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Note

# On the double competition number ☆

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**Abstract**

It is known, that for *almost all* $n$-vertex simple graphs one needs $\Omega(n^{4/3}(\log n)^{-4/3})$ extra vertices to obtain them as a double competition graph of a digraph. In this note a construction is given to show that $2n^{4/3}$ are always sufficient. © 1998 Elsevier Science B.V. All rights reserved.

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## 1. Definitions

The *double competition graph* of a directed graph $D=(V,\mathscr{A})$ is the graph $G=(V,\mathscr{E})$ where $xy\in\mathscr{E}(G)$ if and only if for some $u,v\in V(G)$, the (directed) arcs $ux$, $uy$ and $xv$, $yv\in\mathscr{A}(D)$. The double competition number of a (simple) graph, $\mathrm{dk}(G)$, is the smallest integer $k$ so that $G\cup I_k$ (i.e., $G$ and $k$ isolated vertices) is the double competition graph of some digraph. In most research only acyclic digraphs are investigated, but to obtain the most general result possible, we omit this constraint. Double competition graphs are also called *competition-common enemy graphs*. Generating each edge of $G$ by two new vertices one gets

$$\mathrm{dk}(G)\leqslant 2|\mathscr{E}(G)|. \qquad (1)$$

The set of all (labeled) graphs over the elements $\{1,2,\dots,n\}$ is denoted by $\mathscr{G}^n$. Obviously, $|\mathscr{G}^n|=2^{\binom{n}{2}}$. The statement "almost all graphs have property P" means that there exists a sequence $\varepsilon_1,\varepsilon_2,\dots$ tending to 0 such that the number of graphs $G\in\mathscr{G}^n$ having property P is at least $(1-\varepsilon_n)2^{\binom{n}{2}}$.

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The *neighborhood* of $x$ is denoted by $N_G(x)$. As usual, $\omega(G)$ denotes the size of the largest complete subgraph in $G$. The *chromatic number* of $G$ is denoted by $\chi(G)$, the *complement* of $G$ is $\overline{G}$. The *induced subgraph*, $G|S$, is a graph with the vertex set $S \cap V(G)$ and with the edges of $G$ contained in $S$.

The system $\mathscr{S} = \{S_1, S_2, \ldots, S_t\}$ (where $S_i \subset V$) forms a *clique cover* of the graph $G = (V, \mathscr{E})$ if $\mathscr{E}(G) = \bigcup_{1 \leqslant i \leqslant t} \mathscr{E}(G|S_i)$ and each induced subgraph $G|S_i$ is a complete graph $K(S_i)$. The minimum possible $t$ is called the *clique covering number*, $\mathrm{cc}(G)$. Adding $t+1$ new vertices, $u$ and $v_1, \ldots, v_t$, to $V(G)$, and defining a digraph $D$ by $\mathscr{A} := \{ux\colon x \in V\} \cup \{xv_i\colon x \in S_i, 1 \leqslant i \leqslant t\}$ one gets the following improvement of (1):

$$\mathrm{dk}(G) \leqslant 1 + \mathrm{cc}(G). \tag{2}$$

A *linear transversal system* with parts $\{H_1, H_2, \ldots, H_b\}$ (where these parts are pairwise disjoint sets) is a system of subsets, $\mathscr{L}$, of $H := \bigcup H_i$ such that

1. $|L \cap H_i| \leqslant 1$ for all $L \in \mathscr{L}$ and $1 \leqslant i \leqslant b$;
2. the members of $\mathscr{L} \cup \{H_1, \ldots, H_b\}$ cover every two-element subset of $H$ exactly once.

The existence of projective planes of prime orders (see, e.g., [6]) implies that for every given system $\{H_1, \ldots, H_b\}$, there exists a linear transversal system $\mathscr{L}$ of size

$$|\mathscr{L}| \leqslant p^2, \tag{3}$$

where $p$ is a prime with $p \geqslant \max\{b, |H_1|, \ldots, |H_b|\}$. To prove (3) we simply take a projective plane of order $p$, choose an arbitrary point, $x$, consider $b$ lines $\ell_1, \ldots, \ell_b$ through $x$, put the set $H_i$ to $\ell_i \setminus \{x\}$, and restrict the line system of the projective plane to $\bigcup H_i$.

## 2. Results

Competition graphs, since they were defined in the late 1960s, were a subject of intensive research, see, e.g., the book of Roberts [9]. A number of basic properties of double competition graphs can be found in Lundgren [8].

Kim [7] asked what is the maximum, $\mathrm{dk}(n)$, of the double competition number of graphs $G \in \mathscr{G}^n$. It is proved in [5] that for *almost all* simple graphs over $n$ vertices one needs $\Omega(n^{4/3}(\log n)^{-4/3})$ extra vertices to obtain them as a double-competition graph of a digraph. Hence,

$$\mathrm{dk}(n) \geqslant \Omega(n^{4/3}(\log n)^{-4/3}). \tag{4}$$

The aim of this note is to give a construction showing that $\mathrm{dk}(G) \leqslant 2n^{4/3}$ holds for *every* graph.

Suppose that $G \in \mathscr{G}^n$ is an arbitrary graph, and let $V = V_1 \cup \cdots \cup V_b$ ($b \geqslant 2$) be a partition of its vertex set. Let $G_i$ denote the induced subgraph $G|V_i$. Let $h_i$ be the chromatic number of $\overline{G_i}$, i.e., the minimum number of cliques of $G_i$ covering all the vertices of $V_i$. Finally, suppose that $p$ is a prime with $p \geqslant \max\{b, h_1, \ldots, h_b\}$.

**Theorem 1.** *There exists an acyclic digraph D with vertex set $A \cup V \cup B$, where these three sets are pairwise disjoint, ($A \cap B = \emptyset$ and $V \cap (A \cup B) = \emptyset$), with arcs only going from A to V and from V to B, such that G (with the additional isolated vertices of $A \cup B$) is the double competition graph of D. Moreover,*

$$|A| + |B| \leqslant (b^2/2)\max_i |V_i| + \max_i \mathrm{cc}(G_i) + p^2.$$

The proof, an explicit construction, is given in the next section.

**Corollary 1.**

$$\mathrm{dk}(G) \leqslant 2n^{4/3}$$

*holds for every graph G on n vertices.*

**Proof.** Indeed, by a theorem of Erdős et al. [3], $\mathrm{cc}(G) \leqslant n^2/4$ holds for each $G \in \mathcal{G}^n$. Hence, (2) gives a bound for $\mathrm{dk}(G)$ smaller than $2n^{4/3}$ if $n \leqslant 20$. For $n > 20$ take the largest prime $p$ such that $n^{2/3} \geqslant p > \frac{1}{2}n^{2/3}$. (Such prime exists by Chebisheff's theorem.) Divide the vertex set $V(G)$ into $b := \lceil n/p \rceil$ almost equal sized parts, $V = V_1 \cup \cdots \cup V_b$. (We have $b \leqslant p$.) Use the construction given by Theorem 1. We have $\max |V_i| \leqslant p$, $\mathrm{cc}(G_i) \leqslant p^2/4$ (by [3]), and an easy calculation gives the desired upper bound.

It is very likely, that a more carefully chosen partition of $V(G)$ can yield a better result. We propose the problem to prove $\mathrm{dk}(n) = \mathrm{o}(n^{4/3})$.

**Conjecture 1.** $\mathrm{dk}(n) = \mathrm{O}(n^{4/3}(\log n)^{-2/3})$.

This Conjecture is true for almost all graphs.

**Corollary 2.**

$$\mathrm{dk}(G) \leqslant \mathrm{O}(n^{4/3}(\log n)^{-2/3})$$

*holds for almost all graphs on n vertices.*

**Proof.** Indeed, let $b = \lfloor n^{1/3}(\log n)^{-2/3} \rfloor$. Divide the vertex set $V(G)$ into $b$ parts, $V = V_1 \cup \cdots \cup V_b$, of sizes $(n/b) - 1 < |V_i| \leqslant (n/b) + 1$. The graphs $G_i := G|V_i$ behave exactly as independent random graphs on $|V_i|$ vertices. It is well known, e.g., [1], that the vertex covering number, i.e., the chromatic number of the complement, of a random graph $G_i$ is $\mathrm{O}(|V_i|/\log |V_i|)$ with a very high probability. So it holds for *all* $i$ simultaneously with high probability. We get $\max\{b, \chi(\overline{G_1}), \ldots, \chi(\overline{G_b})\} = \mathrm{O}(n^{2/3}(\log n)^{-1/3})$. Again, using the construction given by Theorem 1, we get $p = \Theta(n^{2/3}(\log n)^{-1/3})$.

The clique covering number of the random graph $G \in \mathcal{G}^k$ has been recently determined by Frieze and Reed [4] to be $\Theta(k^2/(\log k)^2)$. The lower bound is trivial, because $\omega(G) < 2\log k$ holds with an extremely high probability. The previous best upper bound, $\mathrm{cc}(G) \leqslant O(k^2(\log\log k)/(\log k)^2)$, was due to Bollobás et al. [2]. According to Remark 1 in [4], one can see that $\mathrm{cc}(G)$ is close to its mean value with very high probability. Again the independentness of the $G_i$'s implies that $\mathrm{cc}(G_i) \leqslant O(|V_i|^2/(\log|V_i|)^2)$ holds simultaneously for all $i$ with large probability. This implies $\max_i \mathrm{cc}(G_i) \leqslant O(n^{4/3}(\log n)^{-2/3})$, resulting in the desired upper bound for $\mathrm{dk}(G)$. □

It seems to me that there is some room to improve the lower bound (4).

**Conjecture 2.** $\mathrm{dk}(G) = \Theta(n^{4/3}(\log n)^{-2/3})$ *holds for almost all graphs on n vertices.*

## 3. The construction of the acyclic digraph

Here we prove Theorem 1. We can assume that each $V_i$ has at least two vertices as otherwise the theorem follows from (2). We build the digraph $D$ from four types of arcs, $\mathscr{A} := \mathscr{A}_1 \cup \mathscr{A}_2 \cup \mathscr{B}_1 \cup \mathscr{B}_2$.

*First*, take $b$ extra vertices $\{a_1, \ldots, a_b\}$ and let $\mathscr{A}_1$ be the set of arcs connecting $a_i$ to each vertex of $V_i$,

$$\mathscr{A}_1 := \{a_i x\colon\ 1 \leqslant i \leqslant b,\ x \in V_i\}.$$

*Second*, for every $1 \leqslant i < j \leqslant b$ take a $|V_i|$-element set $\{a_{i,j}(x)\colon\ x \in V_i\}$, and connect $a_{i,j}(x)$ to $x$ and to all the neighbors of $x$ in $V_j$, i.e.,

$$\mathscr{A}_2 := \{a_{i,j}(x)x\colon\ 1 \leqslant i < j \leqslant b,\ x \in V_i\}$$
$$\cup \{a_{i,j}(x)y\colon\ 1 \leqslant i < j \leqslant b,\ x \in V_i,\ y \in V_j \cap N_G(x)\}.$$

The set $A := \{a_i\colon\ 1 \leqslant i \leqslant b\} \cup \{a_{i,j}(x)\colon\ 1 \leqslant i < j \leqslant b,\ x \in V_i\}$ has $b + \sum_{1 \leqslant i < b} |V_i|(b-i)$ elements, which is at most $(b^2/2)\max|V_i|$.

*Third*, consider a clique cover $S_{i,\alpha} \subset V_i$ $(1 \leqslant \alpha \leqslant \mathrm{cc}(G_i))$ of $G_i$, take $m := \max_i \mathrm{cc}(G_i)$ extra vertices $\{w_1, \ldots, w_m\}$ and connect every vertex of $\bigcup_{1 \leqslant i \leqslant b} S_{i,\alpha}$ with an arc to $w_\alpha$, i.e.,

$$\mathscr{B}_1 := \{x w_\alpha\colon\ x \in S_{i,\alpha},\ 1 \leqslant i \leqslant b,\ 1 \leqslant \alpha \leqslant \mathrm{cc}(G_i)\}.$$

*Fourth*, consider a clique cover of the *vertices* of $G_i$, $V_i = V_{i,1} \cup \cdots \cup V_{i,h_i}$ and let $\mathscr{L} := \{\ell_1, \ell_2, \ldots\}$ be a linear transversal system with parts $H_i := \{V_{i,1}, \ldots, V_{i,h_i}\}$. Take $|\mathscr{L}|$ new vertices $z_1, z_2, \ldots$ and connect each vertex of the sets $V_{i,s}$ forming the set $\ell_\beta$ to $z_\beta$, i.e.,

$$\mathscr{B}_2 := \{x z_\beta\colon\ x \in V_{i,s} \in \ell_\beta,\ 1 \leqslant i \leqslant b,\ 1 \leqslant \beta \leqslant |\mathscr{L}|\}.$$

The set $B := \{w_i\colon\ 1 \leqslant i \leqslant m\} \cup \{z_\beta\colon\ 1 \leqslant \beta \leqslant |\mathscr{L}|\}$ has at most $\max_i \mathrm{cc}(G_i) + p^2$ elements by (3).

Finally, we show that the above defined directed graph $D$ induces $G$. Obviously, each edge, $xy$, of the double competition graph of $D$ must connect vertices with positive in- and outdegrees, so it must be contained in $V$. Moreover, if $ux$, $uy$, $xw$, and $yw$ are arcs of $\mathscr{A}(D)$, then $u$ should be in $A$ and $w$ in $B$. The set of pairs $\{x,y\} \subset V$ which have a common neighbor $u$ with $ux$, $uy \in \mathscr{A}_1 \cup \mathscr{A}_2$ is formed by the pairs in some $V_i$ and by the edges of $G$:

$$\bigcup_{1 \leqslant i \leqslant b} \mathscr{E}(K(V_i)) \cup \mathscr{E}(G). \tag{5}$$

The set of pairs $\{x,y\}$ in $V$ which have a common neighbor $w$ with $xw$, $yw \in \mathscr{B}_1 \cup \mathscr{B}_2$ is exactly the pairs in some $S_{i,\alpha}$ or in $V_{i,s}$ and all the pairs connecting two parts of the partition $\{V_1, \ldots, V_b\}$:

$$\bigcup_{1 \leqslant i \leqslant b} \mathscr{E}(G_i) \cup \left( \bigcup_{1 \leqslant i < j \leqslant b} V_i \times V_j \right). \tag{6}$$

The intersection of the sets of pairs of (5) and (6) is $\mathscr{E}(G)$, implying that $G \cup I_{A \cup B}$ is the double competition graph induced by $D$. □


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