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Research Problems

In this column *Graphs and Combinatorics* publishes current research problems whose proposer believes them to be within reach of existing methods.

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Sets of Vectors with Many Orthogonal Pairs:

Abstract. What is the most number of vectors in $\mathbf{R}^d$ such that any $k+1$ contain an orthogonal pair? The 24 positive roots of the root system $\mathbf{F}_4$ in $\mathbf{R}^4$ show that this number could exceed dk.

1. Almost Orthogonal Systems

The purpose of this paper is to call attention to the following problem, whose solution might involve different branches of combinatorics. Let $\mathscr{P}$ be a collection of nonzero vectors of the d-dimensional euclidean space. Denote by $\alpha(\mathscr{P})$ the size of the largest subset $A \subset \mathscr{P}$ such that no two of the vectors $v_1, v_2 \in A$ are orthogonal. Finally, $\alpha^{(d)}(k)$ stands for $\max\{|\mathscr{P}| : \mathscr{P} \subset \mathbf{R}^d, \alpha(\mathscr{P}) \leq k\}$. That is, $\alpha^{(d)}(k)$ is the maximum number of vectors such that any $k+1$ of them contain an orthogonal pair. Considering k orthogonal basis we have

$$\alpha^{(d)}(k) \geq dk \tag{1}$$

On the other hand, $\alpha^{(d)}(k)$ is finite, $\alpha^{(d)}(k) \leq R(d+1, k+1) \leq \binom{d+k}{d}$, where R is the usual *Ramsey number* [3]. The problem of determining $\alpha^{(d)}(k)$ is due, of course, to Erdös [5]. He asked whether equality holds in Eq. 1.

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Obviously, $\alpha^{(d)}(1) = d$, and Rosenfeld [5] gave an ingenious algebraic proof for $\alpha^{(d)}(2) = 2d$. The proof of $\alpha^{(2)}(k) = 2k$ is easy (see Sect. 4).

2. The Root System F_4

The following example shows that

$$\alpha^4(5) \geq 24 > 20, \tag{2}$$

i.e., we do not always have equality in Eq. 1. Let e_1, e_2, e_3, e_4 be the four unit vectors, and let $\mathscr{P}$ consist of the following 24 vectors

$$\mathscr{P} := \{e_1, \ldots, e_4\} \cup \{e_i \pm e_j: i < j\} \cup \{e_1 \pm e_2 \pm e_3 \pm e_4\}.$$

We claim that any 6-element subset of $\mathscr{P}$ contains an orthogonal pair. Suppose, on the contrary, that $A \subset \mathscr{P}$ has no orthogonal pair and $|A| = 6$. Split $\mathscr{P}$ into 6 bases:

1	0	0	0
0	1	0	0
0	0	1	0
0	0	0	1

1	1	0	0
1	−1	0	0
0	0	1	1
0	0	1	−1

1	0	1	0
1	0	−1	0
0	1	0	1
0	1	0	−1

1	0	0	1
1	0	0	−1
0	1	1	0
0	1	−1	0

1	1	−1	−1
1	−1	1	−1
1	−1	−1	1
1	1	1	1

1	−1	−1	−1
1	1	1	−1
1	1	−1	1
1	−1	1	1

We obtain that A meets each basis in exactly one element. Observe, that the vector $\mathbf{v} \in \mathscr{P}$ has the same role as $-\mathbf{v}$ and, similarly, all i'th coordinates can be replaced by their opposite ones (i.e., x by $-x$) keeping all the orthogonalities. In this way, the first coordinate has no exceptional role, so we may suppose that $\mathbf{e}_1 \in A$. Simultaneously, we may suppose that $(1,1,1,1) \in A$, too. Then, from the second, third and fourth basis the (non-orthogonal) vector of A is $(1,1,0,0)$, $(1,0,1,0)$ and $(1,0,0,1)$, respectively. However, each vector of the sixth basis is orthogonal to one of $\mathbf{e}_1 + \mathbf{e}_i$, a contradiction. □

3. Asymptotic Results

The next inequalities imply that $\lim_{k\to\infty} \alpha^{(d)}(k)/k$ equals to its supremum, $\alpha^{(d)}$.

$$\alpha^{(d)}(k) + \alpha^{(d)}(l) \leq \alpha^{(d)}(k + l) \tag{3a}$$

$$\alpha^{(d)}(k) + \alpha^{(f)}(k) \leq \alpha^{(d+f)}(k) \tag{3b}$$

We get from Eq. 2

$$\alpha^{(4)}(k) \geq 24\lfloor k/5 \rfloor + \alpha^{(4)}(k - 5\lfloor k/5 \rfloor) > 4.8k - 4.$$

Let $\mathscr{S}$ be an $S(d, 4, 2)$ Steiner system ($d \equiv 1$ or $4 \pmod{12}$, $d \geq 13$), and define $\mathscr{P} \subset \mathbf{R}^d$ as follows. Consider the d unit vectors, the vectors of form $\mathbf{e}_i \pm \mathbf{e}_j$ ($1 \leq i < j \leq d$), and finally the vectors of the form $\mathbf{e}_i \pm \mathbf{e}_j \pm \mathbf{e}_u \pm \mathbf{e}_v$ ($1 \leq i < j < u < v \leq d, \{i, j, u, v\} \in \mathscr{S}$). We have that $|\mathscr{P}| = 1 + (5/3)d(d-1)$, and $\alpha(\mathscr{P}) = 1 + (4/3)(d-1)$. (In case of $|A| > 16$, if A is an orthogonal-free subset of $\mathscr{P}$, then the supports of the vectors from A have a common element, etc.) This implies

$$\alpha^{(d)} > \tfrac{4}{3}d - 3.$$

Even more, Eqs. 2 and 3 imply that there is no equality in Eq. 1 except, eventually, if either $d \leq 3$, or $k \leq 4$.

Blokhuis showed us that the following 8-dimensional example gives $\alpha^{(8)}(8) \geq 120$. Then, using the same procedure as above, for large k or d, one gets better ratios.

$$\mathscr{P} := \{\mathbf{e}_i \pm \mathbf{e}_j \colon 1 \leq i < j \leq 8\} \cup \{\mathbf{e}_1 \pm \mathbf{e}_2 \pm \cdots \pm \mathbf{e}_8\}.$$

4. A Geometric Upper Bound

Let S^{d-1} denote the unit sphere in $\mathbf{R}^d$, and let $C^{d-1} := \{\mathbf{x} \in S^{d-1} \colon |(\mathbf{x}, \mathbf{e}_1)| > 1/\sqrt{2}\}$. This set consists of two spherical caps. Two vectors from the same cap determine an acute angle, and from different caps form an obtuse angle. An averaging argument shows that $\alpha^{(d)}(\mathscr{P}) \geq \mathrm{Area}(C^{d-1})/\mathrm{Area}(S^{d-1})|\mathscr{P}|$. This implies $\alpha^{(2)}(k) \leq 2k$, and

$$\alpha^{(3)}(k) \leq (2 + \sqrt{2})k \sim 3.41\ldots k.$$

In general we obtain

$$\alpha^{(d)}(k)/k \leq (1 + o(1))\sqrt{\frac{\pi d}{8}}\,2^{d/2} \tag{4}$$

Remark. Witsenhausen [1] proposed the following problem. What is $\mu(d)$, the largest area of a measurable subset $C \subset S^{d-1}$ such that no vectors of C are orthogonal? Even the exact value of $\mu(3)$ is unknown. The best known lower bound (and the conjectured exact value) is given by C^{d-1}. The best upper bound ($\mu(d) \leq (1 + o(1))1.13^{-d}$) is due to Frankl [1], and follows from Eq. 5 (see later). Obviously, the essence of Eq. 4 is the upper bound $\alpha^{(d)}(k) \leq (\mathrm{Area}(S^{d-1})/\mu(d))k$.

Considering, e.g., the first orthant of $\mathbf{R}^d$ and its opposite (instead of C^{d-1}), we obtain the simpler but weaker $\alpha^{(d)}(k) \leq 2^{d-1}k$, which holds for all d and k.

5. Further Examples from ± 1 Codes

Let Q^d be the set of vectors of $\mathbf{R}^d$ of the form $(\varepsilon_1, \ldots, \varepsilon_d)$ where ε_i is 1 or -1. We have $|Q^d| = 2^d$. Larman and Rogers [4] raised the question in 1972 that what is the maximum number $m(d)$ of elements of Q^d so that no two are orthogonal, i.e., the determination of $\alpha(Q^d)$. The really interesting case is when the dimension is divisible by 4. Frankl and Rödl [2] proved that $m(4q) < (2 - c)^{4q}$ for some small but positive c. Frankl [1] proved the following exact statement: If q is an odd primepower, then

$$m(4q) = 4\sum_{i=0}^{q-1}\binom{4q-1}{i} < 4^{4q}/3^{3q} = (1.754\ldots)^{4q} \tag{5}$$

This and Eq. 3 imply the exponential lower bound $(27/16)^q < \alpha^{(4q)}$, and in general, for $d > d_0$

$$1.13^d < \alpha^{(d)}.$$

Frankl uses Eq. 5 to estimate the *chromatic number* $\chi(d)$ of $\mathbf{R}^d$, i.e., the chromatic number of the graph with vertex set $\mathbf{R}^d$ and two points are joined if their Euclidean distance is 1. Even the value of $\chi(2)$ is unknown.

6. Sets with More Orthogonal Pairs

Denote by $\alpha_l^{(d)}(k)$ the maximum number of vectors such that any $k + 1$ of them contains $l + 1$ pairwise orthogonal ones. Thus $\alpha_1^{(d)}(k) = \alpha^{(d)}(k)$. All the above questions and examples can be generalized. For example, considering a larger cone $(|(\mathbf{x}, \mathbf{e}_1)| > 1/\sqrt{l+1})$ instead of Eq. 4 we get

$$\alpha_l^{(d)}(k) \leq (1 + o(1))\sqrt{\frac{\pi d}{2l}}\left(\frac{l+1}{l}\right)^{(d/2)-1} k.$$

Frankl and Rödl [2] proved that there exists a positive $c = c(l)$ such that every subset of Q^{4n} of size more than $(2 - c)^{4n}$ contains $l + 1$ pairwise orthogonal vectors. This implies that $\alpha_l^{(d)}(k)/k$ is exponentially large in d.

We conjecture that there is a constant $g = g(l)$ such that $\alpha_l^{(d)}(k) < (dk)^g$.

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