Combinatorics, Probability and Computing (1996) 5, 29–33

An Upper Bound on Zarankiewicz' Problem

ZOLTÁN FÜREDI

Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL 61801-2975, USA
and Mathematical Institute of the Hungarian Academy of Sciences, PO Box 127, Budapest 1364, Hungary

Received 5 January 1995; revised 9 March 1995

Let $\mathrm{ex}(n, K_{3,3})$ denote the maximum number of edges of a $K_{3,3}$-free graph on n vertices. Improving earlier results of Kővári, T. Sós and Turán on Zarankiewicz' problem, we obtain that Brown's example for a maximal $K_{3,3}$-free graph is asymptotically optimal. Hence $\mathrm{ex}(n, K_{3,3}) \sim \frac{1}{2}n^{5/3}$.

1. The Turán problem

Given a graph L, what is $\mathrm{ex}(n, L)$, the maximum number of edges of a graph with n vertices not containing L as a subgraph? This is one of the basic problems of extremal graph theory, the so called Turán problem. The most well-known case is $\mathrm{ex}(n, K_3) = \lfloor n^2/4 \rfloor$ (cf. Mantel [11], Turán [13] and for a survey see Bollobás' book [1]). The Erdős–Stone–Simonovits theorem [5, 6] says that the order of magnitude of $\mathrm{ex}(n, L)$ depends on the chromatic number of L, namely $\lim_{n\to\infty} \mathrm{ex}(n, L)/\binom{n}{2} = 1 - (\chi(L) - 1)^{-1}$. This theorem gives a sharp estimate, except for bipartite graphs.

Until now, the only asymptotics for a bipartite graph which is not a forest, $\mathrm{ex}(n, K_{2,t+1}) = \frac{1}{2}\sqrt{t}(1 + o(1))n^{3/2}$, is due to Erdős, Rényi and Sós [4] and Brown [2] for the case of C_4 (for the most recent results see [8]); the case $t > 1$ can be found in [7]. Brown [2] gave a construction using finite affine geometries showing $\mathrm{ex}(p^3, K_{3,3}) \geqslant (p^5 - p^4)/2$ for all odd primes. Here we prove an upper bound showing that his example is nearly optimal.

Theorem 1. $\mathrm{ex}(n, K_{3,3}) = \frac{1}{2}n^{5/3} + O(n^{5/3-c})$ *for some constant* $c > 0$.

The previous best upper bound, mentioned below as (1), was $(2^{1/3}/2)n^{5/3} + n$.

2. The main theorem

Given m, n, s and t integers, $m \geqslant s \geqslant 1$, $n \geqslant t \geqslant 1$, what is the maximum number, $z = z(m, n, s, t)$, such that there exists a 0–1 matrix M with m rows and n columns containing z 1's without a submatrix with s rows and t columns consisting of entirely of 1's. In 1951

Zarankiewicz [14] posed the problem of determining $z(n,n,3,3)$ for $n \leqslant 6$, and the general problem has also become known as *the problem of Zarankiewicz*. To avoid unnecessary repetitions, from now on, we suppose that $s \geqslant t$. Obviously, $z(m,n,s,1) = (s-1)n$. It is easy to see that $z(n,n,s,2) \leqslant n\sqrt{(s-1)n - s + 1/4} + (n/4)$, and it is known that this bound is asymptotically correct, i.e., $\lim_{n\to\infty} z(n,n,s,2)n^{-3/2} = \sqrt{s-1}$ (Kővári, Sós and Turán [10] for $s = 2$, Hyltén-Cavallius [9] for $s = 3$ and Mörs [12] for all s). For fixed s and t, the best (and simplest) general upper bound

$$z(n,n,s,t) \leqslant (s-1)^{1/t}\, n^{2-1/t} + (t-1)n \tag{1}$$

is believed to give the optimal exponent of n.

Considering the adjacency matrix of a $K_{s,t}$-free graph on n vertices we get $2\mathrm{ex}(n, K_{s,t}) \leqslant z(n,n,s,t)$. Hence Brown's example implies

$$z(n,n,3,3) \geqslant n^{5/3}(1 - o(1)). \tag{2}$$

The probabilistic method [3] gives a lower bound for z of order $\Omega(n^{2-(t+s-2)/(st-1)})$ only. Since 1956 the upper bound (1) was only slightly improved by Znám [15] in the second order term. A proof and further results can be found in [1]. The aim of this note is to present an improvement of (1) yielding that the lower bound of (2) is asymptotically correct and that Brown's construction is asymptotically optimal.

Theorem 2. *$z(m,n,s,t) \leqslant (s-t+1)^{1/t} nm^{1-1/t} + tn + tm^{2-2/t}$ holds for all $m \geqslant s$, $n \geqslant t$, $s \geqslant t \geqslant 1$.*

For fixed $s, t \geqslant 2$ and $n, m \to \infty$ the first term is the largest one for $m = O(n^{t/(t-1)})$. The upper bound in Theorem 2 is asymptotically optimal for $t = 2$ and for $t = s = 3$. It would be interesting to see whether this extends to other values.

3. Lemmata

Define $\binom{x}{k}$ as a real polynomial $x(x-1)\dots(x-k+1)/k!$ of degree k for $x \geqslant k-1$, $k \geqslant 1$ integer. For $k-1 > x \geqslant 0$ let $\binom{x}{k} = 0$, and for all real $x \geqslant 0$ let $\binom{x}{0} = 1$. Note that these functions are convex.

Lemma 1. *Let $v, k \geqslant 1$ be integers, $c, x_0, x_1, \dots, x_k \geqslant 0$ reals. Then*

$$\sum_{1\leqslant i\leqslant v} \binom{x_i}{k} \leqslant c\binom{x_0}{k} \quad \textit{implies} \quad \sum_{1\leqslant i\leqslant v} x_i \leqslant x_0 c^{1/k} v^{1-1/k} + (k-1)v.$$

Proof. Let $S = \sum_{1\leqslant i\leqslant v} x_i$. The case $S < (k-1)v$ is obvious. For $S \geqslant (k-1)v$ Jensen's inequality gives $v\binom{S/v}{k} \leqslant c\binom{x_0}{k}$. Hence

$$\frac{v}{c} \leqslant \frac{x_0}{S/v}\,\frac{x_0-1}{S/v-1}\cdots\frac{x_0-k+1}{S/v-k+1} \leqslant \left(\frac{x_0}{S/v-k+1}\right)^k.$$

Rearranging we get the desired upper bound for S. □

Lemma 2. *Let $t \geqslant 2$, $v \geqslant 1$ be integers, $y_1, \ldots, y_v \geqslant t-2$ reals. Then*

$$\left(\sum_{1\leqslant i\leqslant v}\binom{y_i}{t-2}\right)\left(\sum_{1\leqslant i\leqslant v}(y_i-(t-2))\right) \leqslant v(t-1)\sum_{1\leqslant i\leqslant v}\binom{y_i}{t-1}.$$

Proof. The case $t = 2$ is an identity. For $t \geqslant 3$ and for arbitrary reals $a, b \geqslant t-3$ one has $[a(a-1)\ldots(a-(t-3)) - b(b-1)\ldots(b-(t-3))][(a-(t-2))-(b-(t-2))] \geqslant 0$. This implies

$$\binom{a}{t-2}(b-(t-2)) + \binom{b}{t-2}(a-(t-2)) \leqslant (t-1)\left[\binom{a}{t-1}+\binom{b}{t-1}\right]. \quad (3)$$

Add up (3) with $(a,b) = (y_i, y_j)$ for all $1 \leqslant i, j \leqslant v$. Rearranging we get the desired inequality. □

4. Proof of theorem

The case $t = 1$ is trivial, the case $t = 2$ is known (see [1]), so we suppose that $s \geqslant t \geqslant 3$. (Though inequality (4) below yields the upper bound for $t = 2$, too.) For any $1 \leqslant i \leqslant m$ let $R_i =: \{j : M_{ij} = 1\}$, $C_j := \{i : M_{ij} = 1\}$, $y_j := |C_j|$, i.e., the number of nonzero entries in the j^{th} column. We may suppose that $|R_i| \geqslant t$, $|C_j| \geqslant t$ for all i and j (otherwise we can use induction on $n+m$). Fix $t-2$ rows, $1 \leqslant i_1 < i_2 < \ldots < i_{t-2} \leqslant m$. Consider all t-element subsets of $R_{i_1} \cap \ldots \cap R_{i_{t-2}}$. Any such set T is contained in at most $s-t+1$ further R_x, because M has no $s \times t$ full 1 submatrix. We obtain

$$\sum_{x\neq i_1,\ldots,i_{t-2}} \binom{|R_{i_1}\cap\ldots\cap R_{i_{t-2}}\cap R_x|}{t} \leqslant (s-t+1)\binom{|R_{i_1}\cap\ldots\cap R_{i_{t-2}}|}{t}. \quad (4)$$

Using Lemma 1 (with $v = m-t+2$, $k = t$, $c = s-t+1$, $x_0 = |R_{i_1} \cap \ldots \cap R_{i_{t-2}}|$), one has

$$\sum_{\substack{x\neq i_1,\ldots,i_{t-2}\\ 1\leqslant x\leqslant m}} |R_{i_1}\cap\ldots\cap R_{i_{t-2}}\cap R_x|$$
$$\leqslant (s-t+1)^{1/t}(m-t+2)^{1-1/t}|R_{i_1}\cap\ldots\cap R_{i_{t-2}}| + (t-1)(m-t+2).$$

Add up the above inequality for all the $\binom{m}{t-2}$ choices of $i_1, \ldots, i_{t-2}$. Then in the left-hand side we count $(t-1)$ times each full submatrix of size $(t-1)\times 1$. We obtain

$$\sum_{1\leqslant j\leqslant n}(t-1)\binom{|C_j|}{t-1} \leqslant (s-t+1)^{1/t}(m-t+2)^{1-1/t}\sum_{1\leqslant j\leqslant n}\binom{|C_j|}{t-2}$$
$$+(t-1)(m-t+2)\binom{m}{t-2}.$$

Apply Lemma 2 with $v = n$, $y_j = |C_j|$. We get that the left-hand side is at least

$$\frac{1}{n}\left(\sum_{1\leqslant j\leqslant n}\binom{|C_j|}{t-2}\right)\left[\sum_{1\leqslant j\leqslant n}(|C_j|-(t-2))\right].$$

https://doi.org/10.1017/S0963548300001814 Published online by Cambridge University Press

Thus

$$\left(\sum_{1\leqslant j\leqslant n}|C_j|\right)-n(t-2)\leqslant(s-t+1)^{1/t}(m-t+2)^{1-1/t}n+$$

$$+(t-1)(m-t+2)\frac{n\binom{m}{t-2}}{\sum_{1\leqslant j\leqslant n}\binom{|C_j|}{t-2}}. \qquad (5)$$

If the last fraction is at most $m^{(t-2)/t}$, then (5) implies the desired inequality for $\sum|C_j|$. Finally, we suppose that the fraction exceeds $m^{(t-2)/t}$, i.e.,

$$\sum_{1\leqslant j\leqslant n}\binom{|(C_j|}{t-2}<\frac{n}{m^{1-2/t}}\binom{m}{t-2}.$$

Apply Lemma 1 again (with values $v=n$, $k=t-2$, $c=n/m^{1-2/t}$, $x_0=m$, $x_i=|C_i|$). One gets that

$$\sum_{1\leqslant j\leqslant n}|C_j|<m\left(n/m^{(t-2)/t}\right)^{1/(t-2)}n^{1-1/(t-2)}+(t-3)n$$

$$=nm^{1-1/t}+(t-3)n.$$

We are done. □

Acknowledgements

The author is indebted to M. Simonovits for fruitful conversations and helpful comments. This research was supported in part by the Hungarian National Science Foundation under grants, OTKA 4269, and OTKA 016389, and by a National Security Agency grant No. MDA904-95-H-1045.

References

[1] Bollobás, B. (1978) *Extremal Graph Theory*, Academic Press.

[2] Brown, W.G. (1966) On graphs that do not contain a Thomsen graph. *Canad. Math. Bull.* **9** (1966), 281–289.

[3] Erdős, P. and Rényi, A., (1960) On the evolution of random graphs. *Magyar Tud. Akad. Mat. Kut. Int. Közl.* **5**, 17–61. (Also see *The Art of Counting, Selected Writings of P. Erdős*, J. Spencer ed., pp. 574–617. MIT Press, 1973).

[4] Erdős, P. Rényi, A. and Sós, V.T. (1966) On a problem of graph theory. *Studia Sci. Math. Hungar.* **1**, 215–235.

[5] Erdős, P. and Simonovits, M. (1966) A limit theorem in graph theory. *Studia Sci. Math. Hungar.* **1**, 51–57.

[6] Erdős, P. and Stone, A.H. (1946) On the structure of linear graphs. *Bull. Amer. Math. Soc.* **52**, 1087–1091.

[7] Füredi, Z. (1996) New asymptotics for bipartite Turán numbers. *J. Combin. Th., Ser. A* to appear.

[8] Füredi, Z. (1996) On the number of edges of quadrilateral-free graphs. *J. Combin. Th., Ser. B* to appear.
[9] Hyltén-Cavallius, C. (1958) On a combinatorial problem. *Colloq. Math.* **6**, 59–65.
[10] Kővári, T., Sós, V. T. and Turán P. (1954) On a problem of K. Zarankiewicz. *Colloq. Math.* **3**, 50-57.
[11] Mantel, W. (1907) Problem 28. *Wiskundige Opgaven* **10** (1907), 60–61.
[12] Mörs, M. (1981) A new result on the problem of Zarankiewicz, *J. Combin. Th., Ser. A* **31**, 126–130.
[13] Turán, P. (1941) On an extremal problem in graph theory, *Mat. Fiz. Lapok* **48**, 436–452 (in Hungarian). (Also see On the theory of graphs. *Colloq. Math.* **3**, 19–30).
[14] Zarankiewicz, K. (1951) Problem of P101, *Colloq. Math.* **2**, 301.
[15] Znám, Š. (1963) On a combinatorial problem of K. Zarankiewicz, *Colloq. Math.* **11**, 81–84. (Also see Two improvements of a result concerning a problem of K. Zarankiewicz. *Colloq. Math.* **13** (1965), 255–258).