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The Order Dimension of Two Levels of the Boolean Lattice

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Abstract. Let $\mathcal{B}_n(s,t)$ denote the partially ordered set consisting of all s-subsets and t-subsets of an n-element underlying set where these sets are ordered by inclusion. Answering a question of Trotter we prove that $\dim(\mathcal{B}_n(k, n-k)) = n-2$ for $3 \leqslant k < (1/7)n^{1/3}$. The proof uses extremal hypergraph theory.

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1. Preliminaries

For a positive integer n, let $[n] = \{1, 2, \ldots, n\}$, let 2^S denote the collection of subsets of S, and let $\mathcal{B}_n = (2^{[n]}, \subseteq)$ denote the *Boolean lattice*, the subsets of $[n]$ ordered by inclusion. For a set S, let $\binom{S}{k}$ denote the collection of k-element subsets of S. For $0 \leqslant s < t \leqslant n$ let $\mathcal{B}_n(s,t)$ denote the restriction of $\mathcal{B}_n$ to $\binom{[n]}{s} \cup \binom{[n]}{t}$. Finally, let $\dim(n; s, t)$ denote the (order) dimension of $\mathcal{B}_n(s,t)$. The dimension of a partially ordered set P is the minimum d such that P can be embedded into $\mathbb{R}^d$ in an order preserving way. In other words, it is the minimum number of linear extensions $\pi_1, \ldots, \pi_d$ such that for all $x, y \in P$ there exists an π_i with $x <_i y$ (x precedes y in π_i) except, of course, if $y <_P x$. In the latter case y precedes x in all linear extensions. Additional background material on dimension theory can be found in the monograph [20].

The function $\dim(n; s, t)$ was first studied by Dushnik [5] in 1950. It is easy to see that $\dim \mathcal{B}_n = \dim(n; 1, n-1) = n$. This is the so-called *standard example*, the smallest poset of dimension n. The estimates in general are surprisingly poor, except, in the case $s = 1$. Dushnik determined the exact value for $\dim(n; 1, t)$ when $2\sqrt{n} - 2 \leqslant t < n - 1$. Namely, he proved that $\dim(n; 1, t) = n - j + 1$, where j is the unique integer with $2 \leqslant j \leqslant \sqrt{n}$ for which

$$\left\lfloor \frac{n - 2j + j^2}{j} \right\rfloor \leqslant t < \left\lfloor \frac{n - 2(j-1) + (j-1)^2}{j-1} \right\rfloor. \tag{1}$$

It follows that for all $t^2 \leqslant n$, $t^2/4 < \dim(n;1,t)$. On the other hand, in [11] it was established that $\dim(n;1,t) \leqslant \lceil (t+1)^2 \log n \rceil$ for all $t < n$. The proof is simply a matter of taking d linear orderings of $[n]$, uniformly at random from the set of all possible linear orderings, and noting that the probability that these do not form a family of *preorders* tends to 0.

For fixed t, Spencer [19] proved $\dim(n;1,t) = \Theta(\log\log n)$ establishing the asymptotic behavior. The following asymptotic formula is proved in [10]

$$\dim(n;1,2) = \log\log n + \big(1/2 + o(1)\big)\log\log\log n.$$

Determining $\dim(n;1,t)$ for t a small growing function of n remains an intriguing open problem.

Until recently, very little was known for the case $s > 1$. Brightwell, Kierstead, Kostochka, and Trotter [4] gave a beautiful recursive construction for estimating $\dim(n;s,t)$ if $t-s$ is relatively small. They showed that $\dim(n;s,s+k) \leqslant \dim(n;1,3k) + 18k\log n$, for every positive integer s, with $s+k \leqslant n$. It follows that $\dim(n;s,s+k) = O(k^2 \log n)$. In the most interesting case, for the middle two levels of the Boolean lattice, they gave a better constant factor

$$\log\log n + \big(1/2 + o(1)\big)\log\log\log n < \dim(2s+1;s,s+1) \leqslant (6/\log 3)\log n.$$

(Here $n = 2s+1$ and all log's are of base 2.) These inequalities are far from tight.

Hurlbert [13] showed that for fixed k there is an integer $n_0(k)$ such that for all $n > n_0(k)$

$$\dim(n;k,n-k) \geqslant n - \lfloor (k+1)^2/2 \rfloor + 2.$$

He also showed that $\dim(n;k,n-k) \geqslant \lceil n/k \rceil$ for all $n > 2k$. Hurlbert, Kostochka, and Talysheva in [14] showed that for $n \geqslant 5$, $\dim(n;2,n-2) = n-1$ and $\dim(n;2,n-3) = n-2$. In fact, they proved that if $2\sqrt{n} < t < n-2$ and t is not an integer of the form $j-2+(n-1)/j$ for some positive integer j, then $\dim(n;2,t) = \dim(n;1,t)-1$.

2. Results

PROPOSITION 2.1. *For* $n \geqslant 3k+2$ *one has* $\dim(n;k,n-k) \geqslant n-k$.

Proof. We have $\dim(n;k,n-k) \geqslant \dim(n-k+1;1,n-2k+1)$, because $\mathcal{B}_{n-r}(s-r,t-r)$ can be considered as a subposet of $\mathcal{B}_n(s,t)$ (here $r \leqslant s \leqslant t \leqslant n$). For $n \geqslant 3k+2$ the value of the right hand side is $n-k$ by (1). □

Our next aim is a very short proof for a simple lower bound for all n and k. The proof is based on Lovász–Kneser graph theorem and is postponed to Section 3. As far as the author knows, this is the first application of the Lovász–Kneser theorem besides the chromatic theory of graphs.

PROPOSITION 2.2. *For all $n > 2k$ one has* $\dim(n; k, n-k) \geqslant n - 2k + 2$.

Our main result is the determination of the exact value of $\dim(n; k, n-k)$ for n large. The basic tool of the proof is an application of a new cross-intersecting version of the Erdös–Ko–Rado theorem (Lemma 4 below).

THEOREM 2.3. *For $k \geqslant 3$ and $n > 250k^3$ we have* $\dim(n; k, n-k) = n - 2$.

3. Linear Extensions and Kneser Graphs

Suppose that $1 \leqslant s < t < n$ are integers, and let

$$P = \binom{[n]}{s} \cup \binom{[n]}{t}, \qquad m = |P|.$$

A bijection π: $P \to [m]$ is a *linear extension* of $\mathcal{B}_n(s,t)$ if $X \subset Y$ implies $\pi(X) \leqslant \pi(Y)$. We may call a sequence $\{\pi^{-1}(1), \ldots, \pi^{-1}(m)\}$ *monotone*, because (roughly speaking) the small sets come first, and the larger sets come later. We call a pair (S,T) a *reversed pair* in π (and denote the set of reversed pairs from π by $\mathcal{R}(\pi)$) if $S \subset [n]$, $|S| = s$, $T \subset [n]$, $|T| = t$, and $\pi(T) < \pi(S)$. In that case, S is not a subset of T. Let $\mathcal{R}(s,t)$ be the set of all possible reversed pairs, i.e.,

$$\mathcal{R}(s,t) = \left\{(S,T)\colon\ S \in \binom{[n]}{s},\ T \in \binom{[n]}{t} \text{ and } S \not\subset T\right\}.$$

We have

$$|\mathcal{R}(s,t)| = \binom{n}{s} \times \left(\binom{n}{t} - \binom{n-s}{t-s}\right).$$

The dimension of the poset is the minimum d such that one can find linear extensions $\pi_1, \ldots, \pi_d$ with

$$\bigcup_{1 \leqslant i \leqslant d} \mathcal{R}(\pi_i) = \mathcal{R}(s,t). \tag{3}$$

Indeed, by definition, the poset $(P, <_P) = \pi_1 \cap \cdots \cap \pi_d$ has the underlying set P equipped with the relation:

$$X, Y \in P, \qquad X <_P Y \quad \text{if and only if} \quad \pi_i(X) < \pi_i(Y) \text{ for all } i.$$

Obviously, $X \subset Y$ implies $X <_P Y$. We have to show that $X \not\subset Y$ implies $X \not<_P Y$, i.e., there exists some i with $\pi_i(X) > \pi_i(Y)$. By (3) this holds for all $(X,Y) \in \mathcal{R}(s,t)$. If $(X,Y) \in \binom{[n]}{s} \times \binom{[n]}{s}$, then consider a set $Y' \in \binom{[n]}{t}$ such that $Y \subset Y'$ but $X \not\subset Y'$, and let π_i be a permutation reversing the pair (X, Y'). Then $\pi_i(X) > \pi_i(Y') > \pi_i(Y)$, as desired. The remaining cases i.e.,

$$(X,Y) \in \binom{[n]}{t} \times \binom{[n]}{t} \quad \text{and} \quad (X,Y) \in \binom{[n]}{t} \times \binom{[n]}{s}$$

can be handled in a similar way. □

For $n \geqslant 2k$ the vertex set of the *Kneser graph* $K(n,k)$ is $\binom{[n]}{k}$ and two vertices $A, B \in \binom{[n]}{k}$ are joined by an edge if $A \cap B = \varnothing$. A family of sets $\mathcal{F}$ is called *intersecting* if $A \cap B \neq \varnothing$ hold for all $A, B \in \mathcal{F}$. Consider the families

$$\mathcal{F}_i = \left\{ A \in \binom{[n]}{k} : \min A = i \right\} \quad \text{for } 1 \leqslant i \leqslant n - 2k + 1$$

and let

$$\mathcal{F}_0 = \binom{X}{k}, \quad \text{where } X = \{n - 2k + 2, \dots, n\}.$$

Each $\mathcal{F}_i$ is intersecting, so this partition of $\binom{[n]}{k}$ shows that the chromatic number of the Kneser graph satisfies $\chi(K(n,k)) \leqslant n - 2k + 2$. Kneser conjectured and Lovász [18] proved that here equality holds. Bárány [1] gave a simple proof.

Proof of Proposition 2.2. Let $\pi_1, \dots, \pi_d$ be linear extensions generating $\mathcal{B}_n(k, n - k)$. Define $\mathcal{F}_i$ as the family of k-sets $A \in \binom{[n]}{k}$ such that $(A, [n] \setminus A)$ form a reversed pair in $\pi = \pi_i$. We claim that $\mathcal{F}_i$ is intersecting. Indeed, $A \cap B = \varnothing$ and $\pi([n] \setminus A) < \pi(A)$, $\pi([n] \setminus B) < \pi(B)$ imply $\pi(B) < \pi([n] \setminus A) < \pi(A)$, and $\pi(A) < \pi([n] \setminus B) < \pi(B)$, a contradiction. Thus the families $\mathcal{F}_i$ form a coloring of the Kneser graph $K(n,k)$ implying $d \geqslant \chi(K(n,k)) = n - 2k + 2$. □

4. Cross-Intersecting Families

Let $\mathcal{F} \subset \binom{[n]}{k}$ be an intersecting family. Erdös, Ko, and Rado [6] proved that $|\mathcal{F}| \leqslant \binom{n-1}{k-1}$ holds for $n \geqslant 2k$. Moreover, in case of equality $\bigcap \mathcal{F} \neq \varnothing$ (for $n > 2k$). An intersecting family $\mathcal{G}$ is called *nontrivial* if $\bigcap \mathcal{G} = \varnothing$. Hilton and Milner [12] determined the maximum size of a nontrivial intersecting family $\mathcal{G} \subset \binom{[n]}{k}$ (for a short proof see [8]). Here we recall their result in a weaker form

$$|\mathcal{G}| \leqslant 2k \binom{n-2}{k-2}. \tag{4}$$

Two families $\mathcal{A}$ and $\mathcal{B}$ are called *cross-intersecting* if $A \cap B \neq \varnothing$ hold for all $A \in \mathcal{A}$, $B \in \mathcal{B}$. A pair of cross-intersecting families $(\mathcal{A}, \mathcal{B})$ is called *maximal* $(n; a, b)$-family if $\mathcal{A} \subset \binom{[n]}{a}$, $\mathcal{B} \subset \binom{[n]}{b}$ and $\mathcal{A} \subset \mathcal{A}' \subset \binom{[n]}{a}$, $\mathcal{B} \subset \mathcal{B}' \subset \binom{[n]}{b}$, $\mathcal{A}'$ and $\mathcal{B}'$ cross-intersecting imply that $\mathcal{A}' = \mathcal{A}$, $\mathcal{B}' = \mathcal{B}$.

LEMMA 4. *If* $\mathcal{A} \subset \binom{[n]}{a}$, $\mathcal{B} \subset \binom{[n]}{b}$ *are cross-intersecting*, n *is large*, $n \geqslant ba^2$, $b \geqslant a \geqslant 2$,

$$|\mathcal{A}| > 2b\binom{n-2}{a-2}, \quad \textit{and} \quad |\mathcal{B}| > 2a\binom{n-2}{b-2},$$

then there exists an element $x \in [n]$ *belonging to all members of* $\mathcal{A}$ *and* $\mathcal{B}$. *If, in addition,* $(\mathcal{A}, \mathcal{B})$ *is a pair of maximal cross-intersecting* $(n; a, b)$*-family, then*

$$|\mathcal{A}| = \binom{n-1}{a-1}, \qquad |\mathcal{B}| = \binom{n-1}{b-1}.$$

Proof. First, consider the case that for some element x, $x \in \bigcap \mathcal{A}$. If one can find a $B \in \mathcal{B}$ with $x \notin B$, then each $A \in \mathcal{A}$ contains x and meets B, so we get that $|\mathcal{A}| \leqslant b\binom{n-2}{a-2}$, a contradiction. In the same way, $x \in \bigcap \mathcal{B}$ leads to $x \in (\bigcap \mathcal{A}) \cap (\bigcap \mathcal{B})$. From now on we suppose that $\bigcap \mathcal{A} = \bigcap \mathcal{B} = \varnothing$. This implies that

$$\deg_{\mathcal{A}}(x) \leqslant b\binom{n-2}{a-2}, \quad \text{and} \quad \deg_{\mathcal{B}}(x) \leqslant a\binom{n-2}{b-2}$$

hold for all x.

Suppose that there exists a two element set $T = \{x, y\}$ such that $T \cap A \neq \varnothing$ holds for every $A \in \mathcal{A}$. Then we have that

$$|\mathcal{A}| \leqslant \deg_{\mathcal{A}}(x) + \deg_{\mathcal{A}}(y) \leqslant 2b\binom{n-2}{b-2},$$

a contradiction. So from now on we suppose that for every pair T there exists an $A \in \mathcal{A}$ (and similarly, a $B \in \mathcal{B}$) such that $T \cap A = \varnothing$ ($T \cap B = \varnothing$). Then the number of sets from $\mathcal{A}$ containing T, $\deg_{\mathcal{A}}(T)$, is at most $b\binom{n-3}{a-3}$ (and similarly $\deg_{\mathcal{B}}(T) \leqslant a\binom{n-3}{b-3}$).

Finally, we may suppose that there are two disjoint sets $A_1, A_2 \in \mathcal{A}$. Indeed, if $\mathcal{A}$ itself is intersecting, then (4) gives the upper bound

$$|\mathcal{A}| \leqslant 2a\binom{n-2}{a-2} \leqslant 2b\binom{n-2}{a-2},$$

a contradiction. Considering all possible pairs meeting both A_1 and A_2 we have

$$|\mathcal{B}| \leqslant \sum_{x_i \in A_i} \deg_{\mathcal{B}}(x_1, x_2) \leqslant a^2 a\binom{n-3}{b-3}.$$

Here the right hand side is less than $2a\binom{n-2}{b-2}$ for $n > a^2 b$, a final contradiction. □

We remark that using much more extremal hypergraph theory we can determine the best bound in Lemma 4 (see [9]), but that has no significant effect on the bound $250k^3$ in Theorem 2.3.

5. Cross-Intersecting Sequences

Let $\mathcal{A}$, $\mathcal{B}$ be two families of finite sets. A bijection π: $\mathcal{A} \cup \mathcal{B} \to [|\mathcal{A}| + |\mathcal{B}|]$ is called *cross-intersecting* if $\pi(A) > \pi(B)$, $A \in \mathcal{A}$, $B \in \mathcal{B}$, imply $A \cap B \neq \varnothing$. If $\mathcal{A} = \binom{[n]}{a}$, $\mathcal{B} = \binom{[n]}{b}$, then the triple $(\mathcal{A}, \mathcal{B}, \pi)$ is called a cross-intersecting $(n; a, b)$*-sequence.* There is a 1–1 correspondence between cross-intersecting $(n; a, b)$-sequences and linear extensions of $\mathcal{B}_n(a, n-b)$ (here $a + b < n$). Namely, replacing each $B \in \mathcal{B}$ in the sequence by its complement $[n] \setminus B$ we get a linear extension, and vice versa. We frequently use the same letter, π, denoting the linear extension, and the corresponding intersecting sequence. For a set $F \subset [n]$ we denote the complement by F^{C}. For the family $\mathcal{F} \subset 2^{[n]}$, however, $\mathcal{F}^{\mathrm{C}}$ means the family of complements, $\mathcal{F}^{\mathrm{C}} = \{[n] \setminus F\colon F \in \mathcal{F}\}$. The notion of reversed pair corresponds to the *crossed pair*, i.e., let

$$\mathcal{C}(\pi) := \{(A, B)\colon \pi(A) > \pi(B)\} \subset \binom{[n]}{a} \times \binom{[n]}{b}.$$

A linear extension π is called *maximal,* if $\mathcal{R}(\pi)$ is maximal, i.e., $\mathcal{R}(\pi) \subset \mathcal{R}(\pi')$ implies $\mathcal{R}(\pi) = \mathcal{R}(\pi')$. A cross-intersecting sequence is maximal if the corresponding linear extension is maximal. Define $\mathcal{B}(\pi, < A)$ (and $\mathcal{A}(\pi, < A)$) as the set of all members of $\mathcal{B}$ (or $\mathcal{A}$, respectively) preceding A in the sequence π. Let $\mathcal{A}(\pi, A \leqslant)$ denote the set of those members of $\mathcal{A}$ which follow all members of $\mathcal{B}(\pi, < A)$. This set obviously contains all members of $\mathcal{A}$ after A (i.e., the family $\mathcal{A}(\pi, A <)$), but it could be larger. The above definitions naturally extend to linear extensions as well.

It is easy to see that π is a *maximal* linear extension of the poset $\mathcal{B}_n(a, n-b)$, if and only if, $\mathcal{B}^{\mathrm{C}}(\pi, < A)$ and $\mathcal{A}(\pi, A \leqslant)$ form a maximal cross-intersecting pair for all $A \in \mathcal{A}$.

Define the *colex* order of the members of $\mathcal{B}_n$, as usual, by the relation $X < Y$ for $X, Y \subset [n]$ if the maximum element of the symmetric difference $(X \setminus Y) \cup (Y \setminus X)$ belongs to Y. Let $\mathcal{L}_k(m)$ denote the first m members of $\binom{N}{k}$ in the colex order. For example, for $m = \binom{p}{k}$ (with an integer $p \geqslant k$) we have $\mathcal{L}_k(m) = \binom{[p]}{k}$.

For a family $\mathcal{F}$ and integer h, let $\partial_h(\mathcal{F})$ denote its *h-shadow,* i.e.,

$$\partial_h(\mathcal{F}) = \{H\colon |H| = h, \text{ and for some } F \in \mathcal{F} \text{ we have } H \subset F\}.$$

Kruskal [17] and Katona [16] proved (see [3]), that for any family $\mathcal{F}$ of k-element sets one has

$$|\partial_h(\mathcal{F})| \geqslant |\partial_h(\mathcal{L}_k(|\mathcal{F}|))|. \tag{5.1}$$

From now on, we suppose that $a + b < n$ and π is a maximal linear extension of $\mathcal{B}_n(a, n-b)$. This implies, that for any $B \in \binom{[n]}{n-b}$ the set of A's preceding B consists of the a-subsets of B and the a-subsets of all $B' \in \mathcal{B}$ with $\pi(B') < \pi(B)$. Using our notation $\mathcal{A}(\pi, < B) = \partial_a(\mathcal{B}(\pi, < B) \cup \{B\})$. We maximize the number of reversed pairs when minimize the number of A's preceding B. The Kruskal–Katona theorem, (5.1), implies that $\partial_a(\mathcal{B}(\pi, < B) \cup \{B\})$ is minimal when $\mathcal{B}$ is in colex order. So the number of reversed pairs is minimized if $\mathcal{B}$ is in colex order, consequently $\mathcal{A}$ is in colex order, too. Denote this linear extension by $\pi_{\mathcal{L}}$. Note that in this order every $(n-b)$-element subset avoiding the element n precedes every a-subset containing n. Even more, there is a natural partition of the sequence $\pi_{\mathcal{L}}$, all sets avoiding n precede all sets containing n.

Let $r(n; a, n-b)$ denote max $|\mathcal{R}(\pi)|$, where π is any linear extension of $\mathcal{B}_n(a, n-b)$. The above argument implies that $r(n; a, n-b) = |\mathcal{R}(\pi_{\mathcal{L}})|$, and one can obtain the following recursion

$$r(n; a, n-b) = \binom{n-1}{b-1}\binom{n-1}{a-1} + \\ + r(n-1; a, n-b) + r(n-1; a-1, n-b-1). \tag{5.2}$$

Using the easy identities

$$r(n; 0, n-x) = r(n; x, n) = 0,$$

$$r(n; 1, n-x) = r(n; x, n-1) = \binom{n}{x-1},$$

one can easily prove by induction that

$$r(n; a, n-b) = \sum_{i=0}^{a-1}\sum_{j=0}^{b-1}\binom{i+j}{i}\binom{n-1-i-j}{a-1-i}\binom{n-1-i-j}{b-1-j}. \tag{5.3}$$

In the above argument we followed Hurlbert [13] who proved this upper bound for $a = b$. For our purposes the following estimate is more appropriate, which can be proved by induction using (5.2).

LEMMA 5. *For all $n > a + b$ one has*

$$r(n; a, n-b) \leqslant \binom{n}{a-1}\binom{n}{b-1}.$$

6. Understanding the Finer Structure of Linear Extensions

Let π be a linear extension of $\mathcal{B}_n(a, n-b)$, $n > a+b$, $a, b \geqslant 2$. A member $A \in \mathcal{A}$ of the monotone sequence $\{\pi^{-1}(1), \pi^{-1}(2), \ldots\}$ is called of *type* I (with respect to π) if

$$|\mathcal{B}(\pi, < A)| > 2a\binom{n-2}{b-2} \quad \text{and} \quad |\mathcal{A}(\pi, A \leqslant)| > 2b\binom{n-2}{a-2}.$$

The permutation π is called of type I if $\mathcal{A}$ has a member of type I. A reversed pair $(A, B) \in \mathcal{R}(\pi)$ is called of type I if A is of type I. Any other member of $\mathcal{A}$, reversed pair, or linear extension is called of type II.

LEMMA 6.1. *In any linear extension π of $\mathcal{B}_n(a, n-b)$ the number of reversed pairs of type* II *is at most*

$$|\mathcal{R}^{\mathrm{II}}(\pi)| < \frac{a^2+8ab+b^2}{n}\binom{n-1}{a-1}\binom{n-1}{b-1}.$$

Proof. Split $\mathcal{R}^{\mathrm{II}}(\pi)$ into four parts. The pairs with

$$|\mathcal{A}(\pi, A \leqslant)| \leqslant \binom{n-2}{a-2}$$

are put into $\mathcal{R}^1$, the pairs with

$$2b\binom{n-2}{a-2} \geqslant |\mathcal{A}(\pi, A \leqslant)| > \binom{n-2}{a-2}$$

are put into $\mathcal{R}^2$,

$$\mathcal{R}^3 = \left\{(A,B) \in \mathcal{R}(\pi)\colon\ |\mathcal{B}(\pi, < A)| \leqslant \binom{n-2}{b-2}\right\},$$

and

$$\mathcal{R}^4 = \left\{(A,B) \in \mathcal{R}(\pi)\colon\ \binom{n-2}{b-2} < |\mathcal{B}(\pi, < A)| \leqslant 2a\binom{n-2}{b-2}\right\}.$$

Every $A \in \mathcal{A}$ is preceded by at most

$$\binom{n}{b} - \binom{n-a}{b} \leqslant a\binom{n-1}{b-1}$$

members of $\mathcal{B}$, which gives

$$|\mathcal{R}^1| \leqslant a\binom{n-2}{a-2}\binom{n-1}{b-1}.$$

Similarly we get

$$|\mathcal{R}^3| \leqslant b\binom{n-1}{a-1}\binom{n-2}{b-2}.$$

If $(A, B) \in \mathcal{R}^2$, then A is followed by at least $\binom{n-2}{a-2}$ more A's, so the Kruskal–Katona theorem (5.1) gives that it can be preceded by at most

$$\binom{n}{b} - \binom{n-2}{b} \leqslant 2\binom{n-1}{b-1}$$

members of $\mathcal{B}$. This gives

$$|\mathcal{R}^2| \leqslant 2(2b-1)\binom{n-2}{a-2}\binom{n-1}{b-1},$$

and similarly we get

$$|\mathcal{R}^4| \leqslant 2(2a-1)\binom{n-1}{a-1}\binom{n-2}{b-2}. \qquad \square$$

LEMMA 6.2. *If π is a maximal linear extension of $\mathcal{B}_n(a, n-b)$ of type* I, *and $n > ab(a+b)$, then there exists an element $c = c(\pi) \in [n]$ such that all sets from $\binom{[n]}{a} \cup \binom{[n]}{n-b}$ avoiding c precede all sets containing c.*

We will call the element c the *center* of π. Obviously, the center is unique (if it exists).

Proof of Lemma 6.2. Let $A \in \mathcal{A}$ be of type I. We can apply Lemma 4 for the cross-intersecting families $\mathcal{B}^{\mathrm{C}}(\pi, < A)$ and $\mathcal{A}(\pi, A \leqslant)$. We obtain that there exists an element c which is contained in every member of these two families, i.e.,

$$\mathcal{B}^{\mathrm{C}}(\pi, < A) \subset \left\{B^{\mathrm{C}} \in \binom{[n]}{b} : \ c \in B^{\mathrm{C}}\right\}$$

and

$$\mathcal{A}(\pi, A \leqslant) \subset \left\{A \in \binom{[n]}{a} : \ c \in A\right\}.$$

As π is maximal, we get that all B with $c \notin B \in \mathcal{B}$ precede $\mathcal{A}(\pi, A \leqslant)$. (Otherwise, we can put one before $\mathcal{A}(\pi, A \leqslant)$ and create more reversed pairs.) Hence

$$\mathcal{B}(\pi, < A) = \binom{[n]\setminus c}{n-b}.$$

Similarly, maximality implies that all $A \in \mathcal{A}$ with $c \in A$ follow $\mathcal{B}(\pi, < A)$, implying

$$\mathcal{A}(\pi, A \leqslant) = \left\{A \in \binom{[n]}{a} : \ c \in A\right\}. \qquad \square$$

If π is of type I with center c, then there is a natural partition of it into two halves. The lower part can be considered as a linear extension of $\mathcal{B}_{n-1}(a, (n-1)-(b-1))$ with underlying set $[n] \setminus \{c\}$. The upper part, after removing the element c from each set, can be considered as a linear extension of $\mathcal{B}_{n-1}(a-1, (n-1)-b)$ with underlying set $[n] \setminus \{c\}$. If π is maximal, then both halves are maximal, too (of their kind, of course).

The following lemma is the key for the proof of the main Theorem 2.3. Some notation. For a set $Z \subset [n]$, let $\mathcal{R}(Z) = \{(A, B) \in \mathcal{R}\colon\ A \subset Z, B^{\mathrm{C}} \subset Z\}$. For an element $x \in [n]$, let $\mathcal{R}(x)$ denote all pairs $(A, B) \in \binom{[n]}{a} \times \binom{[n]}{n-b}$ with $x \in A \cap B^{\mathrm{C}}$, and let $\mathcal{R}_0(x)$ denote those pairs where $x = A \cap B^{\mathrm{C}}$. Moreover, $\mathcal{R}(\pi, Z)$ denotes

$\mathcal{R}(\pi) \cap \mathcal{R}(Z)$, and $\mathcal{R}_0(\pi, x, Z)$ stands for $\mathcal{R}(\pi) \cap \mathcal{R}_0(x) \cap \mathcal{R}(Z)$, etc. Beware, the maximality of π does not imply in general that the restriction of π to the members of $\mathcal{R}(Z)$ is maximal (as a $(z; a, z-b)$ linear extension).

LEMMA 6.3. *Suppose that π is a maximal linear extension of $\mathcal{B}_n(a, n-b)$, $Z \subset [n]$, $|Z| > (a+b)ab$, and $c(\pi) \notin Z$. Then*

$$\left|\mathcal{R}(\pi, Z)\right| < \frac{a^2+8ab+b^2}{z}\binom{z-1}{a-1}\binom{z-1}{b-1}. \tag{6}$$

Proof. If π is of type I, then Lemma 6.2 says that for all $(A, B) \in \mathcal{R}(\pi)$ one has $c \in A \cup B^{\mathrm{C}}$. So $c(\pi) \notin Z$ implies that $\mathcal{R}(\pi, Z) = \varnothing$. From now on we suppose that π is of type II.

Suppose, on the contrary, that (6) does not hold. Consider the subsequence defined by π and Z, i.e., the sequence consisting of the sets $B \in \binom{[n]}{n-b}$ with $B^{\mathrm{C}} \subset Z$, $A \subset Z$. This can be viewed, after removing $[n] \setminus Z$ from all affected B's, as a linear extension $\pi|Z$ of $\mathcal{B}_z(a, z-b)$ with underlying set Z. Then Lemma 6.1 and the negation of (6) imply that there exists a set $A' \in \binom{Z}{a}$ with type I in the permutation $\pi|Z$. Apply Lemma 4, as we did in the proof of Lemma 6.2, for the families $\mathcal{B}^{\mathrm{C}}(\pi|Z, < A')$ and $\mathcal{A}(\pi|Z, A' \leqslant)$. We obtain that there exists an element $e \in Z$ which is contained in every member of these two families, i.e.,

$$\mathcal{B}' := \left\{B^{\mathrm{C}}\colon\ \pi(B) < \pi(A') \text{ and } ([n] \setminus Z) \subset B\right\} \subset \left\{X \in \binom{Z}{b}\colon\ e \in X\right\},$$

$$\mathcal{A}' := \left\{A\colon\ \pi(A') \leqslant \pi(A) \text{ and } A \subset Z\right\} \subset \left\{A \in \binom{Z}{a}\colon\ e \in A\right\}.$$

We claim that the maximality of π implies c is a center of π. Indeed, $e \in A$ for all $A \in \mathcal{A}(\pi, A' \leqslant)$. If not, then for some $A_2 \in \binom{[n]}{a}$ we have that $e \notin A_2$ and $\pi(A') < \pi(A_2)$. Then A_2 intersects all members of $\mathcal{B}'$, which implies that

$$|\mathcal{B}'| \leqslant \binom{z-1}{b-1} - \binom{z-a-1}{b-1} < 2a\binom{z-2}{b-2}.$$

This contradicts the fact that A' is of type I (in $\pi|Z$). Then maximality implies that all $B \in \binom{[n]}{n-b}$ with $e \notin B$ precede A'. On the other hand, if $\pi(B) < \pi(A')$, then B^{C} intersects all members of $\mathcal{A}'$, this is only possible whenever $e \in B^{\mathrm{C}}$. We obtained that

$$\mathcal{B}(\pi, < A') = \binom{[n] \setminus e}{n-b}.$$

This (and the maximality) gives that

$$\mathcal{A}(\pi, A' \leqslant) = \left\{A \in \binom{[n]}{a}\colon\ e \in A\right\},$$

e is indeed a center with $e \in Z$. This contradicts $c(\pi) \notin Z$, completing the proof. □

REMARK. Call two a-sets A_1, A_2 *equivalent* in π if $\mathcal{B}(\pi, < A_1) = \mathcal{B}(\pi, < A_2)$. In other words, they belong to the same segment in π. Let $A_1, A_2, \ldots, A_m$ be a representative from each interval, $\pi(A_1) < \pi(A_2) < \cdots < \pi(A_m)$. The maximality implies that for each $1 \leqslant i \leqslant m$ there exists a B_i with $\pi(A_i) < \pi(B_i) < \pi(A_{i+1})$, such that $A_i \subset B_i$.

Frankl [7] and Kalai [15] proved the following ordered version of a theorem of Bollobás [2]. If $X_1, \ldots, X_m$ are a-sets and $Y_1, \ldots, Y_m$ are b-sets with the properties $X_i \cap Y_i = \varnothing$ for all i and $X_i \cap Y_j \neq \varnothing$ for all $1 \leqslant i < j \leqslant m$, then $m \leqslant \binom{a+b}{a}$ holds. Applying this theorem for the sequences $A_1, A_2, \ldots, A_m$ and $B_1^C, B_2^C, \ldots, B_m^C$, we obtain that the number of segments in a maximal π is at most $\binom{a+b}{a}$.

7. Proof of the Main Theorem

Here we prove Theorem 2.3. The upper bound $\dim(n; k, n-k) \leqslant \dim(n; 2, n-3) = n-2$ follows from the results of Hurlbert, Kostochka, and Talysheva in [14] mentioned at the end of Section 1. Let $\pi_1, \ldots, \pi_d$ be a system of linear extensions generating $\mathcal{B}_n(k, n-k)$, $n > 250k^3$, $k \geqslant 3$. We may suppose that each of them is maximal. Suppose on the contrary, that $d \leqslant n-3$. Classify the linear extensions as follows, $\pi_1, \ldots, \pi_\ell$ are of type II, while each π_i for $\ell < i \leqslant d$ has a center $c(\pi_i)$. Let $H = \{x \in [n]\colon x \text{ is not a center}\}$, $|H| = h$. Our first aim is to prove that

$$3 + \ell \leqslant |H| \leqslant 2k. \tag{7}$$

The lower bound follows from $h \geqslant n - (d - \ell)$.

To prove the upper bound in (7) let

$$\mathcal{R}' = \bigcup_{x \in H} \mathcal{R}_0(x).$$

We have

$$|\mathcal{R}'| = h\binom{n-1}{k-1}\binom{n-k}{k-1}.$$

Consider $\mathcal{R}' \cap \mathcal{R}(\pi_i)$. For $i \leqslant \ell$, π_i is of type II, so Lemma 6.1 gives that

$$|\mathcal{R}' \cap \mathcal{R}(\pi_i)| \leqslant |\mathcal{R}(\pi_i)| < (10k^2/n)\binom{n-1}{k-1}^2.$$

If $i > \ell$, and $(A, B) \in \mathcal{R}(\pi_i) \cap \mathcal{R}'$, then $c(\pi_i) \notin A \cap B^C$, which implies that (A, B) is reversed by one of the lower or upper parts of π_i (the definitions can be found after the proof of Lemma 6.2). Each half-permutation reverses at most $\binom{n-1}{k-1}\binom{n-1}{k-2}$ pairs, by Lemma 5. Summarizing, we get

$$h\binom{n-1}{k-1}\binom{n-k}{k-1} = |\mathcal{R}'| \leqslant \sum_{1\leqslant i\leqslant d} |\mathcal{R}'\cap\mathcal{R}(\pi_i)|$$
$$\leqslant \ell(10k^2/n)\binom{n-1}{k-1}^2 + 2(d-\ell)\binom{n-1}{k-1}\binom{n-1}{k-2}.$$

Using the facts $\ell < h$, $d < n$, and rearranging, a short calculation gives (7) for $n \geqslant 50k^3$.

As we have seen, each half-permutation obtained from a π_i of type I can be viewed as a maximal $(n-1;k,(n-1)-(k-1))$ or $(n-1;k-1,(n-1)-k)$ linear extension. If a half-permutation is of type I (of its kind) its center is called a *secondary center* of π_i. Altogether, the $(d-\ell)$ permutations of type I have at most $2(d-\ell)$ secondary centers. For an $x \in H$ let $C_x \subset [n]\setminus H$ be the set of those elements y for which one can find a permutation of type I with center y and semicenter x. We obtain, that

$$\sum |C_x| \leqslant 2(d-\ell).$$

This implies that for some $x \in H$ we have $|C_x| \leqslant 2(d-\ell)/h$. Define

$$Z = \{x\} \cup ([n]\setminus C_x \setminus H), \qquad |Z| = z.$$

We obtain that $z \geqslant 1 + (n-h) - (d-\ell)(2/h)$. Using (7) and the fact that n is large we get $z \geqslant (n/3)$.

Consider $\mathcal{R}_0(x,Z)$, it consists of $\binom{z-1}{k-1}\binom{z-k}{k-1}$ pairs. Next we give an upper bound for $|\mathcal{R}_0(x,Z)|$. If π_i is of type II $(1 \leqslant i \leqslant \ell)$, then Lemma 6.3 gives

$$|\mathcal{R}_0(\pi_i,x,Z)| \leqslant |\mathcal{R}(\pi_i,Z)| < (10k^2/z)\binom{z-1}{k-1}^2.$$

If π_i is of type I, and $c(\pi_i) \notin Z$, then $\mathcal{R}_0(x,Z)\cap\mathcal{R}(\pi_i) = \varnothing$. Finally, if π_i is of type I, and $c(\pi_i) \in Z$, then all the reversed pairs from $\mathcal{R}_0(\pi_i,x,Z)$ are reversed by one of the half-permutations. These half-permutations can be viewed as maximal linear extensions of type $(n-1;k,(n-1)-(k-1))$ or $(n-1;k-1,(n-1)-k)$. If the half-permutation has no (secondary) center, or its center does not belong to Z, then Lemma 6.3 implies that the number of its reversed pairs from $\mathcal{R}_0(x,Z)$ is at most $(10k^2/z)\binom{z-1}{k-1}\binom{z-1}{k-2}$. If the half-permutation has a center it can not be x, so all of its reversed pairs are reversed by one of the halves of the half-permutations. Such a quarter of permutation reverses at most $\binom{z-1}{k-2}^2$ pairs from $\mathcal{R}_0(x,Z)$. So altogether, using generous upper bounds on the number of cases, we get

$$\binom{z-1}{k-1}\binom{z-k}{k-1} = |\mathcal{R}_0(x, Z)|$$

$$\leqslant 2k(10k^2/z)\binom{z-1}{k-1}^2 +$$

$$+ 2n(10k^2/z)\binom{z-1}{k-1}\binom{z-1}{k-2} + 4n\binom{z-1}{k-2}^2.$$

A short calculation implies that this could not hold for $3z \geqslant n > 250k^3$, finishing the proof of the theorem. □

Finally, we remark that some of our results (Proposition 2.1 and (5.3)) have been independently discovered by Hurlbert and will appear in [13].

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