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Note

Random Ramsey graphs for the four-cycle

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Abstract

It is shown that there is a graph $\mathscr{G}$ with n vertices and at least $n^{1.36}$ edges such that it contains neither $\mathscr{C}_3$ nor $\mathscr{K}_{2,3}$ but every subgraph with $2n^{4/3}(\log n)^2$ edges contains a $\mathscr{C}_4$, $(n>n_0)$. Moreover, the chromatic number of $\mathscr{G}$ is at least $n^{0.1}$.

1. Results, problems

A graph $\mathscr{G}$ is *Ramsey with respect to* $\mathscr{H}$, $\mathscr{G}\to\mathscr{H}$, if every two-coloration of the edges of $\mathscr{G}$ results in a monochromatic subgraph isomorphic to $\mathscr{H}$. It is easy to see that $\mathscr{K}_6\to\mathscr{C}_4$ and $\mathscr{K}_{3,7}\to\mathscr{C}_4$. Erdős and Faudree [2] asked to find a $\mathscr{K}_{2,3}$-free graph Ramsey with respect to $\mathscr{C}_4$. The graphs $\mathscr{K}_6$, $\mathscr{K}_{3,7}$ are saturated by $\mathscr{C}_4$'s, so it could be not so surprising they arrow $\mathscr{C}_4$. What Erdős and Faudree asked was whether a graph $\mathscr{G}$ exists with $\mathscr{G}\to\mathscr{C}_4$, such that any two four-cycles in $\mathscr{G}$ are either (vertex) disjoint, or share a common vertex, or an edge. The aim of this note is to show that the random method implies the existence of such a graph.

We obtain that for $E>E_0$ and for some $c>0$ there are $\mathscr{K}_{2,3}$-free graphs with E edges such that the largest $\mathscr{C}_4$-free subgraph has only E^{1-c} edges. Given this result Erdős asked for the best exponent. We have $c\geqslant 1/51-o(1)$ which can be easily

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improved to $c \geqslant 1/21 - o(1)$. On the other hand, obviously, every (connected) graph contains a $\mathscr{C}_4$-free subgraph of size $|V(\mathscr{G})|-1$ (namely, a spanning tree). This is at least $E^{2/3}$, as every $\mathscr{K}_{2,3}$-free graph has at most $(1+o(1))n^{3/2}$ edges. Probably, the best exponent is at least 8/9. It seems interesting to consider other graph pairs $(\mathscr{A}, \mathscr{B})$, i.e.

Problem 1. Determine the minimum size of the largest $\mathscr{A}$-free graphs in a $\mathscr{B}$-free graph with E edges.

The most is known about the case $\mathscr{A} = \mathscr{K}_3$. Frankl and Rödl [3] proved, e.g., that there are $\mathscr{K}_4$-free graphs (even with $\Omega(n^2)$ edges) in which every $(\frac{1}{2}+\varepsilon)E$ edges contain a triangle.

Let $f(\mathscr{A}, n)$ be the number of $\mathscr{A}$-free graphs on n vertices. The latest asymptotic results on $f(\mathscr{A}, n)$ for several $\mathscr{A}$'s with $\chi(\mathscr{A}) \geqslant 3$ were given by Prömel and Steger [8]. Here we need a generalization.

Problem 2. Find bounds on $f(\mathscr{A}, n, E)$, the number of labeled $\mathscr{A}$-free graphs on n vertices with E edges.

Having a good bound on $f(\mathscr{A}, n, E)$, we would be able to extend our Ramsey-result for graphs other than $\mathscr{C}_4$. A class of graphs $\mathfrak{B}$ is *Ramsey* if for all $\mathscr{G} \in \mathfrak{B}$ there exists an $\mathscr{H} \in \mathfrak{B}$ with $\mathscr{H} \to \mathscr{G}$. So our result is a modest first step in proving that $\mathrm{Forb}(\mathscr{K}_{2,3})$ the class of $\mathscr{K}_{2,3}$-free graphs is Ramsey. Nešetřil and Rödl [6] proved that $\mathrm{Forb}(\mathscr{C}_4)$ is Ramsey, and announced that $\mathrm{Forb}(\mathscr{C}_i)$ for $i = 5, 6, 7$ are Ramsey-classes, too. Recently, Nešetřil informed me that their method can be extended. They can decide for each $\mathscr{G}$ whether $\mathrm{Forb}(\mathscr{G})$ is Ramsey or not, and they will return to this problem in a forthcoming book. (The bipartite case can be found in [7].) However, their method, using Hales-Jewett theorem and the partite lemma, generally is unable to yield density theorems.

Another recent density result is due to Łuczak [5], who obtained by counting the $\mathscr{C}_{2k}$-free graphs on n vertices that there exists a graph $\mathscr{G}$ with at most $2^{6l}(kl)^{100k^2}$ vertices of girth $2k$ such that any $|E(\mathscr{G})|/l$ edges of it contains a $\mathscr{C}_{2k}$.

2. The properties of $\mathscr{G}(n, 3n^{-0.64})$

To prove the theorem stated in the abstract we use standard probabilistic methods. Consider the random graph $\mathscr{G}(n, p)$ (with $p = 3/n^{0.64}$) where the edges are chosen independently with probability p, and suppose that n is sufficiently large, $n \geqslant n_0$. With probability $1-o(1)$ $\mathscr{G}(n, p)$ has

$$(1+o(1))\binom{n}{2}p \sim \Theta(n^{1.36})$$

edges, and it contains

$$(1+o(1))\binom{n}{3}p^3 \sim \Theta(n^{1.08})$$

triangles and

$$(1+o(1))10\binom{n}{5}p^6 \sim \Theta(n^{1.16})$$

copies of the complete bipartite graph $\mathcal{K}_{2,3}$. Delete the edges of the appearing triangles and $\mathcal{K}_{2,3}$'s. The obtained graph, $\mathcal{G}^n$, still has more than $n^{1.36}$ edges.

The expected number of $\mathcal{C}_4$-free subgraphs with $T := 2n^{4/3}(\log n)^2$ edges in $\mathcal{G}(n, p)$ is exactly $f(\mathcal{C}_4, n, T)p^T$, where, as above, $f(\mathcal{C}_4, n, T)$ denotes the number of distinct $\mathcal{C}_4$-free graphs over n (labeled) vertices with T edges. This expected number is $o(1)$, as shown by the next lemma.

Lemma. *For all* $T \geqslant 2n^{4/3}(\log n)^2$

$$f(\mathcal{C}_4, n, T) < \left(4\frac{n^3}{T^2}\right)^T. \tag{1}$$

The proof of (1) is the only nontrivial part of this note, and is postponed to the next section.

Finally, to prove $\chi(\mathcal{G}^n) > n^{0.1}$ we use the fact that (with probability $1-o(1)$) for $v > 20(\log n)/p$ *every* v-subset of the vertices of $\mathcal{G}(n, p)$ contains at least $\frac{1}{2}\binom{v}{2}p$ edges. (This is a consequence of the Chernoff inequality, see [1]). It follows that every subset of size $n^{0.9}$ contains more edges in $\mathcal{G}(n, p)$ than we have deleted, so $\alpha(\mathcal{G}^n) < n^{0.9}$ yielding $\chi(\mathcal{G}^n) > n^{0.1}$.

3. The number of $\mathcal{C}_4$-free graphs

Here we prove (1). We extend the ideas of Kleitman and Winston [4] who established

$$f(\mathcal{C}_4, n) < (2.15\cdots)^{n^{3/2}}.$$

Their key lemma is as follows. If $\mathcal{G}$ is a $\mathcal{C}_4$-free graph on $n-1$ vertices with minimum degree at least $d-1$, then there are at most

$$n\binom{n}{z}\binom{x}{d-z} \tag{2}$$

ways to extend it to a $\mathcal{C}_4$-free graph by adding a new vertex of degree d. In (2) x is defined as

$$x = \left\lfloor n\left(1-\frac{d^2}{n+3d}\right)^z + \frac{n}{d}\right\rfloor, \tag{3}$$

and z can be any integer, $0 \leqslant z \leqslant d$. As a $\mathcal{C}_4$-free graph has at most $\frac{1}{2}n(\sqrt{n}+1)$ edges we have $d \leqslant \sqrt{n}+1$. (In [4] there is an unimportant error. Instead of $d^2/(n+3d)$ they simply write d^2/n.) One can build a $\mathcal{C}_4$-free graph on n vertices with T edges by starting with a single point (step one) and adding new points of minimum degree d_i (in step i, $2 \leqslant i \leqslant n$). For $d_i \leqslant n^{1/3} \log n$ we set $z=0$ and get

$$n\binom{n}{d} < \exp(n^{1/3}(\log n)^2) \quad \text{for } n > n_0.$$

For the other terms set $z = \lfloor n^{1/3} \rfloor$. In (3) we get $x = \lfloor n/d \rfloor$, hence the last binomial coefficient in (2) is not more than $(en/d^2)^d$. Collecting all factors we get

$$f \leqslant \exp\left(n^{4/3}(\log n)^2 + \sum d_i + (\log n) \sum d_i - 2 \sum_{i=2}^{n} d_i \log d_i \right). \tag{4}$$

Here $\sum_{2 \leqslant i \leqslant n} d_i = T$, the function $x \log x$ is convex, so we get

$$\sum d_i \log d_i \geqslant n(T/n) \log (T/n)$$

by Jensen's inequality. Then (4) gives the desired upper bound. □

For $T = \lfloor 2n^{4/3}(\log n)^2 \rfloor$ (1) gives the upper bound

$$f(\mathscr{C}_4, n, T) < n^{T(1+o(1))/3}$$

while considering all T-subsets of a $\mathscr{C}_4$-free graph with $\frac{1}{2}n^{3/2}(1+o(1))$ edges we get

$$f(\mathscr{C}_4, n, T) > n^{T(1-o(1))/6}.$$

Having no other reasonable example one can think that the 1/6 is the correct exponent.

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