Intersection Representations of the Complete Bipartite Graph

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Summary. A p-representation of the complete graph $\mathcal{K}_{n,n}$ is a collection of sets $\{S_1, S_2, \ldots, S_{2n}\}$ such that $|S_i \cap S_j| \geq p$ if and only if $i \leq n < j$. Let $\vartheta_p(\mathcal{K}_{n,n})$ be the smallest cardinality of $\cup S_i$. Using the Frankl-Rödl theorem about almost perfect matchings in uncrowded hypergraphs we prove the following conjecture of Chung and West. For fixed p while $n \to \infty$ we have $\vartheta_p(\mathcal{K}_{n,n}) = (1 + o(1))n^2/p$. Several problems remain open.

1. The p-intersection number of $\mathcal{K}(n, n)$

One of the important topics of graph theory is to represent graphs, or an interesting class of graphs, using other simple structures. One approach is to represent the vertices by sets so that vertices are adjacent if and only if the corresponding sets intersect (line graphs). More generally, the p-intersection number of a graph is the minimum t such that each vertex can be assigned a subset of $\{1, \ldots, t\}$ in such a way that vertices are adjacent if and only if the corresponding sets have at least p common elements. Such a system is called a *p-representation* (or p-intersection representation) of the graph $\mathcal{G}$, and the minimum t is denoted by $\vartheta_p(\mathcal{G})$.

For any graph of v vertices Erdős, Goodman, and Pósa [8] showed that $\vartheta_1 \leq \lfloor v^2/4 \rfloor$, and here equality holds for $\mathcal{K}_{\lfloor v/2 \rfloor, \lceil v/2 \rceil}$. Myung S. Chung and D. B. West [5] conjectured that the complete bipartite graph also maximizes ϑ_p. Their lower bound for $p > 1$ is

$$\vartheta_p(\mathcal{K}_{n,n}) \geq (n^2 - n)/p + 2n. \tag{1.1}$$

In this note we determine ϑ_p for these and a few more graphs. The *complete k-partite graph*, $\mathcal{K}^{(k)}_{n,\ldots,n}$, has kn vertices, k disjoint independent sets of sizes n and all the $\binom{k}{2}n^2$ edges between different classes. $\mathcal{K}(n \times k)$ denotes a graph with vertex set $V^1 \cup \ldots \cup V^k$, $V^\ell = \{v_1^\ell, \ldots, v_n^\ell\}$ and v_i^ℓ is joined to v_j^m if and only if $i \neq j$ and $\ell \neq m$. So $\mathcal{K}(n \times 2)$ is obtained from $\mathcal{K}_{n,n}$ by deleting a one factor.

Theorem 1.1. *For fixed p and k, the p-intersection number of the complete k-partite graph* $\mathcal{K}^{(k)}_{n,\ldots,n}$ *is* $(1 + o(1))\, n^2/p$.

Note that the asymptotic is independent from the fixed value of k. In Section 4 we will give a partial proof for Theorem 1.1 using classical design theory and obtain a better error term. An *Hadamard matrix* of order n is a square matrix M with ± 1 entries such that $MM^t = nI_n$. It is conjectured that it exists for all $n \equiv 0 (\mathrm{mod}\ 4)$. The smallest undecided case is larger than 184. An $S_\lambda(v, l, t)$ *block design* is a l-uniform (multi)hypergraph on v vertices such that each t-subset is contained in exactly λ hyperedges (blocks). Block designs with $\lambda = 1$ are called *Steiner systems.* All notions we use about designs can be found, e.g., in Hall's book [12]. Wilson [19] proved that for any l there exists a bound $v_0(l)$ such that for all $v \geq v_0$ there exists a Steiner system $S(v, l, 2)$ if $\binom{v}{2}/\binom{l}{2}$ and $(v-1)/(l-1)$ are integers.

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Theorem 1.2. (a) *If there exists an Hadamard matrix of size* $4p$*, and a Steiner system* $S(n, 2p, 2)$*, then* $\vartheta_p(\mathcal{K}(n \times 2)) = (n^2 - n)/p$.

(b) *If* $p = q^d$ *where* q *is a prime power,* d *a positive integer,* $k \leq q$*, and there exists a Steiner system* $S(n, q^{d+1}, 2)$*, then* $\vartheta_p(\mathcal{K}(n \times k)) = (n^2 - n)/p$.

Corollary 1.1. *For all* p *in Theorem 1.2, and* $n > n_0(p)$

$$n^2/p < \vartheta_p(\mathcal{K}_{n,n}) \leq \vartheta_p(\mathcal{K}(n \times 2)) + pn \leq (n^2/p) + 4pn$$

This covers all cases $p \leq 46$. A construction from the finite projective space is given in Section 3. In Section 5 we list a few open problems.

2. A random construction

A hypergraph $\mathcal{H}$ with edge set $\mathcal{E}(\mathcal{H})$ and vertex set $V(\mathcal{H})$ is called r*-uniform* (or an r-graph) if $|E| = r$ holds for every edge $E \in \mathcal{E}(\mathcal{H})$. The *degree*, $\deg_{\mathcal{H}}(x)$, of the vertex $x \in V$ is the number of edges containing it. The degree of a pair, $\deg_{\mathcal{H}}(x, y)$, is the number of edges containing both vertices x and y. The *dual*, $\mathcal{H}^*$, of $\mathcal{H}$ is the hypergraph obtained by reversing the roles of vertices and edges keeping the incidencies, i.e., $V(\mathcal{H}^*) = \mathcal{E}(\mathcal{H})$. A *matching* $\mathcal{M} \subset \mathcal{E}(\mathcal{H})$ is a set of mutually disjoint edges, $\nu(\mathcal{H})$ denotes the largest cardinality of a matching in $\mathcal{H}$.

We are going to use a theorem of Frankl and Rödl [9]. The following slightly stronger form is due to Pippenger and Spencer [17]: For all integer $r \geq 2$ and real $\varepsilon > 0$ there exists a $\delta > 0$ so that: If the r-uniform hypergraph $\mathcal{H}$ on z vertices has the following two properties (i) $(1-\delta)d < \deg_{\mathcal{H}}(x) < (1+\delta)d$ holds for all vertices, (ii) $\deg_{\mathcal{H}}(x, y) < \delta d$ for all distinct x and y, then there is a matching in $\mathcal{H}$ almost as large as possible, more precisely

$$\nu(\mathcal{H}) \geq (1 - \varepsilon)(z/r). \tag{2.1}$$

Far reaching generalizations of (2.1) has been recently proved by Kahn [15].

Suppose $\mathcal{G}$ is a graph and $\mathcal{F} = \{F_1, ...F_t\}$ is a family of subsets of the vertex set $V(\mathcal{G})$, repetition allowed. Such a system $\mathcal{F}$ is called a *p-edge clique cover* if every edge of $\mathcal{G}$ is contained in at least p members of $\mathcal{F}$ and the non-edge pairs are covered by at most $p-1$ F_i's. A p-edge clique cover is the dual of a p-representation (and vice versa), so the smallest t for which there is a p-edge clique cover is $\vartheta_p(\mathcal{G})$. This was the way Kim, McKee, McMorris and Roberts [16] first defined and investigated $\vartheta_p(\mathcal{G})$.

Proof (of Theorem 1.1). To construct a p-edge clique cover of the complete k-partite graph with n-element classes $V^1, \ldots, V^k$ consider the following multigraph $\mathcal{M}$. Every edge contained in a class V^i has multiplicity $p-1$, and all edges joining distinct classes (crossing edges) have multiplicity p. The total number of edges is

$$|\mathcal{E}(\mathcal{M})| = (p-1)k\binom{n}{2} + p\binom{k}{2}n^2 = (1+o(1))\frac{n^2}{p}\left(k\binom{p}{2} + \binom{k}{2}p^2\right) \tag{2.2}$$

Let $r = |\mathcal{E}(\mathcal{K}^{(k)}_{p,\ldots,p})| = k\binom{p}{2} + \binom{k}{2}p^2$. Define the r-uniform hypergraph $\mathcal{H}$ with vertex set $\mathcal{E}(\mathcal{M})$ as follows. The hyperedges of $\mathcal{H}$ are those r-subsets of $\mathcal{E}(\mathcal{M})$ which form a complete k-partite subgraph with p vertices in each V^i. The number of such subgraphs is

$$|\mathcal{E}(\mathcal{H})| = \binom{n}{p}^k (p-1)^{k\binom{p}{2}} p^{\binom{k}{2}p^2}.$$

Let $e \in \mathcal{E}(\mathcal{M})$ be an edge contained in a class V^i. The number of $\mathcal{K}^{(k)}_{p,\dots,p}$'s, i.e., the number of hyperedges of $\mathcal{H}$ containing e is exactly

$$\deg_{\mathcal{H}}(e) = \binom{n-2}{p-2}\binom{n}{p}^{k-1} (p-1)^{k\binom{p}{2}-1} p^{\binom{k}{2}p^2}. \tag{2.3}$$

For any crossing edge $f \in \mathcal{E}(\mathcal{M})$ connecting two distinct classes we have

$$\deg_{\mathcal{H}}(f) = \binom{n-1}{p-1}^2\binom{n}{p}^{k-2} (p-1)^{k\binom{p}{2}} p^{\binom{k}{2}p^2-1}. \tag{2.4}$$

The ratio of the right hand sides of (2.3) and (2.4) is $n/(n-1)$, so the hypergraph $\mathcal{H}$ is nearly regular, it satisfies the first condition in the Frankl-Rödl theorem for any $\delta > 0$ if n is sufficiently large. For two distinct edges, $e_1, e_2 \in \mathcal{E}(\mathcal{M})$, obviously $\deg_{\mathcal{H}}(e_1, e_2) = O(n^{kp-3})$, so condition (ii) is fulfilled, too. Apply (2.1) to $\mathcal{H}$. We get a system $\mathcal{F} = \{F_1, \dots, F_\nu\}$ of kp-element subsets of $\cup V^i$ such that every pair e contained in a class V^i is covered at most $p-1$ times, every pair f joining two distinct classes is covered at most p times. Moreover, $\nu = (1-o(1))n^2/p$, by (2.2). It follows that almost all edges of $\mathcal{K}^{(k)}_{n,\dots,n}$ are covered exactly p times, so the system $\mathcal{F}$ can be extended to a p-edge clique cover by adding sufficiently many (but only $o(n^2)$) edges.

3. Exact results from the finite projective space

Proposition 3.1. *For all $p \geq 1$, $\vartheta_p(\mathcal{K}(n \times 2)) \geq (n^2 - n)/p$.*

Proof. Let V^1 and V^2 be the two parts of the vertex set of the graph, $|V^1| = |V^2| = n$, let $\{A_i \cup B_i : 1 \leq i \leq t\}$ be a p-edge clique cover, their average size on one side is $\ell := \sum_i (|A_i| + |B_i|)/(2t)$. Using the inequalities (3.1a) $\sum_i \binom{|A_i|}{2} \leq (p-1)\binom{n}{2}$, and (3.1b) $\sum_i \binom{|B_i|}{2} \leq (p-1)\binom{n}{2}$, and the fact that (3.1c) all the $n^2 - n$ crossing edges are covered at least p times we have

$$p(n^2-n) \leq \sum_i |A_i||B_i| \leq \sum_i \left(\binom{|A_i|}{2} + \binom{|B_i|}{2}\right) + \ell t \leq (p-1)(n^2-n) + \ell t.$$

This gives $\ell t \geq (n^2 - n)$. On the other hand, (3.1a-b) give $2t\binom{\ell}{2} \leq 2(p-1)\binom{n}{2}$, hence $\ell \leq p$ and $t \geq (n^2-n)/p$ follows.

Replacing (3.1c) by $pn^2 \leq \sum_i |A_i||B_i|$ the above proof gives

$$\vartheta_p(\mathcal{K}_{n,n}) \geq (n+p-1)^2/p, \tag{3.1}$$

which is better than (1.1) for $n < (p-1)^2$.

Consider a $\mathcal{K}(n \times k)$ with classes $V^1, \dots, V^k$, $V^\ell = \{v^\ell_1, \dots, v^\ell_n\}$. We call the p-edge clique cover $\mathcal{F} = \{F_1, \dots, F_t\}$ *perfect* if the sets $\{F_i \cap V^\ell : 1 \leq i \leq n\}$ form an $S_{p-1}(n, p, 2)$ design for all ℓ. It follows, that $|F_i| = kp$ for all F_i, every edge of $\mathcal{K}(n \times k)$ is contained in exactly p sets, every pair from V^ℓ is covered $p-1$ times, every pair of the form $\{v^\ell_\alpha, v^m_\alpha\}$ is uncovered, and $t = (n^2-n)/p$.

Proposition 3.2. *If $p = q^d$, where q is a prime power, d a positive integer and $k \leq q$, then there exists a perfect p-edge clique cover of $\mathcal{K}(q^{d+1} \times k)$. Hence, in this case, $\vartheta_p(\mathcal{K}(q^{d+1} \times k)) = q^{d+2} - q$.*

Proposition 3.3. *If $p = q^d + q^{d-1} + \ldots + 1$, where q is a prime power, d a positive integer and $k \leq q + 1$, then $\vartheta_p(\mathcal{K}^{(k)}_{q^{d+1},\ldots,q^{d+1}}) = q^2(q^d + q^{d-1} + \ldots + 1)$.*

Proof. The lower bounds for ϑ_p are implied by Proposition 3.1 and (3.2), respectively. The upper bounds are given by the following construction. Let X be the point set of a $(d+2)$-dimensional projective space of order q, $PG(d+2, q)$, let $Z \subset X$ be a subspace of dimension d, and let $Y^1, \ldots, Y^{q+1}$ be the hyperplanes containing Z, $V^\ell = Y^\ell \setminus Z$. The sets V^ℓ partition $X \setminus Z$ into q^{d+1}-element classes. Choose a point $c \in V^{q+1}$ and label the vertices of $V^\ell = \{v_i^\ell : 1 \leq i \leq q^{d+1}\}$ in such a way that $\{v_i^\ell : 1 \leq \ell \leq q\} \cup \{c\}$ form a line for all i.

The hyperplanes not containing Z and avoiding c induce a perfect p-edge clique cover of $\mathcal{K}(p^{d+1} \times k)$ with classes $V^1, \ldots, V^k$ $(k \leq q)$. Indeed, $PG(d+2, q)$ contains $q^{d+2} + q^{d+1} + \ldots + q + 1$ hyperplanes and they cover each pair of points exactly p times. The point c is contained by exactly $q^{d+1} + \ldots + q + 1$ of the hyperplanes, Z is contained in $q + 1$ of them, and $Z \cup \{c\}$ is contained in a unique one. So the above defined cover consists of $q^{d+2} - q$ sets. These sets still cover each pair of the form $\{v_i^\ell, v_j^m\}$, $i \neq j$ exactly p times. However, the pairs of the form $\{v_\alpha^\ell, v_\alpha^m\}$ are uncovered, because any subspace containing these two points must contain the line through them, so it must contain the element c.

Similarly, considering all the hyperplanes not containing Z, we get a p-edge clique cover of $\mathcal{K}^{(k)}_{n,\ldots,n}$ with classes $V^1, \ldots, V^k$ where $n = q^{d+1}$ and $k \leq q + 1$.

The dual (perfect) p-representation of $\mathcal{K}(q^{d+1} \times k)$ can be obtained by considering a line, L, in the affine space of dimension $d+2$, and assigning all sets of the form $Y \setminus \{v^\ell\}$ to the vertices of the ℓ'th color class, where $v^\ell \in L$, and Y is a hyperplane with $Y \cap L = \{v^\ell\}$. Similarly, the dual p-representation of $\mathcal{K}^{(k)}_{n,\ldots,n}$ on the underlying set $X \setminus L$ can be obtained by assigning the sets $Y \setminus \{v^\ell\}$ to the ℓ'th color class. There might be more optimal constructions using higher dimensional spaces.

4. Constructions from Steiner systems

Proposition 4.1. *If there exists an Hadamard matrix of size $4p$, then there exists a perfect p-edge clique cover of $\mathcal{K}(2p \times 2)$, so its $\vartheta_p = 4p - 2$.*

Proof. We are going to give a perfect p-intersection representation with underlying set $\{1, \ldots, 4p-2\}$. Its dual is a perfect p-edge clique cover. Let M be an Hadamard matrix of order $4p$. We may suppose that the last row contains only $+1$'s. The ± 1's in any other row define a partition of $\{1, \ldots, 4p\}$ into two $2p$-element sets $P_i^+ \cup P_i^-$. We may also suppose that the last two entries are $M_{i,4p-1} = 1$, $M_{i,4p} = -1$ for $1 \leq i \leq 2p$. Finally, assign the set $P_i^+ \setminus \{4p-1\}$ to the vertex v_i^1, $P_i^- \setminus \{4p\}$ to v_i^2.

Note that both in Proposition 3.2 and here we have got the perfect p-edge clique cover from a resolvable $S_\lambda(sp, p, 2)$ design, where s is an integer, $\lambda = (p-1)/(s-1)$.

Proof (of Theorem 1.2(b)). First, we consider a perfect p-edge clique cover, $\mathcal{F}$, in the case $n = q^{d+1}$ given by Proposition 3.2. Consider k identical copies of a Steiner system $S(n, q^{d+1}, 2)$ over the n-element sets V^ℓ. Replace each block and its corresponding pairs by a copy of $\mathcal{F}$. Then we obtain a system is a perfect p-edge clique cover.

The proof of the case (a) is similar, we put together a perfect p-edge clique cover using a building block of size $2p$ supplied Proposition 4.1 and a Steiner system $S(n, 2p, 2)$. Taking the sets $\{v_i^1, v_i^2, \dots, v_i^k\}$ p times, we get that $\vartheta_p(K^{(k)}_{n,\dots,n}) \leq \vartheta_p(\mathcal{K}(n \times k)) + pn$. As $\vartheta_p(K^{(k)}_{n,\dots,n})$ is a monotone function of n we got Corollary 1.3.

Conjecture 4.1. If $(n^2 - n)/p$ is an integer and $n > n_0(p,k)$, then there exists a perfect p-edge clique cover of $\mathcal{K}(n \times k)$, hence its p-intersection number $\vartheta_p = (n^2 - n)/p$.

The case $p = 1$ corresponds to the fact that there are transversal designs $T(n,k)$ (i.e., mutually orthogonal Latin squares of sizes n) for $n > n_0(k)$ (Chowla, Erdős, Straus [4], also see Wilson [20]).

Chung and West [5] proved the case $k = p = 2$. They showed $\vartheta_2(\mathcal{K}(n \times 2)) = (n^2 - n)/2$ by constructing a perfect 2-edge cover (they call it a perfect 2-generator) for the cases $n \equiv 1, 2, 5, 7, 10$, or $11 \pmod{12}$. This and (1.1) imply that

$$\vartheta_2(\mathcal{K}_{n,n}) = (n^2 + 3n)/2 \tag{4.1}$$

holds for these cases. Their conjecture about the so-called orthogonal double covers (a conjecture equivalent to the existence of a perfect 2-edge cover of $\mathcal{K}(n \times 2)$) which conjecture had appeared in [6], too, is true for all $n > 8$. This was proved by Ganter and Gronau [11] and independently by Bennett and Wu [1]. So $n_0(2,2) = 8$ and (4.1) holds for all $n > 8$.

There are two more values proved in [5], namely the special cases $q = 2$ and $q = 3$ of the following conjecture. $\mathcal{K}(q^2 + q + 1 \times 2)$ has a perfect q-edge cover whenever a projective plane of order q exists. This would imply that equality holds in (1.1) for $(p, n) = (q + 1, q^2 + q + 1)$.

5. Further problems, conjectures

The first nontrivial lower bound for $\vartheta_p(\mathcal{K}_{n,n})$ was proved by Jacobson [13]. He and Kézdy and West [14] also investigated $\vartheta_2(\mathcal{G})$ for other classes of graphs, like paths and trees.

How large $\vartheta_p(\mathcal{K}(n \times k))$ and $\vartheta_p(\mathcal{K}^{(k)}_{n,\dots,n})$ if n is fixed and $k \to \infty$?

Estimate ϑ_p for complete bipartite graph with parts of sizes a and b when $a \to \infty$, p is fixed and a/b goes to a finite limit.

Another interesting graph where one can expect exact results is a cartesian product, its vertex set is $I_1 \times \dots \times I_\ell$, and $(i_1, \dots, i_\ell)$ is joined to $(i'_1, \dots, i'_\ell)$ if and only if $i_\alpha \neq i'_\alpha$ for all $1 \leq \alpha \leq \ell$.

For a matching, $\mathcal{M}$, of size n it easily follows that $\vartheta_p(\mathcal{M}) = \min\{t : \binom{t}{p} \geq n\}$.

One can ask the typical value of $\vartheta_p(\mathcal{G})$, i.e., the expected value of ϑ_p for the random graph of n vertices. The case of $p = 1$ was proposed in [10], and the best bounds are due to Bollobás, Erdős, Spencer, and West [3]: For almost all graphs its edge set can be covered by $O(n^2 \log\log n / \log n)$ cliques. The conjecture is, that here the term $\log\log n$ can be deleted. A counting argument gives the lower bound $E(\vartheta_1(\mathcal{G})) \geq (2 - o(1))n^2/(2\log_2 n)^2$. Obviously,

$$\vartheta_p(\mathcal{G}) \leq \vartheta_{p-1}(\mathcal{G}) + 1 \leq \vartheta_1(\mathcal{G}) + p - 1,$$

so the order of magnitude of $E(\vartheta_p)$ is at most that of $E(\vartheta_1)$. Until there is such a large gap between the lower and upper bounds of $E(\vartheta_1)$, one cannot expect better bounds for $E(\vartheta_p)$.

The notion of ϑ_p was generalized from the study of the *p-competition* graphs. Another generalization, also having several unsolved questions, is the clique coverings by *p rounds.* Let $\mathcal{G}$ be a simple graph and let $\varphi_p(\mathcal{G})$ be the minimum of $\sum_{1\leq i\leq p} n_i$ such that there are families $\mathcal{A}_1, \dots, \mathcal{A}_p$, $|\mathcal{A}_i| = n_i$, such that each edge $e \in \mathcal{E}(\mathcal{G})$ is covered by each family (i.e., there exists an $A \in \mathcal{A}_i$ with $e \subset A$), but this does not hold for the non-edges. It is known [10], that for all graphs on n vertices $\varphi_2 \leq 3n^{5/3}$ and for almost all graphs $\varphi_2 > 0.1n^{4/3}/(\log n)^{4/3}$. For further problems and questions, see [10].

Bollobás [2] generalized the Erdős-Goodman-Pósa result as follows. The edge set of every graph on n vertices can be *decomposed* into $t(k-1, n)$ parts using only $\mathcal{K}_k$'s and edges, where $t(k-1, n)$ is the maximum number of edges in a $(k-1)$-colored graph on n vertices, e.g., $t(2, n) = \lfloor n^2/4 \rfloor$. There are many beautiful results and problems of this type, the interested reader can see the excellent survey by Pyber [18]. Most of the problems can be posed to multigraphs, obtaining new, interesting, non-trivial problems.

Let $\vartheta_p^*(\mathcal{G})$ the minimum t such that each vertex can be assigned a subset of $\{1, \dots, t\}$ in such a way that the intersection of any two of these sets is at most p, and vertices of $\mathcal{G}$ are adjacent if and only if the corresponding sets have *exactly* p common elements. Note that in all of the results in this paper were proved an upper bound for ϑ_p^*. If $\mathcal{G}^n$ is a graph on n vertices with with $2n-3$ edges such that two vertices are connected to all others, then one can show that $\lim_{n\to\infty} \vartheta_p^*(\mathcal{G}^n) - \vartheta_p(\mathcal{G}^n) = \infty$ for any fixed p. What is $\max(\vartheta_p^*(\mathcal{G}) - \vartheta_p(\mathcal{G}))$ and $\max(\vartheta_p^*/\vartheta_p)$ for different classes of graphs? Is it true that $\vartheta_p^*(\mathcal{G}) \leq (1 + o(1))n^2/(4p)$ for every n-vertex graph?

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One of our main constructions (Theorem 1.1) along with the lower bound (3.2) was independently discovered by Eaton, Gould and Rödl [6]. They also considered 2-representations of bounded degree trees.

References

1. F. E. Bennett and Lisheng Wu, On the minimum matrix representation of closure operations, *Discrete Appl. Math.* **26** (1990), 25–40.
2. B. Bollobás, On complete subgraphs of different orders, *Math. Proc. Cambridge Philos. Soc.* **79** (1976), 19–24.

3. B. Bollobás, P. Erdős, J. Spencer, and D. B. West, Clique coverings of the edges of a random graph, *Combinatorica* **13** (1993), 1–5.
4. S. Chowla, P. Erdős, E. Straus, On the maximal number of pairwise orthogonal Latin squares of a given order, *Canad. J. Math.* **12** (1960), 204–208.
5. M. S. Chung and D. B. West, The p-intersection number of a complete bipartite graph and orthogonal double coverings of a clique, *Combinatorica* **14** (1994), 453–461.
6. J. Demetrovics, Z. Füredi, and G. O. H. Katona, Minimum matrix representation of closure operations, *Discrete Appl. Math.* **11** (1985), 115–128.
7. N. Eaton, R. J. Gould, and V. Rödl, On p-intersection representations, manuscript, 1993.
8. P. Erdős, A. Goodman, and L. Pósa, The representation of a graph by set intersections, *Canad. J. Math.* **18** (1966), 106–112.
9. P. Frankl and V. Rödl, Near-perfect coverings in graphs and hypergraphs, *European J. Combin.* **6** (1985), 317–326.
10. Z. Füredi, Competition graphs and clique dimensions, *Random Structures and Algorithms* **1** (1990), 183–189.
11. B. Ganter and H.-D. O. F. Gronau, Two conjectures of Demetrovics, Füredi, and Katona concerning partitions, *Discrete Mathematics* **88** (1991), 149–155.
12. M. Hall, Jr., *Combinatorial Theory*, Wiley–Interscience, New York, 1986.
13. M. S. Jacobson, On the p-edge clique cover number of complete bipartite graphs, *SIAM J. Disc. Math.* **5** (1992), 539–544.
14. M. S. Jacobson, A. E. Kézdy, and D. B. West, The 2-intersection number of paths and bounded-degree trees, manuscript
15. J. Kahn, On a theorem of Frankl and Rödl, in preparation.
16. Suh-ryung Kim, T. A. McKee, F. R. McMorris, and F. S. Roberts, p-competition graphs, *DIMACS Technical Report* 89-19, to appear in *SIAM J. Discrete Math.*
17. N. Pippenger and J. Spencer, Asymptotic behavior of the chromatic index for hypergraphs, *J. Combin. Th., Ser. A* **51** (1989), 24–42.
18. L. Pyber, Covering the edges of a graph by, *Colloq. Math. Soc. J. Bolyai* **60** pp. 583–610. *Graphs, Sets and Numbers*, G. Halász et al. Eds., Budapest, Hungary, January 1991.
19. R. M. Wilson, An existence theory for pairwise balanced designs I–III, *J. Combinatorial Th., Ser. A* **13** (1972), 220–273 and **18** (1975), 71–79.
20. R. M. Wilson, Concerning the number of mutually orthogonal Latin squares, *Discrete Math.* **9** (1974), 181–198.