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Note

Cross-Intersecting Families of Finite Sets

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It is proved that if $\mathscr{A}$ is a family of a-element sets and $\mathscr{B}$ is a family of b-element sets on the common undelying set $[n]$, and $A \cap B \neq \emptyset$ for all $A \in \mathscr{A}$, $B \in \mathscr{B}$ (i.e., cross-intersecting), and $n \geqslant a+b$, $|\mathscr{A}| \geqslant \binom{n-1}{a-1} - \binom{n-b-1}{a-1} + 1$, and $|\mathscr{B}| > \binom{n-1}{b-1} - \binom{n-a-1}{b-1} + 1$, then there exists an element $x \in [n]$ such that it belongs to all members of $\mathscr{A}$ and $\mathscr{B}$. This is an extension of a result of Hilton and Milner who generalized the Erdős–Ko–Rado theorem for non-trivial intersecting families. Several problems remain open.

1. NON-TRIVIAL CROSS-INTERSECTING FAMILES

For a positive integer n, let $[n] = \{1, 2, ..., n\}$, for integers $1 \leqslant a \leqslant b$ let $[a, b] = \{a, a+1, ..., b\}$. For a set S let $\binom{S}{k}$ denote the collection of k-element subsets of S and let 2^S denote the collection of all subsets of S. A family of sets $\mathscr{F}$ is called *intersecting* if $A \cap B \neq \emptyset$ hold for all $A, B \in \mathscr{F}$.

Let $\mathscr{F} \subset \binom{[n]}{k}$ be an intersecting family. Erdős, Ko, and Rado [3] proved that $|\mathscr{F}| \leqslant \binom{n-1}{k-1}$ holds for $n \geqslant 2k$. Moreover, in case of equality $\bigcap \mathscr{F} \neq \emptyset$ (for $n > 2k$). An intersecting family $\mathscr{G}$ is called *non-trivial* if $\bigcap \mathscr{G} = \emptyset$. Define the following non-trivial families. $\mathscr{G}^1 = \{G \in \binom{[n]}{k} : 1 \in G, G \cap [2, k+1] \neq \emptyset\} \cup \{[2, k+1]\}$ and $\mathscr{G}^2 = \{G \in \binom{[n]}{k} : |[3] \cap G| \geqslant 2\}$. For $k=2$, $\mathscr{G}^1 \equiv \mathscr{G}^2$; for $k=3$, $|\mathscr{G}^1| = |\mathscr{G}^2|$; while for $k \geqslant 4$, $n > 2k$, $|\mathscr{G}^1| > |\mathscr{G}^2|$. Hilton and Milner [10] proved the following generalization of the Erdős–Ko–Rado theorem. If $n > 2k$ and $\mathscr{G} \subseteq \binom{[n]}{k}$ is a non-trivial intersecting family then

$$|\mathscr{G}| \leqslant |\mathscr{G}^1| = \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1, \tag{1}$$

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Moreover, equality is possible only for $\mathscr{G} = \mathscr{G}^1$ or $\mathscr{G}^2$. A short proof was given in [6].

Two families $\mathscr{A}$ an $\mathscr{B}$ are called *cross-intersecting* if $A \cap B \neq \emptyset$ hold for all $A \in \mathscr{A}$, $B \in \mathscr{B}$. Here we extend (1) for two families.

THEOREM. *If* $\mathscr{A} \subset \binom{[n]}{a}$, $\mathscr{B} \subset \binom{[n]}{b}$, $A \cap B \neq \emptyset$ *for all* $A \in \mathscr{A}$, $B \in \mathscr{B}$ *(i.e., cross-intersecting), and* $n \geqslant a+b$, $|\mathscr{A}| > \binom{n-1}{a-1} - \binom{n-b-1}{a-1}$, *and* $|\mathscr{B}| > \binom{n-1}{b-1} - \binom{n-a-1}{b-1}$, *then one of the following two cases holds*:

(i) *there exists an element* $x \in [n]$ *such that* x *belongs to all members of* $\mathscr{A}$ *and* $\mathscr{B}$; *or*

(ii) $|\mathscr{A}| = \binom{n-1}{a-1} - \binom{n-b-1}{a-1} + 1$ *and* $|\mathscr{B}| = \binom{n-1}{b-1} - \binom{n-a-1}{b-1} + 1$.

We can describe the extremal families in case (ii). Namely, either

(ii/1) $n = a+b$, $|\mathscr{A}| = \binom{a+b-1}{a-1}$, $|\mathscr{B}| = \binom{a+b-1}{b-1}$, and for every partition of $X \cup Y = [n]$ with $|X| = a$, $|Y| = b$, either $X \in \mathscr{A}$ or $Y \in \mathscr{B}$; or

(ii/2) $a = b = k$, $\mathscr{A} = \mathscr{B} \cong \mathscr{G}^i$ for some $i \in \{1, 2\}$ (see (1)); or

(ii/3) $a, b \geqslant 2$ and for some a-set A_0 and b-set B_0 with $A_0 \cap B_0 \neq \emptyset$, and for some element $x \notin A_0 \cup B_0$, we have $\mathscr{A} = \{A : x \in A \in \binom{[n]}{a}, A \cap B_0 \neq \emptyset\} \cup \{A_0\}$ and $\mathscr{B} = \{B : x \in B \in \binom{[n]}{b}, B \cap A_0 \neq \emptyset\} \cup \{B_0\}$.

An easy corollary of the theorem was used to answer a problem of Trotter about the order dimension of two levels of the Boolean lattice; see [9].

2. PROOF

We prove the theorem by induction on $a+b$. The cases $a=1$ or $b=1$ are trivial.

The case $n = a+b$ is easy. Indeed, consider all the $\binom{a+b}{a}$ (ordered) partitions of $[n]$ into $X \cup Y = [n]$, with $|X| = a$, $|Y| = b$. For each such partition either $X \notin \mathscr{A}$ or $Y \notin \mathscr{B}$, implying $|\mathscr{A}| + |\mathscr{B}| \leqslant \binom{a+b}{a}$. (The case $a = b$ requires a little more care). The lower bounds for $|\mathscr{A}|$ and $|\mathscr{B}|$ give $|\mathscr{A}| \geqslant \binom{n-1}{a-1} - \binom{n-b-1}{a-1} + 1 = \binom{a+b-1}{a-1}$, and $|\mathscr{B}| \geqslant \binom{a+b-1}{a}$. So both inequalities hold with equality and we get case (ii/1). From now on, we suppose that $n > a+b$, $b \geqslant a \geqslant 2$.

Consider the case when for some $x \in [n]$ we have $x \in \bigcap \mathscr{A}$. If there exists a $B \in \mathscr{B}$ with $x \notin B$, then $\mathscr{A} \subset \{F \in \binom{[n]}{a} : x \in F, \ F \cap B \neq \emptyset\}$, implying $|\mathscr{A}| \leqslant \binom{n-1}{a-1} - \binom{n-b-1}{a-1}$, a contradiction. We obtain that $x \in \bigcap \mathscr{B}$, leading to case (i). From now on, we suppose that $\bigcap \mathscr{A} = \emptyset$. By a similar argument this implies that $\bigcap \mathscr{B} = \emptyset$ holds, too.

Consider the case when $\mathscr{A}$ itself is an intersecting family. Then (1) implies that $|\mathscr{A}| \leqslant \binom{n-1}{a-1} - \binom{n-a-1}{a-1} + 1$, which is not more than $\binom{n-1}{a-1} - \binom{n-b-1}{a-1}$

for $b > a \geqslant 2$, a contradiction. We obtain that $a = b$ and $|\mathscr{A}| = \binom{n-1}{a-1} - \binom{n-a-1}{a-1} + 1$. If there exists a set $B \in \mathscr{B} \backslash \mathscr{A}$, then $\mathscr{A} \cup \{B\}$ is a non-trivial intersecting family of size larger than the bound given by the Hilton–Milner theorem (1), a contradiction. So $\mathscr{B} \subset \mathscr{A}$, imlying $\mathscr{A} = \mathscr{B}$; we obtain case (ii/2).

From now on, we suppose that there are two members $A_1, A_2 \in \mathscr{A}$ disjoint to each other, $A_1 \cap A_2 = \varnothing$. Without loss of generality we may suppose that $A_1 = [a]$, $A_2 = [a+1, 2a]$. We are going to obtain the sharp upper bounds of (ii) for the sizes of $|\mathscr{A}|$ and $|\mathscr{B}|$.

Following Erdős, Ko, and Rado [3] we define a compression operation P_{ij} for all $1 \leqslant i < j \leqslant n$. However, here we will apply it to two families simultaneously, as it was first done for a similar problem in [8] (also see [5]). Then the rest of the proof is an extension of the ideas of the short proof for the Hilton–Milner theorem given in [6]. For a family $\mathscr{G} \subset 2^{[n]}$ let $P_{ij} : \mathscr{G} \to 2^{[n]}$ as

$$P_{ij}(G) = \begin{cases} (G \backslash \{j\}) \cup \{i\}, & \text{if } \ i \notin G, j \in G, (G \backslash \{j\}) \cup \{i\} \notin \mathscr{G}, \\ G, & \text{otherwise.} \end{cases}$$

Let us set $P_{ij}(\mathscr{G}) = \{P_{ij}(G) : G \in \mathscr{G}\}$. Obviously, $|P_{ij}(\mathscr{G})| = |\mathscr{G}|$.

We claim that if $\mathscr{F}$ and $\mathscr{G}$ are cross-intersecting, then $P_{ij}(\mathscr{F})$ and $P_{ij}(\mathscr{G})$ are cross-intersecting, too. Suppose, on the contrary, that $P_{ij}(F) \cap P_{ij}(G) = \varnothing$ for some $F \in \mathscr{F}$, $G \in \mathscr{G}$. As $F \cap G \neq \varnothing$ the only possibility is that one of these sets, say F, is unchanged, $P_{ij}(F) = F$, but the other one is new, $P_{ij}(G) = (G \backslash \{j\}) \cup \{i\}$. As F and G are unchanged outside $\{i, j\}$, we get that $F \cap G = \{j\}$, $i \notin P_{ij}(F) = F$. Then the only reason that F is unchanged is that $F' = (F \backslash \{j\}) \cup \{i\} \in \mathscr{F}$. This leads to $F' \cap G = \varnothing$, a contradiction.

Apply repeatedly P_{ij} for $\mathscr{A}$ and $\mathscr{B}$ simultaneously for all pairs (i, j) with $1 \leqslant i \leqslant a+b < j \leqslant n$ until we get two families $\mathscr{A}^*$ and $\mathscr{B}^*$ having the property $P_{ij}(\mathscr{A}^*) = \mathscr{A}^*$ and $P_{ij}(\mathscr{B}^*) = \mathscr{B}^*$ for every such pair (i, j). This can be reformulated as

$$\begin{aligned} &\text{If } A \in \mathscr{A}^*, i \notin A, j \in A, i \leqslant a+b < j \text{ then } (A \backslash \{j\}) \cup \{i\} \in \mathscr{A}^* \text{ as well.} \\ &\text{If } B \in \mathscr{B}^*, i \notin B, j \in B, i \leqslant a+b < j \text{ then } (B \backslash \{j\}) \cup \{i\} \in \mathscr{B}^* \text{ as well.} \end{aligned} \tag{2}$$

We claim that $\mathscr{A}^*$ and $\mathscr{B}^*$ are not simply cross-intersecting, but that they are cross-intersecting even on the first $a+b$ elements; i.e., for all $A \in \mathscr{A}^*$ and $B \in \mathscr{B}^*$ we have

$$A \cap B \cap [a+b] \neq \varnothing. \tag{3}$$

Proof of (3). Suppose, on the contrary, that $A \in \mathscr{A}^*$, $B \in \mathscr{B}^*$ with $A \cap B \cap [a+b] = \varnothing$, and suppose (A, B) is such that $|A \cap B|$ is minimal.

Thus there exist a $j \in A \cap B$ (hence, $j > a+b$) and an $i \in [a+b]$ such that $i \notin A \cup B$. Then (2) implies that $A \backslash \{j\} \cup \{i\} = A' \in \mathscr{A}^*$. However, $|A' \cap B| < |A \cap B|$, a contradiction. ∎

As $A_1 = [a]$ and $A_2 \subset [a+b]$ are unchanged during the above compressions, we have that $\bigcap \mathscr{A}^* = \varnothing$. This implies that $\bigcap \mathscr{B}^* = \varnothing$, too. Indeed, suppose, on the contrary, that $x \in B$ for all $B \in \mathscr{B}^*$. There exists an $A \in \mathscr{A}^*$ avoiding x, so $\mathscr{B}^* \subset \{F \in \binom{[n]}{b},\ x \in F,\ F \cap A \neq \varnothing\}$, implying $|\mathscr{B}^*| \leqslant \binom{n-1}{b-1} - \binom{n-a-1}{b-1}$. This contradicts the lower bound condition on $|\mathscr{B}|$. So from now on we may suppose that $\bigcap \mathscr{A}^* = \bigcap \mathscr{B}^* = \varnothing$.

Define the families of traces $\mathscr{A}_u = \{A \cap [a+b] : A \in \mathscr{A}^*,\ |A \cap [a+b]| = u\}$, $\mathscr{B}_v = \{B \cap [a+b] : B \in \mathscr{B}^*,\ |B \cap [a+b]| = v\}$. If $|\mathscr{A}_u| \leqslant \binom{l-1}{u-1} - \binom{l-b-1}{u-1}$ holds for all u, where $l = a+b$, then we get

$$|\mathscr{A}| = |\mathscr{A}^*| \leqslant \sum_{u=1}^{a} |\mathscr{A}_u| \binom{n-l}{a-u} \tag{4}$$

$$\leqslant \sum_{u=1}^{a} \left(\binom{l-1}{u-1} - \binom{l-b-1}{u-1} \right) \binom{n-l}{a-u}$$

$$= \binom{n-1}{a-1} - \binom{n-b-1}{a-1}, \tag{5}$$

a contradiction. So for some $1 \leqslant u \leqslant a$ we have $|\mathscr{A}_u| > \binom{l-1}{u-1} - \binom{l-b-1}{u-1}$. In the same way, we can prove that there exists a $1 \leqslant v \leqslant b$ such that $|\mathscr{B}_v| > \binom{l-1}{v-1} - \binom{l-a-1}{v-1}$. We may also suppose that both u and v are chosen to be minimal; i.e.,

$$|\mathscr{A}_i| \leqslant \binom{l-1}{i-1} - \binom{l-b-1}{i-1}, \qquad |\mathscr{B}_j| \leqslant \binom{l-1}{j-1} - \binom{l-a-1}{j-1} \tag{6}$$

hold for all $1 \leqslant i < u$, $1 \leqslant j < v$. By (3) $\mathscr{A}^* \cup \mathscr{A}_u$ and $\mathscr{B}^* \cup \mathscr{B}_v$ are cross-intersecting families.

If $\mathscr{A}_1 \neq \varnothing$, $\{x\} \in \mathscr{A}_1$, then $x \in B$ for all $B \in \mathscr{B}^*$, contradicting $\bigcap \mathscr{B}^* = \varnothing$. So from now on we may suppose that $\mathscr{A}_1 = \varnothing$ and, similarly, $\mathscr{B}_1 = \varnothing$. If $x \in \bigcap \mathscr{A}_u$, then in the same way as above, we get $x \in \bigcap \mathscr{B}^*$, a contradiction. So from now on, we may suppose that $\bigcap \mathscr{A}^* = \bigcap \mathscr{A}_u = \varnothing$ and, similarly, $\bigcap \mathscr{B}^* = \bigcap \mathscr{B}_v = \varnothing$. Moreover, $u, v \geqslant 2$.

Apply the induction hypothesis for the cross-intersecting families $\mathscr{A}_u$ and $\mathscr{B}_v$ with values l, u, v in place of n, a, b. Only case (ii) can occur, so we get $|\mathscr{A}_u| = \binom{l-1}{u-1} - \binom{l-v-1}{u-1} + 1$. But $|\mathscr{A}_u| \geqslant \binom{l-1}{u-1} - \binom{l-b-1}{u-1} + 1$, which implies $v = b$ (because $u - 1 \geqslant 1$). Similarly, we get $u = a$. Now use (6) and the upper bound $|\mathscr{A}_a| \leqslant \binom{l-1}{a-1} - \binom{l-b-1}{a-1} + 1$ in (4). Instead of (5) we get $|\mathscr{A}| \leqslant \binom{n-1}{a-1} - \binom{n-b-1}{a-1} + 1$, as desired. Analogously, we get $|\mathscr{B}| \leqslant \binom{n-1}{b-1} - \binom{n-a-1}{b-1} + 1$, finishing the case (ii). ∎

3. The Case of Equality

For brevity, we characterize the extremal families only if $b \geqslant 4$ and $a \geqslant 3$, the cases $a = b = 3$ and $a = 2$ are left to the reader. From Section 2 the only unsettled case is the relation of $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{A}^*$, $\mathscr{B}^*$ at the very end of the proof of case (ii). Equalities must hold in (4) and in (6) for all i and j for which $\binom{n-l}{a-i}$ and $\binom{n-l}{b-j}$ are positive, in particular, for $i = a-1$ and $j = b-1$. We get $|\mathscr{A}_{a-1}| \geqslant \binom{l-1}{a-2} - \binom{l-(b-1)-1}{a-2} + 1$ and $|\mathscr{B}_{b-1}| > \binom{l-1}{b-2} - \binom{l-(a-1)-1}{b-2} + 1$. Apply the induction hypothesis for $\mathscr{A}_{a-1}$ and $\mathscr{B}_{b-1}$. We get that there exists an element x belonging to all members of $\mathscr{A}_{a-1}$ and $\mathscr{B}_{b-1}$. Suppose that $x \notin A_2$; then the cross-intersecting property (and the size of $\mathscr{B}_{b-1}$) imply that $\mathscr{B}_{b-1}$ consists of all $(b-1)$ element subsets of $[a+b]$ containing x and meeting A_2. It is easy to see that A_2 is the only member of $\mathscr{A}^*$ avoiding x. Then, necessarily, $\mathscr{A}^*$ and $\mathscr{B}^*$ have the structure described in (ii/3).

We claim that $\mathscr{A}$ and $\mathscr{B}$ must have had the same property before the compressions. Consider the way we got $\mathscr{A}^*$ from $\mathscr{A}$ by compressions, $\mathscr{A} = \mathscr{A}^0 \to \mathscr{A}^1 \to \cdots \to \mathscr{A}^s = \mathscr{A}^*$. During the compressions for each family $\bigcap \mathscr{A}^\alpha = \varnothing$ holds, because A_1 and A_2 remained unchanged. This implies, in the familiar way, that at each step $\bigcap \mathscr{B}^\alpha = \varnothing$.

It is easy to prove that if there is a vertex v contained in all but one of the sets of $\mathscr{A}^\alpha$, then the pair $(\mathscr{A}^\alpha, \mathscr{B}^\alpha)$ has the structure described in (ii/3). We claim that $\max_{v \leqslant n} \deg(v, \mathscr{A}^\alpha) = |\mathscr{A}| - 1$ is true for all $0 \leqslant \alpha \leqslant s$.

Suppose that $\max_{v \leqslant n} \deg(v, \mathscr{A}^{\alpha+1}) = |\mathscr{A}| - 1$. Then the pair $\mathscr{A}^{\alpha+1}$, $\mathscr{B}^{\alpha+1}$ has the structure of (ii/3) with special element c and special sets A' and B'. If for the compression P_{ij} for which $P_{ij}(\mathscr{A}^\alpha) = \mathscr{A}^{\alpha+1}$, we have $c \notin \{i, j\}$, then the degree of c is unchanged, $\deg(c, \mathscr{A}^\alpha) = \deg(c, \mathscr{A}^{\alpha+1}) = |\mathscr{A}| - 1$. Suppose that $i = c$ (the case $j = c$ is impossible). We claim that either $\deg(c, \mathscr{A}^\alpha) = |\mathscr{A}| - 1$ or $\deg(j, \mathscr{A}^\alpha) = |\mathscr{A}| - 1$. For brevity we discuss only the case $c, j \notin A' \cup B'$.

Let $\mathscr{X} = \{X \subset [n] : |X| = a-1,\ X \cap B' \neq \varnothing,\ c, j \notin X\}$ and similarly $\mathscr{Y} = \{Y \subset [n] : |Y| = b-1, Y \cap A' \neq \varnothing, c, j \notin Y\}$. Define four families, $\mathscr{X}^c = \{X \in \mathscr{X} : X \cup \{c\} \in \mathscr{A}^\alpha\}$, $\mathscr{X}^j = \{X \in \mathscr{X} : X \cup \{j\} \in \mathscr{A}^\alpha\}$ and $\mathscr{Y}^c = \{Y \in Y: Y \cup \{c\} \in \mathscr{B}^\alpha\}$, $\mathscr{Y}^j = \{Y \in Y: Y \cup \{j\} \in \mathscr{B}^\alpha\}$. The families $\mathscr{X}^c$ and $\mathscr{X}^j$ form a partition of $\mathscr{X}$. A similar statement holds for the $\mathscr{Y}$'s.

We claim that if $X_1, X_2 \in \mathscr{X}$ and they differ only in one element (and $X_1 \cup X_2 \neq A'$), then they both belong to the same part of $\mathscr{X}$, implying that $\mathscr{X}$ is equal to either $\mathscr{X}^c$ or $\mathscr{X}^j$; the other part is empty. Indeed, find a $Y \in \mathscr{Y}$

with $Y \cap (X_1 \cup X_2) = \emptyset$. Then Y belongs to some of the $\mathscr{Y}^z$'s and then both of the X_i's must belong to the same $\mathscr{X}^z$.

Note that in the above argument about the extremal families we corrected a small error in [6] (the families $\mathscr{A}_2$ and $\mathscr{B}_2$ could be empty for small n, $a+b<n<2a+b-2$).

The bounds in the theorem are best possible in the following sense. Let $\mathscr{A}_0 = \{A \in \binom{[n]}{a} : 1 \in A,\ [2, b+1] \cap A \neq \emptyset\}$, and let $\mathscr{B}_0 = \{B \in \binom{[n]}{b} : 1 \in B\} \cup \{[2, b+1]\}$. Then $\mathscr{A}_0$ and $\mathscr{B}_0$ are cross-intersecting, $\bigcap \mathscr{B}_0 = \emptyset$, $|\mathscr{A}_0| = \binom{n-1}{a-1} - \binom{n-b-1}{a-1}$, and $|\mathscr{B}_0| = \binom{n-1}{b-1} + 1$, which is much larger than $\binom{n-1}{b-1} - \binom{n-a-1}{b-1}$. Another analogous example is $\mathscr{A}_1 = \{A \in \binom{[n]}{a} : 1 \in A\} \cup \{[2, a+1]\}$ and $\mathscr{B}_1 = \{B \in \binom{[n]}{b} : 1 \in B,\ B \cap [2, a+1] \neq \emptyset\}$.

The above proof of the theorem can be easily modified in such a way that it includes the proof of the Hilton–Milner theorem, too. (Induction on $n+a+b$, and a more careful choice of the operations P_{ij}). Another short proof for the Hilton–Milner theorem, based on the Kruskal–Katona theorem [12, 11], was given by Alon [1]. Other powerful applications of compressions can be found in the survey of Frankl [4], and in the book of Bollobás [2].

4. Problems

In this section we always suppose that $\mathscr{A} \subset \binom{[n]}{a}$, $\mathscr{B} \subset \binom{[n]}{b}$ are cross-intersecting families with $b \geqslant a$, $n \geqslant a+b$. The maximum of $|\mathscr{A}|\,|\mathscr{B}|$ was studied earlier, Pyber [14] proved that for $n \geqslant 2b+2a$

$$|\mathscr{A}|\,|\mathscr{B}| \leqslant \binom{n-1}{a-1}\binom{n-1}{b-1}. \tag{7}$$

This result was extended by Matsumoto and Tokushige [13] for all $n \geqslant \max\{2a, 2b\}$.

Conjecture 1. The inequality (7) holds for all $n > a+b$.

In the case $n = a+b$ one can construct cross-intersecting families $\mathscr{A}_2$ and $\mathscr{B}_2$ of sizes $|\mathscr{A}_2| = \lfloor \frac{1}{2}\binom{a+b}{a} \rfloor$ and $|\mathscr{B}_2| = \lceil \frac{1}{2}\binom{a+b}{a} \rceil$. Then $|\mathscr{A}_2|\,|\mathscr{B}_2|$ exceeds the right-hand side of (7) for $b > a$. Considering the example $\mathscr{A}_1$, $\mathscr{B}_1$ given at the end of the previous section we propose the following stronger form of Conjecture 1.

Conjecture 2. If $|\mathscr{A}|\,|\mathscr{B}| > |\mathscr{A}_1|\,|\mathscr{B}_1|$ and $n > a+b$, then $\bigcap \mathscr{A} = \bigcap \mathscr{B} \neq \emptyset$.

If we have individual lower bounds, like in our theorem, then we might get more. Define the cross-intersecting families $\mathscr{A}_3 = \{A \in \binom{[n]}{a} : 1 \in A,$

$[2, a+1] \cap A \neq \emptyset\} \cup \{[2, a+1]\}$ and $\mathscr{B}_3 = \{B \in \binom{[n]}{b} : 1 \in B, [2, a+1] \cap B \neq \emptyset\} \cup \{B \in \binom{[n]}{b} : 1 \notin B, [2, a+1] \subset B\}$.

Conjecture 3. If $|\mathscr{A}| \geqslant |\mathscr{A}_3| = \binom{n-1}{a-1} - \binom{n-a-1}{a-1} + 1$ and $|\mathscr{B}| \geqslant |\mathscr{B}_3| = \binom{n-1}{b-1} - \binom{n-a-1}{b-1} + \binom{n-1-a}{b-a}$ and $n > a+b$, then either $\bigcap \mathscr{A} \neq \emptyset$, or $|\mathscr{A}| = |\mathscr{A}_3|$, $|\mathscr{B}| = |\mathscr{B}_3|$. Moreover, for $a+b > 6$, $|\mathscr{A}| \cong \mathscr{A}_3$, $\mathscr{B} \cong \mathscr{B}_3$ are the only extrema.

I can settle the last two conjectures for $n > n_0(a, b)$. The proof is a simple application of the delta-system method. However, it would be interesting to lower $n_0(a, b)$ to $a+b$ (if it is true). The case $a = b$ seems to be especially interesting.

The maximum of $|\mathscr{A}| + |\mathscr{B}|$ was determined by Hilton and Milner [10] (see also Simpson [15]). Their result was extended by Frankl and Tokushige [7] as follows. For $n \geqslant a+b$, $b \geqslant a$, one has $|\mathscr{A}| + |\mathscr{B}| \leqslant \binom{n}{b} - \binom{n-a}{b} + 1$. They also proved a number of interesting inequalities; for example, if we also suppose that $|\mathscr{A}| \geqslant \binom{n-1}{a-1}$, then in the case $b > a$, one can get the stronger bound $|\mathscr{A}| + |\mathscr{B}| \leqslant \binom{n-1}{a-1} + \binom{n-1}{b-1}$. The method of proof in [7] is completely different from ours, and those results and our theorem do not seem to imply one another. Let $\mathscr{A}_4 = \binom{[a+1]}{a}$ and $\mathscr{B}_4 = \{B \in \binom{[n]}{b} : |[1, a+1] \cap B| \geqslant 2\}$.

Conjecture 4. Suppose that $\mathscr{A}$ and $\mathscr{B}$ are cross-intersecting and $\bigcap \mathscr{A} = \bigcap \mathscr{B} = \emptyset$. Then for $n \geqslant a+b$, $b \geqslant a$ one has $|\mathscr{A}| + |\mathscr{B}| \leqslant |\mathscr{A}_4| + |\mathscr{B}_4|$.

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Note added in proof. Very recently P. Frankl and N. Tokushige established Conjecture 3 and found counterexamples for the others.

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