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Minimal Oriented Graphs of Diameter 2*

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This paper is dedicated to the memory of Štefan Znám.

Abstract. Let $f(n)$ be the minimum number of arcs among oriented graphs of order n and diameter 2. Here it is shown for $n > 8$ that $(1 - o(1))n \log n \le f(n) \le n \log n - (3/2)n$.

1. Oriented chromatic number

An *oriented graph* is a digraph without opposite arcs, i.e., every pair of vertices is connected by at most one arc. An *oriented colouring* of an oriented graph D is a colouring of its vertices so that every colour class is an independent set, moreover for any two colour classes U and V, all the arcs between them have the same orientation, i.e., either all the arcs of D joining U and V go from U to V or all the arcs go from V to U. The *oriented chromatic number* of D is the minimum number of colours in such colourings. For an unoriented graph G, the *oriented chromatic number*, $\chi_o(G)$, of G is defined as the maximum oriented chromatic number of the orientations of G. The notion of the oriented chromatic number has been introduced by Courcelle [5]. It was noted in [12] that the oriented chromatic number of the complete bipartite graph, $K_{k,k}$, is $2k$, the order of the graph. In this paper, the following question is studied.

What is the minimum number of edges of a graph G on n vertices with the property $\chi_o(G) = n$?

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Suppose, x, y are vertices of an oriented graph D such that x, y are neither adjacent nor connected by a directed path of length 2. Then the colouring of D which colours x, y by one colour, and colours every other vertex by a distinct colour is an oriented colouring of D. On the other hand, if x, y are adjacent or connected by a directed path of length 2, then they cannot be coloured by the same colour. Therefore, an oriented graph D has oriented chromatic number $|V(D)|$ if and only if every pair of vertices of D is connected by an arc or by a directed path of length 2. In other words, for every pair of vertices of D, at least one vertex can be reached from the other in one or two steps by walking along the arcs of D.

2. Graphs of diameter 2

Let us define the *diameter* of an oriented graph D to be the least integer d, such that every pair of vertices is connected by a directed path of length at most d. Therefore, the question in the previous section is equivalent to the problem of determining the minimum number of arcs in an oriented graph of order n and diameter 2. We denote this number by $f(n)$, i.e., $f(n) = \min\{|E(D)| : |D| = n, \operatorname{diam}(D) = 2\}$.

In fact, the problem of determining the function $f(n)$ was originally posed by Erdős, Rényi and Sós [7] in 1966 and later by Znám [15] and Dawes and Meijer [6]. For unoriented graphs the answer to the question of determining the minimum number of edges among all graphs of order n and diameter 2 is trivial. Such a graph has $n-1$ edges and the star is the only extremal graph. Katona and Szemerédi [11] showed that

$$\frac{n}{2}\log\frac{n}{2} \leq f(n) \leq n\lceil \log n \rceil. \tag{1}$$

All logs in this formula and all over in this paper are of base 2.

The main result of this paper is, that $\lim f(n)/(n \log n) = 1$ for $n \to \infty$. We also provide a slight improvement of the upper bound.

Theorem 1. *For any $n \geq 9$,*

$$(1 - o(1))n \log n \leq f(n) \leq n \log n - \frac{3}{2}n.$$

The corresponding constructions suggest that characterizing the extremal graphs is probably a difficult problem. At the end of the paper it is shown that our asymptotic result applies also to, $\bar{f}(n)$, the minimum number of edges of oriented graphs of strong diameter 2.

3. Recursive constructions

To show that $f(n) \leq n \log n - n$ for $n > 3$, we consider the following operation: Let G, H be two vertex disjoint oriented graphs and let v be a vertex of G. The oriented graph, $G_v(H)$, is obtained from G by replacing the vertex v by the graph H. More formally, the vertex set of $G_v(H)$ is $V(G) \cup V(H) - v$ and the arc $a = (x, y)$ belongs to $G_v(H)$ if one of the following holds:

(i) a is an arc in either $G - v$ or H,
(ii) $x \in V(H)$, and $y \in V(G) - v$ and (v, y) is an arc of G,
(iii) $x \in V(G) - v$, and $y \in V(H)$ and (x, v) is and arc of G.

It is easy to see, that if both G and H are oriented graphs of diameter 2, then $G_v(H)$ is an oriented graph of diameter 2, too.

Now, we construct a sequence $\{H(n)\}_{n=1}^{\infty}$ of oriented graphs of diameter 2, where $H(n)$ is of order n. First set $H(1)$ to be a graph consisting of a single vertex and $H(2)$ to be a graph on two vertices and an arc. Let G be an oriented path of length 2 with initial and terminal vertices u and v, respectively. For $n > 2$, the graph $H(n)$ is obtained from G by first substituting the vertex u by $H\left(\left\lfloor \frac{n-1}{2} \right\rfloor\right)$ and then v by $H\left(\left\lceil \frac{n-1}{2} \right\rceil\right)$. Let $h(n)$ be the number of arcs of $H(n)$. Then $h(n) = h\left(\left\lfloor \frac{n-1}{2} \right\rfloor\right) + h\left(\left\lceil \frac{n-1}{2} \right\rceil\right) + n - 1$. It can be easily proved by induction that

$$h(n) = (n+1)t - 2^{t+1} + 2$$

where $t = \lfloor \log n \rfloor$. We have that $h(n) - n \log n = n(t - \log n) - 2^{t+1} + (t+2)$, which is clearly negative for $n \geq 4$. Even more, we have that

$$f(n) \leq n \log n - 1.913 \cdots n + t + 2, \tag{2}$$

where the number 1.913 ... stands for $1 + \log e - \log \log e$. This gives the upper bound in the Theorem for all $n > 12$. One can further improve the constant 1.913 ... by about another 0.03 observing that $f(5) = 5$ (obtained from an oriented cycle), and again applying the recursion

$$f(n) \leq \min_i \{f(i) + f(n-1-i) + n - 1\}.$$

Especially, this yields $f(9) \leq 15$, $f(10) \leq 18$, $f(11) \leq 20$, $f(12) \leq 24$ giving the upper bound in the Theorem for all $n > 8$.

However, all these efforts have no effect on the linearity of our error term.

4. Lower bound by the method of crossinter secting pairs

Now, we shall prove the lower bound in Theorem 1. To get the lower estimate in (1) Katona and Szemerédi [11] derived, first, a lower bound on the number of vertices of bipartite graphs which cover the edges of the complete graph. (The same result was, independently but later, obtained by A. Moon [13], too). The method they used is known now as the method of crossintersecting families and it was first applied successfully by Bollobás [4] in 1965. Actually, the method was rediscovered several times (e.g., Alspach, Ollmann and Reid [2] in 1975), for developments see, e.g., Tuza [14], and the survey [8]. To be able to improve on (1) by about a factor of 2, the crucial observation of our proof is that in an extremal oriented graph most of the edges are concentrated to a few high-degree vertices. Thus applying the method of crossintersecting families at the low degree endpoints *only*, we are able to avoid the double counting. To do this we have to use a version of the method allowing a bit more (but bounded) disjointness among the cross-intersecting pairs. (A similar approach was used in [10].)

Let $G = (V, E)$ be an oriented graph on n vertices of diameter 2. Let $d = \lceil (\log n)^2 \rceil$, and let A be the set of vertices of G of degree less than d. Suppose $A = \{x_1, x_2, \ldots, x_k\}$ and that the vertex x_i is adjacent to d_i vertices in $V - A$. Let $s = |V - A|$, and assume that $V - A = \{v_1, v_2, \ldots, v_s\}$. For each vertex x of A, we associate a set $U(x)$ of 0–1-sequences of length s as follows:

$$U(x) = \{(a_1, a_2, \ldots, a_s) : a_i = 0 \text{ if } xv_i \in E \text{ and } a_i = 1 \text{ if } v_i x \in E\}.$$

Then $|U(x_i)| = 2^{s-d_i}$, as a_ℓ could be either 0 or 1 in case of x_i is not adjacent to v_ℓ. Next we show that each 0–1-sequence of length s can appear in at most $1 + d^2$ sets $U(x)$. Indeed, if $\mathbf{a} = (a_1, a_2, \ldots, a_s) \in U(x) \cap U(y)$ then there is no vertex $v \in V - A$ such that $xv, vy \in E(G)$ or yv, $vx \in E(G)$. Since G has diameter 2, we conclude that x, y are either adjacent or connected by a path of length 2 contained in A. Since the subgraph of G induced by A has maximum degree d, there are at most d^2 vertices of A that are connected to x by a directed path (of either direction) of length 1 or 2. Therefore the 0–1-sequence $\mathbf{a}$ appears in at most d^2 other sets $U(y)$. Hence, by taking the sum $\sum_{1 \le i \le k} |U(x_i)|$, each 0–1-sequence of length s is counted at most $1 + d^2$ times. It follows that

$$\sum_{1 \le i \le k} 2^{-d_i} \le 1 + d^2.$$

Let $e = \sum_{1 \le i \le k} d_i$. Since the function 2^{-x} is convex, Jensen's inequality implies that

$$k 2^{-e/k} \le 1 + d^2.$$

Hence $e \ge k \log k - k \log(1 + d^2)$.

If $|V - A| \ge 2n/\log n$ then the number of edges of G is at least $n \log n$. If $|V - A| < 2n/\log n$, then $k = |A| > n - (2n/\log n)$. This implies that

$$|E(G)| \ge e \ge k \log k - k \log(1 + \lceil \log n \rceil^4) = n \log n - O(n \log \log n). \qquad \square$$

5. Strong diameter and other problems

Strong diameter. Define the *strong diameter* of a digraph as the least integer d such that for any pair of vertices, u and v, there exist two directed paths of length at most d, one from u to v and the other from v to u. Then one can ask the following question:

What is the minimum number of arcs, $\bar{f}(n)$, of an oriented graph on n vertices with strong diameter 2?

It turns out that $\bar{f}(n)$ and $f(n)$ are asymptotically the same.

Theorem 2. $\bar{f}(n) = n \log n + O(n \log\log n)$.

Proof. The lower bound follows from the fact that $\bar{f}(n) \geq f(n)$. For the upper bound, we construct the digraph G as follows:

Let A be a set of size $2k$, and let B be a set of size at most $\binom{2k}{k}$, where we choose k to be the minimum integer with $n \leq 2k + 2 + \binom{2k}{k}$. Then the vertex set of G is $V = A \cup B \cup \{x, y\}$ and there is an arc from x to each element of A, an arc from each element of A to y, an arc from y to each element of B, an arc from each element of B to x. Moreover associate each vertex b of B a distinct k-subset $S(b)$ of A. Then put an arc from b to each element of $S(b)$ and an arc from each element of $A - S(b)$ to b. It is straightforward to verify that G indeed has strong diameter 2, and that the number of arcs as it was claimed. (The only extra care we need is, that for every $a, a' \in A$ we have to choose an $S(b)$ with $a \in S(b)$ but $a' \notin S(b)$. This can be done by less than $k(k-1)$ sets, then the rest of the choices for $S(b)$'s could be arbitrary.) □

We believe, though, that the above construction is close to the optimal, and there must be a relatively large gap between $f(n)$ and $\bar{f}(n)$.

Conjecture 1. $\bar{f}(n) \geq n \log n + (\frac{1}{2} + o(1))n \log\log n$.

Larger diameters. Let $f(n,d)$ ($\bar{f}(n,d)$) denote the minimum number of arcs in a simple n-vertex directed graph of diameter (strong diameter, respectively) at most d. For fixed $d > 2$, the order of magnitude is only linear in n. This and some conjectures of Znám [15] and Dawes and Meijer [6] are the subject of a forthcoming manuscript [9].

Homomorphisms of edge coloured graphs. Very recently, N. Alon and T. H. Marshall [1] discussed the following problem. A *homomorphism* of an edge coloured graph $G_1 = (V_1, E_1)$ to another edge coloured graph $G_2 = (V_2, E_2)$ is a mapping $\varphi : V_1 \to V_2$ such that for every edge uv of G_1, $\varphi(u)\varphi(v)$ is an edge of G_2, and that the colour of the edge $\varphi(u)\varphi(v)$ is the same as that of uv. For an edge coloured graph G, let $\lambda(G)$ be the minimal order of an edge coloured graph H such that there is a homomorphism of G to H. For an uncoloured graph G, we may define $\lambda(G,k)$ to be the maximum of $\lambda(G')$ where G' runs over all edge colourings of G by k colours. Given an uncoloured, undirected graph G, it is easy to see that $\lambda(G,1) = \chi(G)$. It turns out that $\lambda(G,2)$ and $\chi_o(G)$ are closely related, although they are different. One can ask the following question:

What is the minimum number of edges, $g(n)$, of a graph G on n vertices such that $\lambda(G,2) = n$?

Similarly to the case of the oriented chromatic number, this question is equivalent to the question of finding the minimum number of edges of an edge coloured graph with 2 colours such that any pair of vertices are either adjacent or connected by a path of length two whose two edges are coloured by distinct colours. In general $g(n)$ is different from $f(n)$, for example $f(5) = 5$ and $g(5) = 6$. However, it is straightforward to modify the argument in this paper to show that

$$(1 - o(1))n \log n \leq g(n) \leq n \log n. \quad (3)$$

The details are omitted.

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