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Note

New Asymptotics for Bipartite Turán Numbers

ZOLTÁN FÜREDI*

Department of Mathematics, University of Illinois at Urbana-Champaign, 1409 West Green Street, Urbana, Illinois 61801-2917; and Mathematical Institute of the Hungarian Academy of Sciences, P.O.B. 127, Budapest 1364, Hungary

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An algebraic construction implies $\lim_{n\to\infty} \mathrm{ex}(n, K_{2,t+1})\, n^{-3/2} = \sqrt{t}/2$.

1. THE TURÁN PROBLEM

Given a graph F, what is $\mathrm{ex}(n, F)$, the maximum number of edges of a graph with n vertices not containing F as a subgraph? Until now, the only asymptotic for a bipartite graph which is not a forest, $\mathrm{ex}(n, C_4) = \frac{1}{2}(1 + o(1))\, n^{3/2}$, is due to Erdős, Rényi and T. Sós [ERS] and (simultaneously and independently) to Brown [B].

THEOREM. *For any fixed* $t \geqslant 1$ $\mathrm{ex}(n, K_{2,t+1}) = \frac{1}{2}\sqrt{t}\, n^{3/2} + O(n^{4/3})$.

Let G be a graph on n vertices with e edges such that any two vertices have at most t common neighbors. The inequality $\sum_{x\in V}\binom{d(x)}{2} \leqslant t\binom{n}{2}$ (Kővári, T. Sós, Turán [KST]) implies the upper bound $e < \frac{1}{2}\sqrt{t}\, n^{3/2} + (n/4)$. To prove the Theorem we obtain a matching lower bound from a construction closely related to the examples from [ERS] and [B], and inspired by an example of Hyltén–Cavallius [H] and Mörs [M] given for Zarankiewicz's problem $z(n, n, 2, t+1)$ (see later in Section 3). The topic is so short of constructions that about 20 years ago, as a first step, Erdős [E67, E75] even proposed the problem whether $\lim_t(\liminf_n \mathrm{ex}(n, K_{2,t+1})\, n^{-3/2})$ goes to ∞ as $t \to \infty$. For a survey see Bollobás' book [Bo].

* E-mail: z-furedi@math.uiuc.edu; furedi@math-inst.hu.

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2. A Large Graph with no $K_{2,t+1}$

Let q be a prime power such that $(q-1)/t$ is an integer. We will construct a If $K_{2,t+1}$-free graph G on $n=(q^2-1)/t$ vertices such that every vertex has degree q or $q-1$. Then G has more than $(1/2)\sqrt{t}\,n^{3/2}-(n/2)$ edges. The lower bound for the Turán number for all n then follows from the fact for every sufficiently large n there exists a prime q satisfying $q\equiv 1 \pmod t$ and $\sqrt{nt}-n^{1/3}<q<\sqrt{nt}$ (see [HI]).

Let $\mathbf{F}$ be the q-element finite field, and let h be an element of order t, i.e., $h^t=1$ and the set $H=\{1, h, h^2, ..., h^{t-1}\}$ form a t-element subgroup of $\mathbf{F}\backslash\{0\}$. We say that $(a, b)\in \mathbf{F}\times\mathbf{F}$, $(a, b)\neq(0, 0)$ is equivalent to (a', b'), in notation $(a, b)\sim(a', b')$, if there exists some $h^\alpha\in H$ such that $a'=h^\alpha a$ and $b'=h^\alpha b$. The elements of the vertex set V of G are the t-element equivalence classes of $\mathbf{F}\times\mathbf{F}\backslash(0, 0)$. The class represented by (a, b) is denoted by $\langle a, b\rangle$. Two (distinct) classes $\langle a, b\rangle$ and $\langle x, y\rangle$ are joined by an edge in G if $ax+by\in H$. This relation is symmetric, and compatible to the equivalence classes, i.e., $ax+by\in H$, $(a, b)\sim(a', b')$, and $(x, y)\sim(x', y')$ imply $a'x'+b'y'\in H$.

For any given $(a, b)\in\mathbf{F}\times\mathbf{F}\backslash(0, 0)$ (say, $b\neq 0$) and for any given x and h^α, the equation $ax+by=h^\alpha$ has a unique solution for y. This implies that there are exactly tq solutions (x, y) with $ax+by\in H$. The solutions come in equivalence classes, one of these might coincide with $\langle a, b\rangle$ so the degree of the vertex $\langle a, b\rangle$ in G is either q or $q-1$.

We claim that G is $K_{2,t+1}$-free. First we show, that for $(a, b), (a', b')\in\mathbf{F}\times\mathbf{F}\backslash(0, 0)$, $(a, b)\nsim(a', b')$ the equation system

$$\begin{aligned} ax+by&=h^\alpha \\ a'x+b'y&=h^\beta \end{aligned} \tag{1}$$

has at most one solution $(x, y)\in\mathbf{F}\times\mathbf{F}\backslash(0, 0)$. Indeed, the solution is unique if the determinant det $\left(\begin{smallmatrix} a & b \\ a' & b'\end{smallmatrix}\right)$ is not 0. Otherwise, there exists a c such that $a=a'c$ and $b=b'c$. If there exists a solution of (1) at all, then multiplying the second equation by c and subtracting it from the first one we get on the right hand side $h^\alpha-ch^\beta=0$. Thus $c\in H$, contradicting the fact that (a, b) and (a', b') are not equivalent.

Finally, there are t^2 possibilities for $0\leqslant\alpha, \beta<t$ in (1). The set of solutions again form t-element equivalence classes, so there are at most t classes $\langle x, y\rangle$ joint simultaneously to $\langle a, b\rangle$ and $\langle a', b'\rangle$.

The sets $N\langle a, b\rangle=\{\langle x, y\rangle: ax+by\in H\}$ almost form a $q+1$-uniform, symmetric t-design. This means that, they mutually intersect in exactly t elements except if $(a, b)=(ca', cb')$ holds for some c when they are disjoint. It seems to me that this structure, unfortunately, cannot be extended to a proper t-design.

3. Corollaries for Zarankiewicz's Problem

Given m, n, s and t, what is the maximum number, $z = z(m, n, s, t)$, such that there exists a $0-1$ matrix with m rows and n columns containing z 1's without a submatrix with s rows and t columns consisting of entirely of 1's. This question has become known as the problem of Zarankiewicz [Z]. For a bipartite graph F define the bipartite Turán number, $\mathrm{ex}(m, n, F)$, as the maximum number of edges in an F-free bipartite graph with m and n vertices in its color classes. Considering the adjacency matrix of a $K_{s,t}$-free graph on n vertices as a matrix of a bipartite graph on $n+n$ vertices (cf. [Bo] p. 310) we get

$$2\,\mathrm{ex}(n, K_{s,t}) \leqslant \mathrm{ex}(n, n, K_{s,t}) \leqslant z(n, n, s, t). \tag{2}$$

The determination of $z(m, n, s, t)$ is equivalent to a unidirectional Turán problem, when we label the two color classes of F and only those copies of F are forbidden in which the entire first color class is contained in the m-element set and the second color class lies in the n-element set.

It is easy to see that $z(n, n, 2, t+1) \leqslant n\sqrt{tn - t + 1/4} + (n/2)$, and it is known that this bound is asymptotically correct, i.e., $\lim_{n\to\infty} z(n, n, 2, t+1)\, n^{-3/2} = \sqrt{t}$ ([KST] for $t=1$, [H] for $t=2$ and [M] for all t). Our Theorem and the lower bound in (2) gives

Corollary. *For any fixed* $t \geqslant 1$ $\mathrm{ex}(n, n, K_{2,t+1}) = \sqrt{t}\, n^{3/2} + O(n^{4/3})$.

Thus we have a new near optimal construction for $z(n, n, 2, t+1)$. The gap between the lower and upper bounds in the case $n = (q^2 - 1)/t$ is only $O(\sqrt{n})$.

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