|c_\ell|\geqslant 1/\sqrt{t}$. Now $|\mathbf{f}\cdot\mathbf{h}_\ell|=|c_\ell t|\geqslant\sqrt{t}$. ∎

Proof of Lemma 5.2. Since G is a regular bipartite graph with parts $C\cup\{a^+,a^-\}$ and $B\cup D\cup\{d^*\}$ we obtain from (5.1) that

$$tf(a^+)+tf(C)+tf(a^-)=\sum_{v\in B\cup D\cup\{d^*\}} f(N(v))\geqslant 2t\beta.$$

This gives

$$f(C)+f(a^+)+f(a^-)\geqslant 2\beta. \tag{5.3}$$

Let $\mathbf{f}$ be the restriction of f to B. Because the edges between $a^+\cup C$ and B come from the Hadamard matrix H, (5.2) implies that (at least) one of the following two cases holds.

Case 1. $\ell=1$ and $|f(B)|=|\mathbf{f}\cdot\mathbf{h}_1|\geqslant\sqrt{t}$.

Case 2. There is an $\ell>1$ such that $|f(N(C_\ell^+, B))-f(N(c_\ell^-, B))|=|\mathbf{f}\cdot\mathbf{h}_\ell|\geqslant\sqrt{t}$.

In Case 1, as $|\beta|<\sqrt{t}$ and $f(B)=f(N(a^+))$, (5.1) implies that $f(B)\leqslant-\sqrt{t}$ is impossible, so we have $f(B)\geqslant\sqrt{t}$. Using this, and (5.3), and $f(D\cup d^*)=f(N(a^-))\geqslant\beta$ we get

$$f(V)=f(B)+f(C\cup\{a^+,a^-\})+f(D\cup\{d^*\})\geqslant\sqrt{t}+3\beta\geqslant\sqrt{t}/2+4\beta$$

and we are done.

In Case 2, i.e., when $\ell>1$ define $I=N(c_\ell^+, B)$ and $J=N(c_\ell^-, B)$ (see Fig. 1). We have

$$|f(I)-f(J)|=|\mathbf{f}\cdot\mathbf{h}_\ell|\geqslant\sqrt{t}. \tag{5.4}$$

This is not enough to give a lower bound of $\sqrt{t}/2$ on $f(B)$ since, for example, we could have $f(I)=\frac{1}{2}\sqrt{t}$ and $f(J)=\beta-\frac{1}{2}\sqrt{t}$. We use the edges between C and D to obtain the required lower bound. Let $K=N(c_\ell^+, D)$ and $L=D-K$. Because $N(c_\ell^+)=I\cup K\cup d^*$,

$$2(f(I)+f(K)+f(d^*))\geqslant 2\beta. \tag{5.5}$$

Choose j such that $N(c_j^-, D)=L$. Then

$$f(I)+f(J)+2f(L)-f(d_{\ell-1})=f(N(c_j^+))+f(N(c_j^-))\geqslant 2\beta. \tag{5.6}$$

The sum of the left-hand-sides of (5.5) and (5.6) together with the double of (5.3) is exactly $2f(V)+f(I)-f(J)-f(d_{\ell-1})$. We get

$$2f(V)\geqslant f(J)-f(I)+8\beta+f(d_{\ell-1}). \tag{5.7}$$

On the other hand, since $N(c_\ell^-)=J\cup K\cup d_{\ell+t/2-1}$, we obtain

$$2(f(J)+f(K)+f(d_{\ell+t/2-1}))\geqslant 2\beta. \tag{5.5$'$}$$

Adding (5.5′) to (5.6) and to the double of (5.3) we get

$$2f(V) \geqslant f(I) - f(J) + 8\beta + f(d_{\ell-1}) + 2f(d^*) - 2f(d_{\ell+t/2-1}). \quad (5.7')$$

Thus (5.7), (5.7′), and (5.4) give

$$f(V) \geqslant (|f(I) - f(J)|)/2 + 4\beta - 5/2 \geqslant \tfrac{1}{2}\sqrt{t} + 4\beta - 5/2. \quad \blacksquare$$

6. LOWER BOUNDS

In this section we use Construction 5.1 to obtain lower bounds for the signed domination numbers, e.g., for $\gamma_s(n, r)$.

COROLLARY 6.1. *Suppose that* r, t *are positive integers such that an Hadamard matrix of order* t *exists and* $t \leqslant r \leqslant t + 2\sqrt{t}$, $t > 3$. *Suppose that* $-\sqrt{t} < \alpha - 1 - (r - t) \leqslant \sqrt{t}/2$. *Then there exists an* r*-regular bipartite graph* F *on* $4t$ *vertices such that*

$$\operatorname{dom}_\alpha(F) \geqslant \tfrac{1}{2}\sqrt{r} + 4\alpha - 4(r - t) - 8. \quad (6.1)$$

Proof. Consider the graph G^{4t} defined in the previous section. Since it is regular and bipartite, by the König–Frobenius theorem [14] one can add $r - t$ perfect matchings to G to obtain F. Let g be an α-domination function of F. Since $g(N(v)) \geqslant \alpha - 1 - (r - t)$ for all $v \in V(F)$, we can apply Lemma 5.2. ∎

Considering vertex disjoint copies of graphs one can see that

$$\gamma_s(n_1, r) + \gamma_s(n_2, r) \geqslant \gamma_s(n_1 + n_2, r).$$

This, and the obvious upper bound $\gamma_s(G) \leqslant n$, imply that the limit

$$c_r := \lim_{n \to \infty} \gamma_s(n, r)/n$$

exists and is equal to its supremum (Fekete's Lemma, see, e.g., in [19]). The same is true for

$$C_r := \lim_{n \to \infty} \gamma_s(n, \geqslant r)/n,$$

and for

$$S_{k,\alpha} := \lim_{v \to \infty} \operatorname{dom}_\alpha(v, k)/v,$$

where $\mathrm{dom}_\alpha(v, k) = \max_{n+m \leqslant v}\{\mathrm{dom}_\alpha(\mathscr{H})$: $\mathscr{H}$ is an n vertex hypergraph with at most m edges each having at least k elements$\}$. Theorem 1.1 implies

$$c_r \leqslant C_r \leqslant 2S_{r+1,1} \leqslant O\left(\sqrt{\frac{\log r}{r}}\right).$$

It is conjectured (see e.g. Hall's book [11]) that Hadamard matrices exist for all t divisible by 4 (and for $t = 1, 2$). Let $\mathbf{H}$ denote the set of orders of Hadamard matrices. It is known, that $2(p+1) \in \mathbf{H}$ for all odd prime powers p, moreover $h_1, h_2 \in \mathbf{H}$ implies $h_1 h_2 \in \mathbf{H}$. Therefore (as H. Diamond [8] pointed out for us), the sequence $\mathbf{H}$ has positive density. For these values (6.1) implies

Corollary 6.2. $(1/8 - o(1))\, r^{-1/2} \leqslant c_r$.

We would get this lower bound asymptotically for all r, if we knew that the largest gap in $\mathbf{H} \cap \{1, 2, 3, ..., x\}$ is at most $o(\sqrt{x})$. It is known [8] that the largest gap between numbers having at most 3 prime divisors is less than $O(x^{4/11})$. The sieve method might give this for the numbers of the form $8(p_1+1)(p_2+1)(p_3+1)$, too.

Note that, according to (6.1) (say for $|r-t| \leqslant \sqrt{r}/100$) we have that $\mathrm{dom}_\alpha(F)$ is positive (and of order $O(\sqrt{r})$) even for α as small as $-\sqrt{r}/10$.

One can get a slightly better lower bound for C_r from the following graph G' with minimum degree r: Let $t = 2r$, $t \in \mathbf{H}$ and let G' be the restriction of the construction G^{4t} in Section 4 to $B \cup C$. Thus $G' := G[B, C] \in \mathscr{G}^{6r-2}(\geqslant r)$. This graph gives

$$\left(\frac{1}{3\sqrt{2}} - o(1)\right) r^{-1/2} \leqslant C_r.$$

Indeed, using again (5.2), one can show that $\mathrm{dom}_\alpha(G') \geqslant \sqrt{2r} + \alpha - O(1)$ (for $|\alpha| < \sqrt{r/2}$). The details are omitted.

7. A MATCHING UPPER BOUND FROM A DISCREPANCY CONJECTURE

Let us call the set T an ℓ-transversal of a hypergraph $\mathscr{H}$ if $|A \cap T| \geqslant \ell$ for all edges $A \in \mathscr{H}$. The minimum cardinality of such a T is denoted by $\tau_\ell(\mathscr{H})$. Let $\mathscr{H}$ be a hypergraph with m edges and n vertices such that $|A| \geqslant k$ for all $A \in \mathscr{H}$. Alon [1] proved that

$$\tau_1(\mathscr{H}) \leqslant \frac{\log k}{k} n + \frac{1}{k} m, \tag{7.1}$$

and this is best possible in the following sense. Define

$$\tau_{k,\ell} =: \sup_{(n+m)\to\infty} \tau_\ell(\mathscr{H})/(n+m),$$

then $\tau_{k,1} \geqslant (\log k/k)(1-(\log\log k/\log k))$ for k sufficiently large.

Our first aim is to extend (7.1) for all ℓ in the following convenient form.

PROPOSITION 7.1. *Let $\mathscr{H}$ be a k-uniform hypergraph with m edges and n vertices. Then for all $(k/2) > \ell \geqslant 1$*

$$\tau_\ell(\mathscr{H}) \leqslant \frac{2\ell}{k} n + \frac{\ell}{\exp[\ell/4]} m. \tag{7.2}$$

Proof. It is an easy application of the probabilistic method. We proceed as in [1], and as in the proof of Theorem 1.1. First select from each edge $A \in \mathscr{H}$ an ℓ-subset $L(A) \subset A$. Then let us pick, randomly and independently each vertex $v \in S := V(\mathscr{H})$ with probability $p = 2\ell/k$. Let $P \subset S$ be the random set of the vertices thus picked. Denote by $\mathscr{U} = \mathscr{U}(P)$ the set of edges $A \in \mathscr{H}$ that intersect P in fewer than ℓ elements, and call them *uncovered* by P. Let $Q = Q_P$ be the union of $L(A)$ over all uncovered edges. Then, $P \cup Q$ is obviously an ℓ-transversal of $\mathscr{H}$.

Now we are going to estimate the expected value of $|P \cup Q|$. Clearly, $\mathrm{E}[|P|] = pn$. Moreover, $|Q| \leqslant \sum_{A \in \mathscr{U}} |L(A)|$, hence the linearity of expectation gives

$$\mathrm{E}[|Q|] \leqslant \sum_{\text{for all } A \in \mathscr{H}} \ell \,\mathrm{Prob}[|A \cap P| < \ell].$$

Using (3.1) one can estimate this probability.

$$\mathrm{Prob}[|A \cap P| < \ell] \leqslant \exp[-(\ell - |A|\, p)^2/(2\,|A|\, p)] < \exp[-\ell/4]. \quad \blacksquare$$

Note that (substituting $\ell = 4\log k$) one gets Alon's bound, too, up to a constant factor.

Recall that $\mathscr{D}(\mathscr{H})$ is the discrepancy of a hypergraph $\mathscr{H}$. Spencer [18] proved that if $\mathscr{H}$ has m edges, then

$$\mathscr{D}(\mathscr{H}) < 6\sqrt{m}. \tag{7.3}$$

The next Proposition shows that one cannot expect to improve essentially the lower bound for c_r and C_r using relatively small graphs, like we did using F^{4r} and G^{4t} defined in the previous sections.

Proposition 7.2. *Suppose that $G \in \mathcal{G}^n(\geqslant r)$. If $n/r \leqslant K$, where K is fixed, then*

$$\gamma_s(G) \leqslant 12\sqrt{K}\frac{n}{\sqrt{r}} + O(1).$$

Proof. First, define $\ell = 6\sqrt{n+1}$. As $\ell \leqslant (r+1)/2$ one can apply (7.2) (for the neighborhood hypergraph, $\mathcal{H}$, of G) and obtain an ℓ-transversal T of size at most $12\sqrt{n+1}(n/(r+1)) + O(1)$. Consider the restriction of $\mathcal{H}$ to $S \backslash T$ and add the hyperedge $S \backslash T$. Apply (7.3) for this hypergraph (with $m = n+1$) to get a partition $U \cup V = S \backslash T$ such that $||U \cap N[v]| - |V \cap N[v]|| < \ell$ for all $v \in S$. Suppose that $|U| \leqslant |V|$. Then the function g: $S \to \{+1, -1\}$ with value 1 exactly at the points of $T \cup U$ is a domination function with $g(S) \leqslant |T|$. ▮

Let $\Delta = \Delta(\mathcal{H})$ denote the maximum degree of a hypergraph $\mathcal{H}$. Beck and Fiala [6] proved that $\mathcal{D}(\mathcal{H})$ is bounded by $2\Delta - 1$, and conjecture that for some constant K

$$\mathcal{D}(\mathcal{H}) < K\sqrt{\Delta} \qquad (?). \tag{7.4}$$

Proposition 7.3. *If Conjecture* (7.4) *is true, then* $c_r < 2Kr^{-1/2} + O(1/r)$.

Proof. As in the previous proof, let $G \in \mathcal{G}^n(r\text{-reg})$. First apply (7.2) for the neighborhood hypergraph with $\ell = K\sqrt{r+2}$ to obtain an ℓ-cover, then apply (7.4) to the restricted-extended hypergraph. The details are omitted. ▮

8. FURTHER PROBLEMS, GENERALIZATIONS

The determination of the limits c_r, C_r, S_k, and $\tau_{k,\ell}$ could be very interesting, but may be difficult. There might be some hope to find them for some small values, we only know $c_3 = 4/5$. Is $C_3 = 4/5$?

Is it true that $c_r < C_r$, at least for $r > 3$?

We cannot even prove, that the sequences $\{c_r\}$, $\{C_r\}$, $\{S_k\}$ are (strictly) monotone decreasing.

A general method in discrepancy theory is to obtain an estimation from the discrepancy of the (of a) random structure. What is $\gamma_s(G)$ for a random graph (in different graph models), especially for random r-regular graphs? What is $\gamma_s(G)$ for random hypergraphs?

Other reasonable graph classes, like line-graphs, and graphs with large girth should be investigated. Is it true that their signed domination number is significantly smaller?

Instead of considering $+1$, -1 labelings, i.e., 2-colorings one can investigate c-colorings.

In [13] and further papers, the *minus domination* function is introduced, i.e., $g\colon S \to \{-1, 0, 1\}$ with $g(A) \geqslant 1$ for all $A \in \mathscr{H}$; $\gamma^-(\mathscr{H}) = \min g(S)$. Since $\gamma^-(\mathscr{H}) \leqslant \tau(\mathscr{H})$ Alon's theorem (7.1) implies that

$$\gamma^-(G) \leqslant O\left(\frac{\log r}{r}\right) n \tag{8.1}$$

holds for every $G \in \mathscr{G}^n(\geqslant r)$. We think this gives the best upper bound.

If one allows only nonnegative real weights, then one obtains the well-studied notion of *fractional matching* of $\mathscr{H}$.

Actually, the problem of $\gamma_s(\mathscr{H})$ can be considered as an integer programming problem. Its real relaxation(s) is (are) a linear programming problem (for example, having real weights on the vertices such that each weight is at least -1, and the sum along each edge is at least 1, and then we want to minimize the total sum). This gives a lower bound, but, in general, as (8.1) shows, this lower bound seems to be very far from the actual value of γ_s.

One can extend the notion of an α-dominating function of $\mathscr{H}$ by imposing an upper bound $g(A) \leqslant \beta$, too. Such (α, β) colorings are not unknown, especially in number theory, to mention one recent result, see Mathias [15]. This upper bound β could be a function of the size $|A|$.

Instead of minimizing $g(S)$ one might want to minimize $\sum_{A \in \mathscr{H}} g(S)/m$. One is tempted to conjecture that in general this quantity is much smaller than one can obtain from dom$(\mathscr{H})$. (Though for the regular, uniform case, like $\mathscr{G}^n(r\text{-reg})$, this coincides with γ_s).

ACKNOWLEDGMENTS

This research was supported in part by the Hungarian National Science Foundation Grant OTKA 016389, and by a National Security Agency Grant MDA904-98-I-0022. The authors are indebted to H. Diamond and H. Halberstam (Urbana, IL) for helpful comments and to Kevin O'Bryant for help with Mathematica.

REFERENCES

1. N. Alon, Transversal numbers of uniform hypergraphs, *Comb.* **6** (1990), 1–4.
2. N. Alon, J. Spencer, and P. Erdős, "The Probabilistic Method," p. 238, Wiley, New York, 1992.
3. J. Beck, Roth's estimate of the discrepancy of integer sequences is nearly sharp, *Combinatorica* **6** (1981), 319–325.

4. J. Beck and W. W. L. Chen, "Irregularities of Distribution," Cambridge Tracts in Mathematics, Vol. 89, Cambridge Univ. Press, Cambridge/New York, 1987.
5. J. Beck and T. Fiala, "Integer-making" theorems, *Discrete Appl. Math.* **3** (1981), 1–8.
6. J. Beck and V. T. Sós, Discrepancy theory, *in* "Handbook of Combinatorics" (R. L. Graham *et al.*, Eds.), Vols. 1 and 2, pp. 1405–1446, Elsevier, Amsterdam, 1995.
7. H. Chernoff, A measure of asymptotic efficiency for tests of a hypothesis based on the sum of observations, *Ann. Math. Statist.* **23** (1952), 493–509.
8. H. Diamond and H. Halberstam, personal communication.
9. J. E. Dunbar, S. T. Hedetniemi, M. A. Henning, and P. J. Slater, Signed domination in graphs, *in* "Graph Theory, Combinatorics, and Algorithms, Proceedings of the Seventh International Conference in Graph Theory, Combinatorics, Algorithms, and Applications, Kalamazoo, MI, 1992" (Y. Alavi and A. Schwenk, Eds.), pp. 311–321, Wiley–Interscience, New York, 1995.
10. O. Favaron, Signed domination in regular graphs, *Discrete Math.* **158** (1996), 287–293.
11. M. Hall, Jr., "Combinatorial Theory," Wiley–Interscience, New York, 1986.
12. M. A. Henning, Domination in regular graphs, *Ars Combin.* **43** (1996), 263–271.
13. M. A. Henning and P. J. Slater, Irregularities relating domination parameters in cubic graphs, *Discrete Math.* **158** (1996), 87–98.
14. L. Lovász, "Combinatorial Problems and Exercises," Problem 7.10, Akadémiai Budapest, North-Holland, Amsterdam, 1979.
15. A. R. D. Mathias, On a conjecture of Erdős and Čudakov, *Combin. Probab. Comput.*, in press.
16. J. E. Olson and J. H. Spencer, Balancing families of sets, *J. Combin. Theory Ser. A* **25** (1978), 29–37.
17. V. T. Sós, Irregularities of partitions, *in* "Surveys in Combinatorics" (E. K. Lloyd, Ed.), London Math. Soc. Lecture Notes, Vol. 82, pp. 201–245, Cambridge Univ. Press, Cambridge, UK, 1983.
18. J. Spencer, Six standard deviations suffice, *Trans. Amer. Math. Soc.* **289** (1985), 679–706.
19. J. H. van Lint and R. M. Wilson, "A Course in Combinatorics," Cambridge Univ. Press, Cambridge, UK, 1992, page 85.
20. B. Zelinka, Some remarks on domination in cubic graphs, *Discrete Math.* **158** (1996), 249–255.