QUADRILATERAL-FREE GRAPHS WITH MAXIMUM NUMBER OF EDGES

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ABSTRACT. Let q be an integer and $\mathbf{G}$ be a graph on q^2+q+1 vertices without a cycle of lengh 4. Here it is proved that the number of edges $|E(\mathbf{G})| \leq \frac{1}{2}q(q+1)^2$. Moreover, equality holds (for $q > q_0$) if and only if $\mathbf{G}$ is obtained from a projective plane via a polarity with $q+1$ absolute elements.

1. DEFINITIONS

A *hypergraph* $\mathbf{H}$ is a pair $(V, \mathcal{E})$ (or $(V(\mathbf{H}), \mathcal{E}(\mathbf{H}))$ if it is necessary to denote $\mathbf{H}$), where V, the set of *vertices*, is a finite set, and $\mathcal{E}$, the set of *edges*, is a collection of subsets of V. If every edge contains exactly two vertices, then $\mathbf{H}$ is called a *graph*. A (hyper)graph is called *simple* if it has no multiple edges. Let $\mathcal{E}[x]$ denote the set of edges containing $x \in V$. $\deg(\mathbf{H}, x)$ stands for $|\mathcal{E}[x]|$ (i.e. the *degree* of x). Set $D(\mathbf{H}) = \max\{\deg(\mathbf{H}, x) : x \in V\}$, the maximum degree. If all the degrees are d, then $\mathbf{H}$ is called *d-regular*. A 1-regular subgraph is called a 1-factor. For a vertex $x \in V$, $N_{\mathbf{H}}(x)$ (or briefly $N(x)$) denotes its neighborhood, i.e., $N_{\mathbf{H}}(x) = \{y \in V : x \neq y$ and there exists an edge $E \in \mathcal{E}$ with $\{x, y\} \subset E\}$. If all the edges have k elements, then $\mathbf{H}$ is *k-uniform*. The hypergraph $\mathbf{H}$ is called *intersecting* if $E \cap E' \neq \emptyset$ for all edges $E, E' \in \mathcal{E}$. Morover it is called 1-*intersecting* if $|E \cap E'| = 1$ holds for all distinct edges. If $|E \cap E'| \leq 1$ holds for all distinct edges, then $\mathbf{H}$ is called a *linear* (or 0–1 intersecting) hypergraph. The *restriction* $\mathbf{H}|X$ stands for the hypergraph $(V \cap X, \{E \cap X : E \in \mathcal{E}$ and $|E \cap X| \geq 1\})$. The *induced* subhypergraph $\mathbf{H}\|X$ has vertex set X and edge set $\{E \in \mathcal{E} : E \subset X\}$. The *dual* hypergraph $\mathbf{H}^*$ is obtained by interchanging the roles of vertices and edges of $\mathbf{H}$ keeping the incidences, i.e. $V(\mathbf{H}^*) = \mathcal{E}$ and $\mathcal{E}(\mathbf{H}^*) = \{\mathcal{E}[x] : x \in V\}$. The *incidency matrix* M (or $M_{\mathbf{H}}$) is an $|\mathcal{E}|$ by $|V|$ matrix with 0-1 entries, $M(E, x) = 1$ if $x \in E$ and 0 otherwise. For a simple graph $\mathbf{G}$ its *adjacency* matrix A (or $A_{\mathbf{G}}$) is a symmetric $|V| \times |V|$ matrix with $A(x, y) = 1$ if $\{x, y\} \in \mathcal{E}(\mathbf{G})$ and 0 otherwise.

A *finite projective plane* of order q (or briefly, a $PG(2, q)$) is a $(q+1)$-uniform, $(q+1)$-regular, 1-intersecting hypergraph $\mathbf{H} = (P, \mathcal{L})$, where the vertex set P has

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q^2+q+1 elements. The elements of P are called *points*, and the members of $\mathcal{L}$ are called *lines*. It follows that $|\mathcal{L}| = q^2+q+1$ and every pair of points u, v is contained in a unique line $L[u, v]$. The desarguesian projective plane, $DPG(2, q)$, is obtained from the finite field, $\mathbf{F}_q$, as follows. The points of this plane are the equivalence classes in $\mathbf{F}_q^3 \setminus \{(0,0,0)\}$ of the relation "$\sim$" defined by $(x, y, z) \sim (x', y', z')$ if there exists $c \neq 0$ in $\mathbf{F}_q$ such that $(x', y', z') = (cx, cy, cz)$. The line $[a, b, c]$ of $DPG(2, q)$ has the equation of the form $ax + by + cz = 0$ in $\mathbf{F}_q$ with $(a, b, c) \neq (0, 0, 0)$.

A *polarity* π of the projective plane $\mathbf{H}$ is a bijection $\pi : P \leftrightarrow \mathcal{L}$ such that $x \in L$ $(x \in P, L \in \mathcal{L})$ implies $\pi(L) \in \pi(x)$. A point x (line L) is called *absolute* (with respect to π) if $x \in \pi(x)$ $(\pi(L) \in L)$. The number of absolute points is denoted by $a(\pi)$. In other words, let $P = \{x_1, \dots, x_{q^2+q+1}\}$ and $\mathcal{L} = \{L_1, \dots, L_{q^2+q+1}\}$. Then the bijection $x_i \leftrightarrow L_i$ is a polarity if and only if the incidency matrix, M, of the projective plane is symmetric. Moreover, the number of absolute points equals to the number of nonzero entries in the main diagonal of M, i.e.,

$$a(\pi) = \operatorname{trace} M.$$

A theorem of Baer [B] states that for every polarity one has $a(\pi) \geq q+1$, and even more

$$a(\pi) = q + 1 + m\sqrt{q} \tag{1.1}$$

holds for some non-negative integer m. It is easy to see, that for the $DPG(2, q)$ the bijection

$$(a, b, c) \leftrightarrow [a, b, c] \tag{1.2}$$

defines a polarity π with $a(\pi) = q+1$. (A point (a, b, c) is absolute if and only if it is on the conic $a^2 + b^2 + c^2 = 0$.)

The polarity graph. The following definition is due to Erdős and Rényi [ER] from 1962. Consider a $\mathbf{H} = PG(2, q)$ and suppose it has a polarity π. Let M be a symmetric incidency matrix of $\mathbf{H}$ defined by π. Replace the 1's in the main diagonal by 0's. The obtained matrix, A, is an adjacency matrix of a graph $\mathbf{G} = \mathbf{G}(\pi)$, called the *polarity graph*. Obviously,

$$\mathbf{G}(\pi) \text{ is quadrilateral free.} \tag{1.3}$$

If $\mathbf{H}$ is desarguesian, and π is defined by (1.2) then two points (x, y, z) and (x', y', z') are joined in $\mathbf{G}$ if and only if $xx' + yy' + zz' = 0$. A point not on the conic $x^2 + y^2 + x^2 = 0$ is joined to exactly $q+1$ points and each of the $q+1$ points on this conic are joined to exactly q points. Thus

$$|\mathcal{E}(\mathbf{G})| = \frac{1}{2} q(q+1)^2. \tag{1.4}$$

We will use the notations $\lfloor x \rfloor$ and $\lceil x \rceil$ for the lower and upper integer part of the number x, resp. The notation $\binom{x}{2}$ for a real x stands for $x(x-1)/2$.

2. RESULTS

Let $f(n)$ denote the maximum number of edges in a (simple) graph on n vertices not having a cycle of length 4, (i.e., quadrilateral-free). P. Erdős [E38] proposed the problem of determining $f(n)$ more than 50 years ago. It is easy to see (Kővári, T. Sós, Turán [KST], Reiman [R]) that

$$f(n) \le \frac{1}{4}n(1+\sqrt{4n-3}),$$

i.e. for $n = q^2+q+1$

$$f(q^2+q+1) \le \frac{1}{2}(q^2+q+1)(q+1). \tag{2.1}$$

On the other hand, as Erdős, Rényi, T. Sós [ERS] pointed out in 1966, the polarity graph shows that the upper bound (2.1) must be very close to the optimal one. The same result was proved simultaneously and independently by W. G. Brown [Br].

(2.2) If q is a prime or prime power, then $\frac{1}{2}q(q+1)^2 \le f(q^2+q+1)$.

The above two inequalities imply an asymptotic solution, that is

$$f(n) \sim \frac{1}{2}n\sqrt{n}.$$

To determine the exact value of f seems to be hopeless, except in the case $n = q^2+q+1$. Erdős [E74, E75, E76] conjectured that equality holds in (2.2). (1.1) implies that the polarity graph cannot have more edges then the bound in (2.2). On the other hand, Erdős, Rényi, and T. Sós [ERS] proved that equality never holds in (2.1). (This was the so-called Friendship Theorem.)

In [F] the Erdős conjecture was proved whenever q is a power of 2. The aim of this paper is to prove it for all values of q, in the following stronger form.

Theorem (2.3). *Let* $\mathbf{G}$ *be a quadrilateral-free graph on* q^2+q+1 *vertices. Then* $|\mathcal{E}(\mathbf{G})| \le \frac{1}{2}q(q+1)^2$. *Here equality holds only for the polarity graphs, (whenever* $q > q_0$*).*

McCuaig [Mc] and independently Clapham, Flockart and Sheehan [CFS] determined $f(n)$ and all the extremal graphs up to $n \le 21$. For $q = 2$ (and $q = 3$) there are 7 (2, resp.) graphs with maximum number of edges, and only one of them is the polarity graph. So the constrain $q > q_0$ can not be omitted (as it was mistakenly stated in the earlier work of the author [F].) It seems to me that there are no other exceptional cases even for $4 \le q \le q_0$, but eventually in the case $q = 5$ (see the end of the Section 8). The value of q_0 proven in this paper is 190, and probably it can be lowered.

Conjecture 2.4. *(McCuaig [M]) Each extremal graph is a subgraph of a polarity graph.*

It was proven for $n \le 21$.

The proof of the Theorem is lengthy, and we have to distiguish between several subcases. In the next Section we collected further definitions and lemmas. In Section 4 it is proved that for an extremal graph

$$D(\mathbf{G}) \leq q+1.$$

The Section 5 contains the proof of the Theorem for *all* even q's. The last 4 Sections deal with the case q odd. Only in Sections 8 it is proven that

$$f(q^2+q+1) \leq \frac{1}{2}q(q+1)^2. \tag{2.5}$$

Moreover, for the extremal graph all degrees are either $q+1$ or q (whenever $q > q_0$).

3. LEMMAS

We will use the following more exact form of Jensen's inequality. Suppose that $k_1, \ldots, k_m$ and a, r are nonnegative integers such that $\sum k_i \geq am + r$, and $0 \leq r < m$. Then

$$\sum \binom{k_i}{2} \geq r\binom{a+1}{2} + (m-r)\binom{a}{2}. \tag{3.1}$$

Moreover, here equality holds if and only if exactly r of the k_i's equal to $a+1$, and the other k_i's equal to a. □

A *linear space* $\mathbf{L}$ is a pair $(P, \mathcal{L})$ consisting of a set P of *points* and a set of $\mathcal{L}$ of subsets of P called lines with the properties that

(1) any two distinct points x and y are contained in a unique line $L[x,y]$, and
(2) every line contains at least two points.

Note that the dual of a 1-intersecting family is a linear space. The linear space is called *trivial* if it has only one line, $\mathcal{L} = \{P\}$. The linear space is called a *near pencil* if it has a line which contains all but one of the points of P.

In 1948 deBruijn and Erdős [BE] proved that for every nontrivial linear space one has

$$|\mathcal{L}| \geq |P|. \tag{3.2}$$

Moreover, here equality holds if and only if $\mathbf{L}$ is either a near pencil or a finite projective plane $PG(2,q)$. Jim Totten [T] described all the linear spaces with v points and at most $v + \sqrt{v}$ lines. An easy consequence of his classification is the following. Suppose that $(P, \mathcal{L})$ is a nontrivial linear space with

$$|\mathcal{L}| = |P| + 1, \tag{3.3}$$

then $\mathbf{L}$ is obtained as a restriction of a $PG(2,q)$ into q^2+q points.

A hypergraph $\mathbf{H}$ is said to be *embedded* in the linear space $\mathbf{L}$, if $V(\mathbf{H}) \subset P$ and $\mathcal{E}(\mathbf{H}) \subset \mathcal{L}$. Vanstone [V] pointed out that if $\mathbf{H}$ is a $(q+1)$-uniform, 1-intersecting hypergraph with at most q^2+q+1 vertices, moreover $|\mathcal{E}| \geq q^2$, then $\mathbf{H}$ can be embedded into a projective plane of order q. Recently, Metsch [M] improved this to

$$|\mathcal{E}| > q^2 - (q/6). \tag{3.4}$$

Suppose that, as earlier, we have a $(q+1)$-uniform, 1-intersecting hypergraph $\mathbf{H}$ over q^2+q+1 vertices. Such a hypergraph is called a *partial projective plane.* S. Dow [D] proved that if

$$|\mathcal{E}(\mathbf{H})| > q^2 - q + 1, \tag{3.5}$$

and $\mathbf{H}$ can be embedded into a $PG(2,q)$, then this embedding is unique.

Suppose now that $\mathbf{H}$ is a linear hypergraph (not necessarily a linear space) such that every edge has at least k elements. Then, as Corrádi [C] proved

$$|V(\mathbf{H})| \geq \frac{k^2|\mathcal{E}|}{k-1+|\mathcal{E}|}. \tag{3.6}$$

Lemma (3.7). *Let M be the incidency matrix of a projective plane of order q. Suppose that $M_{i,j} = M_{j,i}$ whenever $1 \leq i \leq q^2-q+3$ or $1 \leq j \leq q^2-q+3$, i.e., M is symmetric, except a small submatrix in its lower left corner. Then the whole M is symmetric.*

Proof: If not, then M and its transpose M^T give two different extensions of the partial projective plane defined by the first q^2-q+3 rows of M. However, by (3.5) these two extensions must be the same, appart from the ordering of the rows. But in this case every row, but eventually one exception, contains at least two 1's from the first q^2-q+3 columns, so the ordering of the rows is also determined. □

A *biplane* of rank k is a k-uniform, k-regular hypergraph $(V,\mathcal{E})$, with $\binom{k}{2}+1$ edges, the same number of vertices, such that $|E \cap E'| = 2$ holds for every two distinct edges $E, E' \in \mathcal{E}$. No biplane is known for $k > 13$.

Lemma (3.8). *Suppose that M is an incidence matrix of a biplane of rank $q+1$. Suppose further that M is symmetric, and* $\operatorname{trace} M = 0$. *Then $q = 2$ or 5.*

Proof: A special case of a theorem of Hoffman, Newman, Strauss, and Taussky [HNST] says that for a symmetric biplane of rank $q+1$ one has

$$\operatorname{trace} M = q + 1 + a\sqrt{q-1},$$

where a is a nonnegative integer. (The proof goes in the same way as the proof of the next Lemma, i.e. we have to calculate the eigenvalues of the matix MM^T.) In our case $\operatorname{trace} M = 0$, so $\sqrt{q-1} = t$ must be an integer dividing $q+1 = t^2+2$. This implies that $t = 1$ or 2, i.e., $q = 2$ or 5. □

Lemma (3.9). *Suppose that M is a symmetric $0,1,-1$-matrix with $(q^2+q+2)/2$ rows and columns. Suppose further, that every row contains exactly $q+1$ nonzero entries, the scalar product of every two rows is 0, and the* $\operatorname{trace} M = -q+1$. *Then $q \leq 3$.*

Proof: Consider the product of M by its transpose. We have that $MM^T = M^2 = (q+1)I$. Consequently, every eigenvalue of M is $\pm\sqrt{q+1}$. So

$$-q + 1 = \operatorname{trace} M = a\sqrt{q+1}$$

holds for some integer a. If $q \neq 1$, then $a \neq 0$, and $\sqrt{q+1} = t$ is an integer dividing $-q+1 = -t^2+2$. Again, it follows that t is either 1 or 2, implying $q = 0$ or 3. □

Lemma. *Suppose that* $\mathbf{H} = (P, \mathcal{L})$ *is a non-trivial linear space. Then*

$$\sum_{x \in P} \deg(\mathbf{H}, x) \geq 3(|P| - 1). \tag{3.10}$$

Moreover, if $\mathbf{H}$ *is not the near pencil, then*

$$\sum_{x \in P} \deg(\mathbf{H}, x) \geq 4|P| - 8. \tag{3.11}$$

Proof: Let $k = \max\{|L| : L \in \mathcal{L}\}$, $|L_0| = k$. Then $\deg(\mathbf{H}, x) \geq k$ for all $x \notin L_0$. So

$$\sum \deg(\mathbf{H}, x) \geq 2k + k(|P| - k).$$

Here the right hand side is at least $4|P| - 8$ for $4 \leq k \leq |P| - 2$. The case $k = |P| - 1$ provides the near pencil. Finally, if $k \leq 3$, then $|\mathcal{L}| \geq \binom{|P|}{2}/3$, and the statement easily follows by inspecting the cases $|P| \leq 6$ separately. $\square$

4. THE MAXIMUM DEGREE IS AT MOST $q + 1$

Let $\mathbf{G}$ be a quadrilateral-free graph on $n = q^2 + q + 1$ vertices, $V(\mathbf{G}) = \{v_1, \ldots, v_n\}$. As the cases $q \leq 4$ were settled in [Mc], [CFS], we suppose that $q \geq 5$. (Although, with a little more efforts the results of the next two and the last sections can be extended to the cases $q = 3$ and 4.) Suppose that

$$2|\mathcal{E}(\mathbf{G})| \geq q(q+1)^2. \tag{4.1}$$

We will see that here equality holds, and for $q > q_0$ $\mathbf{G}$ is a polarity graph. Suppose that v_1 has maximum degree D. Clearly, $D \geq q + 1$. The aim of this section is to prove that $D = q + 1$.

Consider the hypergraph $\mathbf{N} = (V(\mathbf{G}), \mathcal{N})$, defined by the neighborhoods of the vertices of $\mathbf{G}$, i.e., $\mathcal{N} = \{N(v) : v \in V(\mathbf{G})\}$. Denote $N_{\mathbf{G}}(v_i)$ by N_i. For $N \in \mathcal{E}(\mathbf{N})$ let $v(N)$ denote the vertex for which $N(v(N)) = N$. (As in the extremal graph every vertex has degree at least 2, the above notation is justified.) Since $\mathbf{G}$ is quadrilateral-free, this hypergraph is linear, so

$$\binom{n}{2} = \binom{|V(\mathbf{G})|}{2} \geq \sum_{v \in V} \binom{|N(v)|}{2}. \tag{4.2}$$

Using the Jensen's inequality we get

$$\binom{n}{2} \geq n \binom{\sum |N(v)|/n}{2},$$

which yields the quite close upper bound

$$2|\mathcal{E}(\mathbf{G})| = \sum |N(v)| \leq \frac{n}{4}(1 + \sqrt{4n - 3}) = (q^2 + q + 1)(q + 1).$$

This proof was given in all the early papers [ERS], [Br], etc.

Next we give a better upper estimate for $|\mathcal{E}(\mathbf{G})|$, than (4.2). The idea we use is fairly simple, basically it is the same as in [F]. We separate the neighborhood of v_1, $N_1 = N_{\mathbf{G}}(v_1)$, and investigate the rest of the graph. Consider the subhypergraph $\mathbf{N}_1 = \mathbf{N}|(V \setminus N_1)$. It is still linear on $n - D$ vertices, so

$$\binom{n-D}{2} \geq \sum_{2\leq i\leq n} \binom{|N_i \setminus N_1|}{2}. \tag{4.3}$$

As $|N_i \setminus N_1| \geq |N_i| - 1$, we have

$$\sum_{2\leq i\leq n} |N_i \setminus N_1| \geq \sum_{i>1} |N_i| - (n-1) \geq 2|\mathcal{E}(\mathbf{G})| - D - (n-1). \tag{4.4}$$

Then (4.1) gives

$$\sum_{2\leq i\leq n} |N_i \setminus N_1| \geq (n-1)q - D. \tag{4.5}$$

Using the Jensen's inequality again, (4.3) and (4.5) imply that

$$\binom{n-D}{2} \geq (n-1)\binom{((n-1)q-D)/(n-1)}{2},$$

or equivalently

$$(n-1)(n-D)(n-D-1) \geq ((n-1)q-D)((n-1)(q-1)-D). \tag{4.6}$$

As a first step we prove that

Proposition. *For $q \geq 3$ one has*

$$D \leq q+2. \tag{4.7}$$

Proof: Suppose on the contrary, that $D \geq q+3$. Then for $q \geq 3$ it is easy to check the following inequalities, (which are linear in D).

$$(q+1)(n-D) < (n-1)q - D \tag{4.8}$$

$$q(n-D-1) \leq (n-1)(q-1) - D \tag{4.9}$$

However, the product of (4.8) and (4.9) contradicts to (4.6). $\square$

From now on suppose that $D = q+2$. Then

$$\binom{n-D}{2} = (q^2-1)\binom{q}{2} + (q+1)\binom{q-1}{2}, \tag{4.10}$$

i.e., the right hand side of (4.10) is at least as large as the right hand side of (4.3). Then (3.1) implies that

$$\sum_{2\leq i\leq n} |N_i \setminus N_1| \leq (q^2-1)q + (q+1)(q-1).$$

This and (4.4) give $2|\mathcal{E}(\mathbf{G})| \leq q(q+1)^2 + 1$. Hence,

$$|\mathcal{E}(\mathbf{G})| = \frac{1}{2}q(q+1)^2. \tag{4.11}$$

Proposition (4.12). *Equality holds in (4.5).*

Proof: Suppose on the contrary. Then $\sum_{i>1} |N_i \setminus N_1| \geq (q^2-1)q+(q+1)(q-1)$. This, (4.10) and (3.1) imply that $\mathbf{N}_1$ covers all pairs of $V \setminus N_1$, i.e., it is a linear space. Moreover, exactly q^2-1 of the edges of $\mathbf{N}_1$ have cardinality q, and the other $q+1$ edges have cardinality $q-1$. It follows that all degree in $\mathbf{N}_1$ is exactly $q+1$. Indeed, for an arbitrary vertex $x \in V \setminus N_1$ we have

$$q^2-1 = |V \setminus N_1| = 1 + \sum_{x \in E \in \mathcal{E}(\mathbf{N}_1)} |E| - 1 \leq 1 + (q-1)\deg(\mathbf{N}_1, x).$$

This gives $\deg(\mathbf{N}_1, x) \geq q+1$. However,

$$\sum_x \deg(\mathbf{N}_1, x) = \sum_{i>1} |N_i \setminus N_1| = (q+1)|V \setminus N_1|,$$

so there is no vertex of $\mathbf{N}_1$ of degree larger than $q+1$. This is a contradiction, because $\deg(\mathbf{N}_1, v_1) = \deg(\mathbf{N}, v_1) = q+2$. □

Now (4.11) and (4.12) imply that

$$\text{(4.13)} \qquad \sum_{N \in \mathcal{E}(\mathbf{N})} |N \cap N_1| = \sum_{x \in N_1} \deg(\mathbf{G}, x) = n - 1 + D.$$

This means that all edges of $\mathbf{N}$ intersect N_1.

Definition (4.14). *Define the graph of uncovered pairs,* $\mathbf{U}$*, as follows.* $V(\mathbf{U}) = V(\mathbf{G})$ *and* $\{x,y\} \in \mathcal{E}(\mathbf{U})$ *if* $\{x,y\}$ *is not covered by any edge of* $\mathbf{N}$*, or, equivalently,* $N(x) \cap N(y) = \emptyset$.

For any $x \in V \setminus N_1$ all $E \in \mathcal{E}(\mathbf{N}_1)$ intersects N_1 in exactly one element. Hence the number of uncovered pairs between x and N_1 is $|N_1| - \deg(\mathbf{N}, x)$. So the total number of uncovered pairs between $V \setminus N_1$ and N_1 is

$$\text{(4.15)} \qquad \sum_{x \in V \setminus N_1} (q + 2 - \deg(\mathbf{G}, x)) = q^2.$$

(Here, in the evaluation of the sum, we used (4.11) and (4.13).) Denote by ν the number of vertices in $\mathbf{G}$ of degree $q+2$. Then (4.5) and (3.1) imply that

$$\sum_{i>1} \binom{|N_i \setminus N_1|}{2} \geq (\nu - 1)\binom{q+1}{2} + (q^2 - 2\nu)\binom{q}{2} + (q+1+\nu)\binom{q-1}{2}$$
$$= \binom{n-D}{2} - (q - \nu).$$

This implies that the number of uncovered pairs in $V \setminus N_1$ is at most

$$\text{(4.16)} \qquad |\mathcal{E}(\mathbf{U} \| (V \setminus N_1))| \leq q - \nu.$$

We claim that $\nu = 1$. Indeed, suppose on the contrary that $|N_2| = q+2$, and let $\{x\} = N_1 \cap N_2$. Then the number of uncovered pairs between N_2 and $V \setminus N_2$, similarly as in (4.15), is also q^2. The number of uncovered pairs between $N_1 \setminus \{x\}$ and $N_2 \setminus \{x\}$ is exactly

$$(q+1)^2 - (|\mathcal{E}| - \deg(x)) = \deg(x) - q \leq 2.$$

But the total number of edges in $\mathbf{U}$ is at most $q^2 + q - \nu$. This implies that $\deg(\mathbf{U}, x) \geq q^2 - q + \nu - 2$. Obviously, the number of covered pairs containing x is at least $2(q+1)$. This altogether is more than $q^2 + q$ pairs coming out from x, a contradiction. So we have obtained that all neighborhood but $N(v_1)$ have at most $q+1$ elements.

Suppose that the vertex $v \in V \setminus N_1$ has degree (in $\mathbf{N}$) at most q. Then

$$\begin{aligned} \deg(\mathbf{U}\|(V \setminus N_1), v) &= |V \setminus N_1| - 1 - \sum_{v \in N \in \mathcal{E}(\mathbf{N})} (|N| - 2) \\ &\geq (q + 1 - \deg(v))(q-1) - 1. \end{aligned} \tag{4.17}$$

This and (4.16) give that every degree in $V \setminus N_1$ is at least q. We know from (4.12) that

$$\sum_{x \in V \setminus N_1} \deg(x) = (n - D)(q+1) - 1.$$

As $\deg(v_1) = q+2$, this implies that there are at least 2 vertex outside N_1 of degree q. Then (4.17) implies that there are at least $2(q-2)-1$ uncovered pairs in $V \setminus N_1$. This contradicts (4.16) for $q > 4$.

5. q IS EVEN

From now on we suppose that $D = q + 1$. In this section we prove the theorem for all even q.

Proposition. *For any vertex v one has*

$$\sum_{x \in N(v)} \deg(x) \leq n - 1 + \deg(v), \tag{5.1}$$

moreover, if here equality holds, then $\deg(v)$ *is an even integer.*

Proof: The number of (v, x, y) paths, is $\sum_{x \in N(v)} (\deg(x) - 1)$, and cannot exceed $n - 1$.

In the case of equality, all vertices (distinct to v) can be reached in (exactly) two steps from v, (i.e., $\deg(\mathbf{U}, v) = 0$). Namely, for all $y \in N(v)$ there exists an $x \in N(v)$ with $\{x, y\} \in \mathcal{E}$. However, the induced subgraph, $\mathbf{G}\|N(v)$, does not contain a vertex of degree larger than 1, so it is a union of disjoint edges. □

The above Proposition holds for all q. From now on we suppose that q is even. This part of the proof was independently settled by McCuaig [Mc], as well.

Corollary (5.2). *Suppose that* $\deg(v) = q + 1$. *Then there exists an $x \in N(v)$ with* $\deg(x) \leq q$.

Proof: Otherwise all degree in $N(v)$ is $q + 1$, implying $\sum_{x \in N(v)} = (q+1)^2 = n - 1 + \deg(v)$. However, $q + 1$ is an odd integer. □

Let $S(i)$ denote the set of vertices of degree i, and let $S = S(q) \cup S(q-1) \cup \dots$, the set of vertices with small degrees. Then (5.2) implies that

$$|S(q+1)| \le \sum_{x \in S} \deg(x) \le q|S|.$$

So $|V| \le (q+1)|S|$, yielding $|S| \ge q+1$. On the other hand we have

$$(5.3)\quad (n-1)(q+1) \le 2|\mathcal{E}(\mathbf{G})| = n(q+1) - \sum_{x \in S}(q+1-\deg(x)) \le n(q+1) - |S|,$$

which gives $|S| \le q+1$. In this way we obtained that $|S| = q+1$, and every degree in S is exactly q.

A vertex v has *property 1* (or sometimes we call it *regular*), if $\deg(v) = q+1$, and it has only one neighbour from S. Denote R the set of vertices of property 1. Then (5.2) implies that

$$|R| + 2|(V \setminus S) \setminus R| \le \text{ the number of edges between } S \text{ and } V \setminus S \le q^2 + q.$$

This gives $|R| \ge q^2 - q > 0$.

Let v be an arbitrary vertex of property 1. Denote its neighbours by $v_1, v_2, \dots, v_{q+1}$. Then, as in (5.1), one can prove that $N(v)$ contains exactly $q/2$ vertex-disjoint edges, say $\{v_{2t-1}, v_{2t}\}$ $(1 \le t \le q/2)$.

Proposition (5.4). $\deg(v_{q+1}) = q$, *i.e., it is an isolated vertex in* $\mathbf{G}\|N(v)$.

Suppose on the contrary, say, v_1 is the only vertex in $N(v)$ of degree q. Let $N_i = N(v_i) \setminus \{v\} \setminus N(v)$. Then $N_1, \dots, N_{q+1}$ is a partition of $V \setminus \{v\} \setminus N(v)$, and $|N_1| = q-2$, $|N_2| = \dots = |N_q| = q-1$, $|N_{q+1}| = q$. Every N_i $(2 \le i \le q)$ contains at most $\lfloor |N_i|/2 \rfloor = (q/2) - 1$ edges. There is no edge between N_{2t-1} and N_{2t}. Moreover there are at most $\min\{|N_i|, |N_j|\}$ edges between N_i and N_j. So taking account all the edges adjacent to N_3, say, we have

$$\sum_{x \in N_3} \deg(x) \le |N_3| + 2\lfloor |N_3|/2 \rfloor + \sum_{1 \le j \le q+1\ j \ne 3,4} \min\{|N_i|, |N_j|\} = (q+1)|N_3| - 2.$$

So N_3 (and every N_i for $3 \le i \le q$) contains at least two vertices of degree q. These vertices together with v_1 and $S \cap N_{q+1}$ makes at least $2q-2$ small degrees. However $|S| = q+1$, a contradiction for $q > 2$. So we obtained that the only possibility is that $S \cap N(v) = \{v_{q+1}\}$. $\square$

Apply Corollary (5.2) for v_i, $1 \le i \le q$. Then $S \cap N_i \ne \emptyset$, implying that each N_i contains exactly one vertex, u_i, of degree q. That is, that every v_i has property 1, too. In more general, we obtained that if a vertex has property 1, then all of its neighbours of degree $q+1$ have property 1, as well. Let $W = S(q+1) \setminus R$, i.e., the set of nonregular vertices (of degree $q+1$). Apply the above argument for all v_i (where all of them are regular vertices, so do their neighbours, $1 \le i \le q$). We obtain that

$$W \subset N(v_{q+1}).$$

Similarly, starting with v_i instead of v, we have that $W \subset N(u_i)$, implying that

$$W \subset \bigcap_{1 \le i \le q} N(u_i),$$

implying that $|W| \leq 1$. On the other hand $W \neq \emptyset$, so $|W| = 1$.

We concluded, that there exists a vertex w ($W = \{w\}$) of degree $q+1$ which is joined to every vertex of degree q. Moreover all the other vertices of degree $q+1$ have property 1. Then $N(w) = S$. Then (5.2) implies that S contains no edge of $\mathbf{G}$. Then (5.4) implies that $N(x)$ does not contain any edge of $\mathbf{G}$ for all $x \in S$.

These last two properties give that the following hypergraph, $\mathbf{H}$, is linear. $\mathcal{E}(\mathbf{H}) = \{E(x) : x \in V\}$, where

$$E(x) = \begin{cases} N(x) \text{ if } \deg(x) = q+1, \\ N(x) \cup \{x\} \text{ if } \deg(x) = q. \end{cases}$$

As $\mathbf{H}$ is a $(q+1)$-uniform and $(q+1)$-regular linear hypergraph, it follows that it is a projective plane. Its incidency matrix obtained from the adjacency matrix of $\mathbf{G}$, is obviously symmetric. So the bijection $x \leftrightarrow E(x)$ is a polarity of this plane, the graph $\mathbf{G}$ is a polarity graph.

6. q IS ODD, THE INTERSECTING CASE

In this section we suppose that $\mathcal{R} = \{N(x) : x \in V, |N(x)| = q+1\}$ is an intersecting family. Denote by $\mathcal{Q}$ the family of the neighborhoods of the vertices with degree at most q. We know that $|\mathcal{Q}| \leq q+1$, even more

$$\sum_{x \in S} (q+1-\deg(x)) \leq q+1, \tag{6.1}$$

where $S = S(q) \cup S(q-1) \cup \dots$, the set of vertices of small degrees. As $\mathcal{R}$ is a 1-intersecting family of size at least q^2, (3.4) can be applied, i.e., there exists a family $\mathcal{P}$ such that $\mathcal{R} \cup \mathcal{P}$ forms the line set of a finite plane. For every $N \in \mathcal{Q}$ the hypergraph $\mathcal{P}|N$ is a linear space (not considering the edges of size less than 2). We distinguish between two cases.

Case I. Suppose first that for all $N \in \mathcal{Q}$ there exists a (unique) $P = P(N) \in \mathcal{P}$, such that $N \subset P$. Then the incidency matrix, M', of $\mathcal{R} \cup \mathcal{P}$ majorates the incidency matrix, M, of $\mathcal{R} \cup \mathcal{Q}$. (We supposed that the ordering of the vertex sets and $\mathcal{R}$ in both matrices are the same, and the for the row $N \in \mathcal{Q}$ we associate the row $P(N)$ in M'.) Where are the new entries? As $\mathcal{R} \cup \mathcal{P}$ is $q+1$ uniform and $q+1$ regular, the new entries must be in the rows corresponding to $\mathcal{P}$, and in the columns corresponding to S. Then M and M' coincide outside of the lower left corner. The matrix M is symmetric, we can apply Lemma (3.7). We obtain that M' is also symmetric.

(1.1) implies that M' has at least $q+1$ nonzero elements in the main diagonal. However, trace $M = 0$, so M' was obtained by adding $q+1$ new elements to the main diagonal of M. That is, $\mathbf{G}$ is the polarity graph. □

Case II Suppose that the edges of $\mathcal{P}|N_0$ with at most two elements form a nontrivial linear space for some $N_0 \in \mathcal{Q}$. Denote this linear space by $\mathbf{N}_0$. Then (3.2) gives that

$$|\mathcal{P}| \geq |N_0|, \tag{6.2}$$

which is equivalent to $|\mathcal{Q}| \geq |N_0|$. Denote $|N_0|$ by m. Then (6.1) and (6.2) give that

$$q+1 \geq \sum_{N \in \mathcal{Q}} (q+1-|N|) \geq (q+1-|N_0|) + (|\mathcal{Q}|-1) \geq q. \tag{6.3}$$

This implies that $|\mathcal{Q}| \leq |N_0| + 1$, or equivalently

$$|\mathcal{P}| \leq |N_0| + 1.$$

Then one can apply (3.3) which says that one of the following three cases holds for $\mathbf{N}_0$.

(i) $\mathbf{N}_0$ is a near-pencil, or
(ii) it is a finite projective plane, or
(iii) it is a finite projective plane with one point deleted.

In all these three cases we can see that the number of intersecting pairs of the lines of $\mathbf{N}_0$ is at least

$$\binom{m}{2}. \tag{6.4}$$

(We remark that this holds for *all* linear spaces, see [FF].) In other words, (6.4) means that the number of pairs $\{P, P'\}$ $(P, P' \in \mathcal{P})$ such that $P \cap P' \cap N_0 \neq \emptyset$ is at least $\binom{m}{2}$. Moreover, at the vertex $v(N_0)$, the degree of $\mathcal{P}$ is at least $q + 1 - m$. So we have a lower bound for the total number of intersecting pairs $\{P, P'\}$.

$$\binom{m+1}{2} \geq \binom{|\mathcal{P}|}{2} \geq \binom{m}{2} + \binom{q+1-m}{2}.$$

This inequality implies that

$$m > q - \sqrt{2q}. \tag{6.5}$$

From (6.3) we obtain that all but eventually one vertex of $S \setminus \{v(N_0)\}$ has degree q, and that exceptional vertex has degree $q - 1$. This implies that

$$\sum_{x \in N_0} \deg(\mathcal{P}, x) = \sum_{x \in N_0} (\deg(\mathcal{Q}, x) + q + 1 - \deg(x)) \leq 2m + 1 + |N_0 \cap S|. \tag{6.6}$$

Here we used that $\mathcal{Q}$ is a 0–1 intersecting family, hence $\sum_{x \in N} \deg(\mathcal{Q}, x) \leq |N| + |\mathcal{Q}| - 1$ holds for every $N \in \mathcal{Q}$. Moreover we used the fact that $v(N_0) \notin N_0$.

In the cases (ii) and (iii) the $\sum_{x \in N_0} \deg(\mathcal{P}, x)$ is at least $m\sqrt{m} - m$, while the right hand side of (6.6) is at most $3m + 1$. This leads to $m \leq 16$, hence (6.5) yields $q \leq 22$.

In the case (i), $\mathbf{N}_0$ is a near-pencil, so the left hand side of (6.6) is at least $3m - 3$. So (6.6) implies that

$$|N_0 \cap S| \geq |N_0| - 4.$$

In other words, $\deg(\mathcal{Q}, v(N_0)) \geq m - 4$, so at least $m - 4$ lines of $\mathcal{P}$ go through $v(N_0)$. However, at least $m - 1$ lines of $\mathcal{P}$ go through a certain point, $x \in N_0$, because $\mathbf{N}_0$ is a near-pencil. As $|\mathcal{P}| \leq m + 1$, we obtain that at least $m - 6$ lines of it have to pass both x and $v(N_0)$. This is impossible for $m > 7$. So $m \leq 6$ and then $q \leq 11$. □

Finally we remark that this part of the proof can be done easily for all odd q's by a slightly more careful calculation.

7. q IS ODD, THE ALMOST INTERSECTING CASE

From now on we suppose that $\mathcal{R} = \{N(x) : x \in V, |N(x)| = q+1\}$ is *not* an intersecting family, say, $F, F' \in \mathcal{R}$, $F \cap F' = \emptyset$. Then for all $x \in F$ at least one of the following cases holds. Either

(7.1a) $\deg(x) \le q$, or

(7.1b) $x \in \cup\mathcal{Q}$, i.e., x is covered by a small edge of $\mathbf{N}$.

This follows from the fact, that if $\mathcal{N}[x] = \{F, N_1, \dots N_q\}$, and all members of it have $q+1$ elements, then the family $\{F, N_1 \setminus \{x\}, \dots N_q \setminus \{x\}\}$ is a $q+1$-partition of V. However, $F \cap F' = \emptyset$ and $|N_i \cap F'| \le 1$ imply $|F'| \le q$, a contradiction. $\square$

Let $|\mathcal{Q}| = q+1-t$, $(t \ge 0)$. Then (7.1) implies that $|F \cap S|$ plus the number of (F, N) pairs with $F \cap N \ne \emptyset$, $N \in \mathcal{Q}$ is at least $q+1$. Hence we have

$$|F \cap S| \ge t. \tag{7.2}$$

Again (7.1) implies that there are at least t sets $F_1', F_2', \dots F_t'$ from $\mathcal{R}$ disjoint to F. Indeed, $\deg(\mathcal{R} \setminus \{F\}, x) \le q-1$ for all $x \in F$, hence F is intersected by at most $q^2 - 1$ sets from $\mathcal{R} \setminus \{F\}$.

The roles of F and F_i' are symmetric, so (7.2) holds for F_i', too. Then the family $\{F \cap S, F_1' \cap S, \dots F_t \cap S\}$ is a linear hypergraph with every member having at least t vertices. Then (3.6) gives that

$$q+1 \ge |S| \ge |(F \cap S) \bigcup_i (F_i' \cap S)| \ge \frac{t^2(t+1)}{2t},$$

which yields

$$t < \sqrt{2q}. \tag{7.3}$$

Hence $|\mathcal{Q}| > q+1-\sqrt{2q}$, which together with (6.1) imply that

$$|S(q)| \ge q+1-2t > q+1-2\sqrt{2q}, \tag{7.4}$$

and

(7.5) every degree in $\mathbf{N}$ and

the cardinality of each member of $\mathcal{Q}$ is at least $q - t > q - \sqrt{2q}$.

Let $\mathcal{I} \subset \mathcal{R}$ be an intersecting subfamily. In this section we deal with the case that there exists an $\mathcal{I}$ with

$$|\mathcal{I}| > q^2 - \frac{1}{6}q. \tag{7.6}$$

Then (3.4) implies that there exists a (q+1)-uniform family $\mathcal{P}$ such that $\mathcal{I} \cup \mathcal{P}$ form the line set of a projective plane. We may suppose that $F \notin \mathcal{I}$. Let $\mathcal{L}$ be the set of at least 2-element edges of the hypergraph $(F, \mathcal{P}|F)$. Then $(F, \mathcal{L})$ is a nontrivial linear space. For an arbitrary vertex x we have

$$\deg(\mathcal{P}, x) = \deg(\mathcal{R} \setminus \mathcal{I}, x) + \deg(\mathcal{Q}, x) + (q+1-\deg(x)). \tag{7.7}$$

Add up (7.7) for all $x \in F$. As $\mathcal{N}$ is a linear hypergraph, and $F \in \mathcal{R} \setminus \mathcal{I}$, the following upper bounds are obtained.

$$\sum_{x\in F} \deg(\mathcal{R} \setminus \mathcal{I}, x) \leq q + |\mathcal{R}| - |\mathcal{I}|,$$

$$\sum_{x\in F} \deg(\mathcal{Q}, x) \leq |\mathcal{Q}|.$$

The above two inequalities, (6.1) and (7.6) give that

$$\sum_{x\in F} \deg(\mathcal{P}, x) \leq 2q + 1 + (|\mathcal{N}| - |\mathcal{I}|) \leq 3.16\ldots q + 2.$$

For $q > 6$ the right hand side is less than $4(q+1) - 8$, so (3.11) implies that the linear space $(F, \mathcal{L})$ is a near pencil. Denote the center of this near pencil by x_0, i.e., $(F \setminus \{x_0\}) \in \mathcal{L}$. Then $\deg(\mathcal{P}, x_0) \geq q$. Apply (7.7) to x_0. As $|\mathcal{R} \setminus \mathcal{I}| < q/6 + t$ by (7.6), and $(q + 1 - \deg(x)) \leq t + 1$ by (7.5), we obtain that

$$\deg(\mathcal{Q}, x_0) > \frac{5}{6}q - 2t - 1. \tag{7.8}$$

This inequality implies that $\sum_x \deg(\mathcal{Q}, x)$ for $x \in F \setminus \{x_0\}$ is at most $(q/6) + t + 2$. Then (7.1) gives that

$$|F \cap S| \geq (q + 1) - (\frac{q}{6} + t + 2).$$

However, (7.8) implies that $|N(x_0) \cap S| > \frac{5}{6}q - 2t - 1$. As $|S| = q + 1 - t$ we obtain that $|F \cap N(x_0) \cap S| > \frac{2}{3}q - 2t - 3$. This is larger than 1 for $q > 30$, by (7.3). We arrived to the contradiction $|F \cap N(x_0)| \geq 2$. $\square$

8. q IS ODD, THE DISJOINT CASE

This is a continuation of the previous section. From now on we suppose the opposite of (7.6), that is, for all intersecting subfamily $\mathcal{I} \subset \mathcal{R}$ one has

$$|\mathcal{I}| \leq q^2 - (q/6). \tag{8.1}$$

Definition (8.2). *A vertex $x \in V$ is called regular if* $\deg(x) = q + 1$ *and* $\mathcal{N}[x]$ *consists of q $(q + 1)$-element sets and one q-element set. Let $R = \{x \in V : x$ is regular $\}$.*

For a regular vertex x we have

$$|\cup \mathcal{N}[x]| = 1 + \sum_{x\in N} (|N| - 1) = |V| - 1,$$

so exactly one other vertex remains uncovered. Denote it by $f(x)$. So x is regular if and only if $\deg(\mathbf{U}, x) = 1$, and then the only uncovered pair through x is $\{x, f(x)\}$. Note that, if $f(x)$ is regular, too, then $f(f(x)) = x$.

Suppose now that x is regular and $x \in Q \in \mathcal{Q}$, (i.e., Q is the q element set through x). Then for all set $N \in \mathcal{R}$ we have

$$N \cap Q = \emptyset \text{ implies } f(x) \in N. \tag{8.3}$$

Indeed, $N \cap Q = \emptyset$ and $|N \cap N_i| \leq 1$ for $N_1, \dots N_q \in \mathcal{N}[x] \setminus \{Q\}$ imply that $|N \cap (\cup\mathcal{N}[x])| \leq q$. Consequently, $N \setminus (\cup\mathcal{N}[x]) \neq \emptyset$. □

Similarly, if $F \cap F' = \emptyset$, F, $F' \in \mathcal{R}$ and $x \in F$ is regular, then

$$f(x) \in F'. \quad \square \tag{8.4}$$

The proof of the following statement is analogous to (5.1). Suppose that x is regular. Then

$$\mathbf{G} \| N(x) \text{ is a 1-factor.} \quad \square \tag{8.5}$$

Suppose that $Q \in \mathcal{Q}$, $|Q| = q$. Then it is impossible that all of its vertices are regular, i.e.,

$$Q \not\subset R. \tag{8.6}$$

Proof: Suppose on the contrary, and let $u \in Q$ arbitrary. Then $\mathbf{G}\|N(u)$ is a 1-factor, by (8.5). So there is a vertex u' such that $\{u', v(Q)\}$ is an edge contained in $N(u)$. Hence, the edge $\{u, u'\}$ is contained in $Q\ (= N(v(Q)))$. This implies that every degree in $\mathbf{G}\|Q$ is at least 1, so it must be a 1-factor. But this is impossible, because q is odd. □

A corollary of (8.6) is the following. For all $Q \in \mathcal{Q}$, $|Q| = q$, one can find at least $t+1$ sets $N \in \mathcal{R}$ such that

$$N \cap Q = \emptyset. \tag{8.7}$$

Indeed, suppose on the contrary, then at least q^2 members of $\mathcal{R}$ intersect Q. Then every degree of Q is regular, a contradiction. □

Let $\mathcal{F}$ be the family of those $(q+1)$-element sets from $\mathcal{R}$ which are disjoint to another large set, i.e., $\mathcal{F} = \{F \in \mathcal{R} : \exists F' \in \mathcal{R},\ F \cap F' = \emptyset\}$. We are going to define a subset

$$\{F_1, F_1', F_2, F_2', \dots F_s, F_s'\} \subset \mathcal{F} \tag{8.8}$$

such that $F_i \cap F_i' = \emptyset$ and $s \geq q/12$. We have at least one disjoint pair $\{F_1, F_1'\}$. In the $(i+1)$'th step consider $\mathcal{R} \setminus \{F_1, F_1', \dots F_i, F_i'\}$. If its cardinality exceeds $q^2 - (q/6)$, then it contains a disjoint pair $\{F_{i+1}, F_{i+1}'\}$, by (8.1). This procedure supplies at least $q/12$ pairs. □

The hypergraph consisting of the sets $\{F_i \cap S\}$ and $\{F_i' \cap S\}$ $(1 \leq i \leq s)$ is linear. So we have an i such that

$$\binom{|F_i \cap S|}{2} + \binom{|F_i' \cap S|}{2} \leq \frac{1}{s}\binom{|S|}{2} \leq 6(q-1). \tag{8.9}$$

This implies that

$$\min_{1 \leq i \leq s} \{|F_i \cap S| + |F_i' \cap S|\} = T \leq \sqrt{24q} + 1. \tag{8.10}$$

Proposition (8.11). *There are at least $q+1-2T$ sets $Q \in \mathcal{Q}$ such that $|Q| = q$ and Q intersects both F_i and F_i' in regular vertices.*

Proof: $\deg(\mathcal{Q}, x) \geq 1$ for all $x \in F_i \cup F_i' \setminus S$, by (7.1). For all $Q \in \mathcal{Q}$ we have $|Q \cap (F_i \cup F_i')| \leq 2$. Let $\mathcal{B}$ ($\mathcal{A}$) be the collection of the traces of the q-element (less than q element) sets on $(F_i \cup F_i') \setminus S$. Then Proposition (8.11) follows from the following easy statement:

Suppose that $\mathcal{A} \cup \mathcal{B}$ is a collection of 1 and 2-element sets over the set X such that

(i) $|X| \geq 2q+2-T$
(ii) $\deg(\mathcal{A} \cup \mathcal{B}, x) \geq 1$ for all $x \in X$
(iii) $|\mathcal{A}| \leq t$
(iv) $|\mathcal{A} \cup \mathcal{B}| = q+1-t$,

then there exist $B_1, B_2, \dots B_{q+1-2T} \in \mathcal{B}$ such that

(α) B_i is disjoint to all other members of $\mathcal{A} \cup \mathcal{B}$
(β) $|B_i| = 2$. □

For $q \geq 98$ we have $T < (q+1)/2$, so (8.11) implies that there exists a q-element set Q_0 such that $Q_0 \cap F_i = \{u\}$ and $Q_0 \cap F_i' = \{u'\}$ are both regular vertices.

Proposition (8.12). *There is exactly one set $N \in \mathcal{R}$ avoiding Q_0.*

Proof: There are at least $t+1$ such sets by (8.7). Suppose that $N \in \mathcal{R}$ and $N \cap Q_0 = \emptyset$. Then $f(u) \in N$ and $f(u') \in N$ hold by (8.3). However, by (8.4), $f(u) \in F_i'$ and $f(u') \in F_i$. Hence $f(u) \neq f(u')$. Then the pair $\{f(u), f(u')\}$ uniquely determines the set N. □

This proof gave that $t = 0$, so every set in $\mathcal{Q}$ has q elements, and $|S| = q+1$.

In the rest of this section we suppose that $\mathcal{Q}$ is totally disjoint, i.e., $Q \cap Q' = \emptyset$ hold for all Q, $Q' \in \mathcal{Q}$. The other case will be investigated in the next section.

As $\deg(\mathcal{Q}, x) \leq 1$ for all vertices, it follows that $|N \cap S| \leq 1$ for all $N \in \mathcal{N}$. So $|S \cap Q| = 1$ for all $Q \in \mathcal{Q}$, by (8.6). Moreover, every pair in S is uncovered. Then the following graph $\mathbf{G}^+$ is quadrilateral free. $V(\mathbf{G}^+) = V \cup \{w\}$ and $\mathcal{E}(\mathbf{G}^+) = \mathcal{E}(\mathbf{G}) \cup \{\{w, x\} : x \in S\}$. So $\mathbf{G}^+$ is obtained by adding a new vertex, w, to $\mathbf{G}$, and joining it to S. Then

$$\mathcal{N}(\mathbf{G}^+) = \mathcal{R} \cup \{S\} \cup \{Q \cup \{w\} : Q \in \mathcal{Q},\, w \in S\}.$$

As $|\cup \mathcal{N}[x]| = q^2+q+1$ for all vertex x of $\mathbf{G}^+$, similarly as in (5.1) and in (8.5), we obtain that

(8.13) $$\mathbf{G}^+ \| N(x) \text{ is a 1-factor.}$$

Call a hypergraph $\mathbf{H}$ *regular 2-packing of order* q if it is $q+1$-regular, $q+1$-uniform and linear over q^2+q+2 elements. The above defined $\mathcal{N}(\mathbf{G}^+)$ is a regular 2-packing. The existence of regular 2-packings is an open question, only finitely many are known. The latest results can be found (or at least a reference to it) in Lamken, Mullin, Vanstone [LMV]. Another simple characterization was given by Ryser [Ry]. But in our case the incidency matrix of $\mathcal{N}(\mathbf{G}^+)$ is symmetric, since it was obtained from a graph. Now we recall a basic property of the struture of a regular 2-packing $\mathbf{H} = (X, \mathcal{E})$.

There exists a partition of the vertices into 2-element sets $X = X(1)\cup\dots X(\frac{1}{2}(q^2+q+2))$, such that $|X_i\cap E|\le 1$ for all $E\in\mathcal{E}$. (These are the uncovered pairs.) There exists a partition of the edge set $\mathcal{E}=\mathcal{E}(1)\cup\dots\mathcal{E}(\frac{1}{2}(q^2+q+2))$ into disjoint pairs. (These are the only disjoint pairs in $\mathcal{E}$.) Moreover, if $\mathcal{E}(i)=\{E,E'\}$ then E and E' intersect the same $q+1$ pairs. These imply that the following hypergraph $\mathbf{B}$ is a biplane. $V(\mathbf{B})=\{X(1),\dots X(\frac{1}{2}(q^2+q+2))\}$ and $\mathcal{E}(\mathbf{B})=\{\{X(a_1),\dots X(a_{q+1})\}:$ there exists an edge $E\in\mathcal{E}$ such that $E\subset\cup_{1\le i\le q+1}X(a_i)\}$.

In our case, the biplane obtained from $\mathcal{N}(\mathbf{G}^+)$ is symmetric and there are no entries in the main diagonal. Indeed, (8.13) implies that if $X_i=\{x,y\}$ then $y\notin N(x)$. Hence the submatrix $X_i\times\{N(x),N(y)\}$ is empty. Lemma (3.8) can be applied, which says that $q=2$ or 5. $\square$

9. q IS ODD, THE DISJOINT CASE (THE END)

This is a continuation of the previous section. Define $\mathcal{Q}^{\text{reg}}\subset\mathcal{Q}$ as the set of those q-element sets which intersect all but one members of $\mathcal{R}$. By (8.11) and (8.12) we have that

(9.1) $$|\mathcal{Q}^{\text{reg}}|\ge q+1-2T.$$

This implies that $\deg(\mathcal{Q},x)\le 2T$ holds for every vertex x. The dual of this inequality says that

(9.2) $$|N\cap S|\le 2T$$

for all $N\in\mathcal{N}$. Moreover for every set $F\in\mathcal{F}$

(9.3) $$|F\cap R|\ge q+1-4T.$$

Claim (9.4). *Suppose that $Q_0\in\mathcal{Q}^{\text{reg}}$, $Q_1\in\mathcal{Q}$, $Q_0\cap Q_1\ne\emptyset$. Then Q_1 is disjoint to at least $q-1$ members of $\mathcal{R}$.*

Proof: Let $x\in Q_0\setminus Q_1$ be an arbitrary vertex. It is regular, its degree is $q+1$, so at least one member of $\mathcal{N}[x]$ (and this member is not Q_0) avoids the q-element Q_1. $\square$

The above claim and the definition of $\mathcal{Q}^{\text{reg}}$ imply that

(9.5) the sets in $\mathcal{Q}^{\text{reg}}$ are pairwise disjoint.

The next step is done only in order to keep the value of q_0 (the bound in the second half of the Theorem) to be low.

Claim (9.6). *For $q\ge 190$ we have $T\le 10$.*

Proof: Let S_0 be the set of vertices $\{v(Q):Q\in\mathcal{Q}^{\text{reg}}\}$. We have $S_0\subset S$. Then $\mathbf{U}\|S_0$ is a complete graph by (9.5), or in other words for all $N\in\mathcal{N}$ one has $|N\cap S_0|\le 1$. So instead of (8.9) we can use the following

$$\binom{|F_i\cap S|-1}{2}+\binom{|F_i'\cap S|-1}{2}\le\binom{|S\setminus S_0|}{2}(q/12)^{-1}.$$

As $|F_i\cap S|+|F_i'\cap S|=T$, and $|S\setminus S_0|\le 2T$, the above inequality yields $T\le 10$ for q is sufficiently large. $\square$

Proposition (9.7). *Let $F \in \mathcal{F}$. Then there is a unique $F' \in \mathcal{F}$ disjoint to it.*

Proof: Suppose on the contrary, and let $F_1, F_2 \in \mathcal{R}$ such that $F \cap F_1 = \emptyset$ and $F \cap F_2 = \emptyset$. Let $x \in F \cap R$. Then (8.4) gives that $f(x) \in F_1$, and similarly, $f(x) \in F_2$. For the vertex $\{u\} = F_1 \cap F_2$ we have that $f(x) = u$ for all $x \in F \cap R$. Then, by the definition (8.2), we have that $\{u, x\} \in \mathcal{E}(\mathbf{U})$, (i.e., an uncovered pair) for all $x \in F \cap R$. This means that the sets through u avoid $F \cap R$. So at least $q - 4T$ members of $\mathcal{N}$ avoid F. This implies that

$$\sum_{x \in F} \deg(x) \le |\mathcal{N}| + (|F| - 1) - (q - 4T) = q^2 + q + 1 + 4T.$$

However,

$$\sum_{x \in F} \deg(x) = (q+1)(q+1) - |F \cap S| \ge (q+1)^2 - 2T,$$

by (9.2). The above two inequalities contradict each other for $T < q/6$. □

The total number of disjoint $\{N, N'\}$ pairs ($N, N' \in \mathcal{N}$) is

$$|\mathcal{E}(\mathbf{U})| = \binom{q^2+q+1}{2} - |\mathcal{R}|\binom{q+1}{2} - |\mathcal{Q}|\binom{q}{2} = q^2 + q. \tag{9.8}$$

Every $Q \in \mathcal{Q} \setminus \mathcal{Q}^{\text{reg}}$ intersect at least $q^2 - 2T$ others by (9.2). So $\deg(\mathbf{U}, v(Q)) \le q + 2T$. As $\mathcal{R}$ contains exactly $|\mathcal{F}|/2$ disjoint pairs, and $\mathcal{Q}^{\text{reg}}$ cannot have more than $\binom{|\mathcal{Q}^{\text{reg}}|}{2}$, we obtain that

$$|\mathcal{E}(\mathbf{U})| \le \frac{|\mathcal{F}|}{2} + |\mathcal{Q} \setminus \mathcal{Q}^{\text{reg}}|(q + 2T) + |\mathcal{Q}^{\text{reg}}| + \binom{|\mathcal{Q}^{\text{reg}}|}{2}.$$

This, (9.1) and (9.8) imply that

$$\frac{|\mathcal{F}|}{2} \ge \frac{q^2 - q}{2} - (6T^2 - 3T + 1) > \binom{q}{2} - 600. \tag{9.9}$$

As $F \cap S \ne \emptyset$ for all $F \in \mathcal{F}$, (9.9) implies that at most $2q + 1200$ F intersect S in more than 1 element. Hence the number of disjoint pairs $\{F, F'\} \subset \mathcal{F}$ intersecting S in exactly 1–1 element is at least

$$\frac{q^2 - 5q}{2} - 1800.$$

Call such a pair *perfect*. There are perfect pairs, (for $q \ge 63$), so in the inequalities (9.1)–(9.3) we can use the value $T = 2$.

From now on we supppose, on the contrary of the previous section, that there are intersecting q-sets. This implies that there exists a $Q_0 \in \mathcal{Q} \setminus \mathcal{Q}^{\text{reg}}$. First we claim that

$$|Q_0 \cap R| \le \frac{q+7}{2}.$$

Indeed, if $|Q_0 \cap R| \ge (q+9)/2$, then (9.9) implies that there exists a disjoint pair $\{F, F'\} \subset \mathcal{F}$, such that $F \cap Q_0$ and $F' \cap Q_0$ are both regular. Then (8.12) implies that $Q_0 \in \mathcal{Q}^{\text{reg}}$, a contradiction.

Q_0 has at least $q-4$ vertices of degree $q+1$, so the above argument gives that at least

$$|Q_0 \setminus (S \cup R)| \geq (q-4) - \frac{q+7}{2} \tag{9.10}$$

of them are contained in another q-element sets. So there are sets $Q \in \mathcal{Q}^{\text{reg}}$ such that $Q \cap Q_0 \neq \emptyset$. (By (9.1) and (9.10), there are at least $(q-21)/2$ such sets.) Then (9.4) gives that Q_0 is avoided by at least $q-1$ sets from $\mathcal{R}$. Hence Q_0 has at least

$$|Q_0 \setminus S| - 5 \geq q - 7$$

vertices where it is intersected by q element sets, especially it is intersected by at least $q-7-2T = q-11$ members of $\mathcal{Q}^{\text{reg}}$.

For a $Q \in \mathcal{Q}^{\text{reg}}$ denote the unique $q+1$-element set disjoint to it by $J(Q)$, and its unique vertex x with $\deg(\mathcal{R}, x) = q-1$ by $c(Q)$. The existence and unicity of $J(Q)$ and $c(Q)$ are proved in (8.6) and (8.12). Let $C = \{c(Q) : Q \in \mathcal{Q}^{\text{reg}}\}$. For every $Q \in \mathcal{Q} \setminus \mathcal{Q}^{\text{reg}}$ we have obtained that

$$|Q_0 \cap C| \geq q - 11. \tag{9.11}$$

This implies that Q_0 is unique, and $|\mathcal{Q}^{\text{reg}}| = q$. Moreover,

$$|\cup \mathcal{Q} \cap S| \leq |Q_0 \cap S| + \text{ the number of sets from } \mathcal{Q}^{\text{reg}} \text{ avoiding } Q_0 \leq 15,$$

$$|V \setminus (\cup\mathcal{Q} \cup S)| \leq |S \cap (\cup\mathcal{Q}^{\text{reg}})| \leq 11. \tag{9.12}$$

Claim (9.13). *$S \cap (\cup\mathcal{Q}) = \emptyset$, i.e., $C = Q_0$.*

Proof: First we prove that $|V \setminus (\cup\mathcal{Q}^{\text{reg}} \cup S)| \leq 1$. Suppose that $x \in V \setminus (\cup\mathcal{Q}^{\text{reg}} \cup S)$. Then for all $Q \in \mathcal{Q}^{\text{reg}}$, $Q \cap Q_0 \neq \emptyset$ implies that $J(Q) \in \mathcal{N}[x]$. If there was another $y \in V \setminus (\cup\mathcal{Q} \cup S)$, $y \neq x$, then $J(Q) \in \mathcal{N}[y]$, too. This implies that the unique set $N \in \mathcal{N}[x] \cap \mathcal{N}[y]$ avoids all $Q \in \mathcal{Q}^{\text{reg}}$ which intersect Q_0. Then $|N \cap S| = |S \setminus \cup\mathcal{Q}^{\text{reg}}|$. Here the left hand side is at most 4, by (9.2), while the right hand side is at least $q-11$, by (9.12), a contradiction.

Suppose now that $\{x\} = V \setminus (\cup\mathcal{Q} \cup S)$. Then $\mathcal{N}[x] \subset \mathcal{R}$, so $N(x) \in R$, and $N(x) \cap S = \emptyset$. Moreover, $x \notin N(x)$, and $|N(x) \cap Q| \leq 1$. These imply that $|N(x)| \leq q$, a contradiction.

Suppose finally that $(V \setminus S) \subset \cup\mathcal{Q}$, but there exists an x such that $\{x\} = V \setminus (S \cup \mathcal{Q}^{\text{reg}})$. Then there exists a unique $Q_1 \in \mathcal{Q}$ such that $Q_1 \cap S \neq \emptyset$. ($Q_1 \neq Q_0$.) Then the vertex $v(Q_1)$ has degree q. As all but one degree in Q_1 is $q+1$, $v(Q_1)$ carries $q-1$ sets from $\mathcal{R}$ and one set from $\mathcal{Q}$. However such a vertex does not exist (except $Q_1 \cap S$). □

Now we are able to describe the exact structure of the graph of uncovered pairs $\mathbf{U}$. As $S = \{v(Q) : Q \in \mathcal{Q}\}$, we have that all pairs in $S \setminus \{v(Q_0)\}$ are uncovered. As $Q \in \mathcal{Q}^{\text{reg}}$ and $J(Q)$ are disjoint, the pair $\{v(Q), v(J(Q))\}$ is uncovered. However, as we have seen in (8.6), $\mathbf{G}\|Q$ is almost a 1-factor, the only isolated vertex is $c(Q)$. This means, that

$$\{v(Q), c(Q)\} \in \mathcal{E}(\mathbf{U}).$$

We obtain, that $c(Q) = v(J(Q))$. As $\deg(\mathcal{Q}, x) = 2$ for all $x \in Q_0$, it follows that all edges of $\mathcal{N}[v(Q_0)]$ intersect S in exactly 2 elements. Hence $\{J(Q) : Q \in \mathcal{Q}^{\text{reg}}\} =$

$\mathcal{N}[v(Q_0)]$. All the other sets from $\mathcal{R}$ intersect S in exactly one element. We claim that $\cup J(Q)$ is disjoint to Q_0, meaning that

$$\{v(Q_0), c(Q)\} \in \mathcal{E}(\mathbf{U}). \tag{9.14}$$

Proof: Let w be a new vertex and define the following graph $\mathbf{G}_1$ over $V \cup \{w\} \setminus \{v(Q_0)\}$. Join w to all vertices of $S \setminus \{v(Q_0)\}$, otherwise $\mathbf{G}_1 \| (V \setminus \{v(Q_0)\}) = \mathbf{G} \| (V \setminus \{v(Q_0)\})$. Then $\mathbf{G}_1$ is quadrilateral free over q^2+q+1 vertices with $\frac{1}{2}q(q+1)^2$ edges. Moreover, $\mathcal{N}(\mathbf{G}_1)$ contains disjoint $q+1$-element sets. Hence, by the above arguments, $\mathbf{G}_1$ has the same structure as $\mathbf{G}$. Denote the set of vertices of degree q of $\mathbf{G}_1$ by S_1. Then $S_1 = \{w\} \cup Q_0$. Hence we obtained that S_1 does not contain any edge from $\mathbf{G}_1$.

Now return to $\mathbf{G}$. As Q_0 does not contain any edge we obtain that all sets of the form $N(c(Q))$, i.e., $J(Q)$, avoids Q_0, proving (9.14). □

Finally, the rest of the edge set of $\mathbf{U}$ is a 1-factor on $V \setminus (S \cup Q_0)$.

Define the following hypergraph, $\mathbf{H}$, over the vertex set $V \cup \{w\}$.

$$\mathcal{E}(\mathbf{H}) = \mathcal{R} \cup \{Q \cup \{w\} : Q \in \mathcal{Q}^{\text{reg}}\} \cup \{Q_0 \cup \{v(Q_0)\}\} \cup \{S \setminus \{v(Q_0)\} \cup \{w\}\}.$$

$\mathcal{E}(\mathbf{H})$ can be obtained as the neighborhood structure of the following graph $\mathbf{G}^+$. Add the new vertex w to $\mathbf{G}$, and join it to each vertex from $S \setminus \{v(Q_0)\}$. Moreover add loops at the vertices w and $v(Q_0)$. (So from now on we allow loops in graphs.) Then $\mathcal{E}(\mathbf{H})$ is a regular 2-packing with symmetric incidency matrix. Consider the disjoint pair decomposition of $V(\mathbf{H})$, $V(\mathbf{H}) = \{w, v(Q_0)\} \cup \{c(Q), v(Q)\} \cup \{x_1, x_1'\} \cup \dots \{x_{(q^2-q)/2}, x'_{(q^2-q)/2}\}$, where $Q \in \mathcal{Q}^{\text{reg}}$. Consider the corresponding disjoint pair decomposition of $\mathcal{E}(\mathbf{H})$, $\mathcal{E}(\mathbf{H}) = \{S \setminus \{v(Q_0)\} \cup \{w\}, Q_0 \cup \{v(Q_0)\}\} \cup \{J(Q), Q \cup \{w\}\} \cup \{N(x_1).N(x_1')\} \dots$. Every 2×2 submatrix of the incidency matrix of $\mathbf{H}$ induced by the above decompositions is either full of 0 or I or $J - I$.

Define the following $\{0, 1, -1\}$ matrix M of size $\frac{1}{2}(q^2+q+2) \times \frac{1}{2}(q^2+q+2)$. Replace each 2×2 submatrix by 0, 1, or -1 if it is empty, I, or $J - I$, respectively. Then $MM^T = (q+1)I$. (8.13) still holds for $x \in V \setminus (S \cup Q_0)$, so the main diagonal of M corresponding to the squares $\{N(x_i), N(x_i')\} \times \{x_i, x_i'\}$ contains only 0's. Moreover, $c(Q) \in Q \cup \{w\}$ and $v(Q) \in J(Q)$, so the corresponding part of the main diagonal consists of -1's. Finally, $w \in (S \setminus \{v(Q_0)\}) \cup \{w\}$ and $v(Q_0) \in Q_0 \cup \{v(Q_0)\}$, hence tha first entry of the main diagonal is 1. We obtained that $\operatorname{trace} M = -q + 1$. Lemma (3.9) can be applied, implying $q \leq 3$.

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