

Singularities of Minimal Surfaces and Networks and Related Extremal Problems in Minkowski Space

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ABSTRACT. This paper describes results on two questions about points in a Minkowski space arising from the study of minimal surfaces and networks with singularities. Let Φ denote a norm on \mathbb{R}^n having unit ball \mathcal{B} . The first question concerns the maximal number of vectors in a Φ -equilateral set both for general norms and for strictly convex norms. A new proof is given of the known result that a Φ -equilateral set has cardinality at most 2^n for a general norm. There exists a strictly convex norm having a Φ -equilateral set of cardinality at least $(1.02)^n$, for large n . The second question concerns the maximal number of Φ -unit vectors such that $\Phi(\mathbf{x}_i + \mathbf{x}_j) \leq 1$ whenever $i \neq j$, both with and without the side condition $\sum_{i=1}^m \mathbf{x}_i = \mathbf{0}$. Here, exponentially large sets exist without the side condition; with it there are at most $2n$ vectors in the set.

1. Introduction

Soap films, grain boundaries in materials, and crystals all tend to minimize energy, and often have interesting singularities. The study of such singularities leads to various auxiliary problems, including questions in combinatorial geometry concerning arrangements of unit vectors in Minkowski space; [16–19; 23, Problem 11; 24–25].

A Minkowski space (\mathbb{R}^n, Φ) is just \mathbb{R}^n with distances measured using a norm Φ . A norm Φ is completely determined by its unit ball

$$\mathcal{B} = \{\mathbf{x} : \Phi(\mathbf{x}) \leq 1\},$$

which is a bounded convex body with nonempty interior, centrally symmetric around $\mathbf{0}$. The *dual norm* Φ^* has unit ball

$$\mathcal{B}^* = \{\mathbf{y} : \langle \mathbf{x}, \mathbf{y} \rangle \leq 1 \text{ for all } \mathbf{x} \in \mathcal{B}\},$$

and $\Phi^{**} = \Phi$. A norm Φ is said to be *strictly convex* (also called *rotund*) if the boundary of \mathcal{B} contains no line segment; it is said to be *smooth*

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(also called *differentiable*) if each point on the boundary of \mathcal{B} has a unique supporting hyperplane. These are dual notions: a norm Φ is strictly convex if and only if Φ^* is smooth [5].

Lawlor and Morgan [13] recently obtained results on the structure of singularities of cones and of networks minimizing energies given by certain functionals involving general norms Φ on \mathbb{R}^n . They show that sets of Φ -equidistant points yield singular hypersurfaces C that minimize an energy $\int_C \Phi^*(\mathbf{n})$, where \mathbf{n} is the unit normal to C , subject to certain boundary conditions.

THEOREM I. *Let Φ be a norm on \mathbb{R}^n with dual norm Φ^* . Suppose there are points $\mathbf{p}_1, \dots, \mathbf{p}_m \in \mathbb{R}^n$ such that*

$$\Phi(\mathbf{p}_j - \mathbf{p}_i) = 1 \quad \text{for } i \neq j.$$

Let $C \subset \mathcal{B}(\mathbf{0}, 1)$ be a hypersurface that divides the Euclidean unit ball $\mathcal{B}(\mathbf{0}, 1)$ into regions R_1, \dots, R_m separated by pieces of hyperplanes H_{ij} with unit normals \mathbf{n}_{ij} dual to $\mathbf{p}_j - \mathbf{p}_i$ (i.e., $\mathbf{n}_{ij} \cdot (\mathbf{p}_j - \mathbf{p}_i) = \Phi(\mathbf{n}_{ij})$). Then C minimizes $\sum_{i,j} \Phi^(H_{ij})$ among hypersurfaces (closed sets that are C^1 manifolds almost everywhere) that separate the fixed boundary regions $R_i \cap S(\mathbf{0}, 1)$ on the sphere $S(\mathbf{0}, 1)$ from each other in $\mathcal{B}(\mathbf{0}, 1)$.*

Lawlor and Morgan also prove a result concerning the singularity at $\mathbf{0}$ of a network C in \mathbb{R}^n that minimizes $\int_C \Phi^*(\mathbf{t})$, where \mathbf{t} is the unit tangent to C .

THEOREM II. *Let Φ be a strictly convex norm on \mathbb{R}^n , and let Φ^* denote the dual norm. Let $\mathbf{a}_1, \dots, \mathbf{a}_k \in \mathbb{R}^n$, normalized so that $\Phi^*(\mathbf{a}_j) = 1$, and let $\mathbf{b}_1, \dots, \mathbf{b}_k$ denote the unique dual vectors such that $\Phi(\mathbf{b}_j) = 1$ and $\mathbf{a}_j \cdot \mathbf{b}_j = 1$. Then the network C consisting of rays from the origin to $\mathbf{a}_1, \dots, \mathbf{a}_k$ is Φ^* -minimizing if and only if every subcollection J of the \mathbf{b}_j has a sum in the Φ -norm of length at most one, i.e.,*

$$(1.1) \quad \Phi \left(\sum_{j \in J} \mathbf{b}_j \right) \leq 1,$$

and in addition

$$(1.2) \quad \mathbf{b}_1 + \dots + \mathbf{b}_k = \mathbf{0}.$$

If Φ is not strictly convex, the dual vectors \mathbf{b}_j are not necessarily uniquely determined, but if (1.1) and (1.2) hold for some choice of such \mathbf{b}_j , then C is Φ^ -minimizing.*

We call the condition (1.2) a *balancing condition*. For the Euclidean norm on \mathbb{R}^2 , these conditions include the well-known conditions for a Steiner point in a Euclidean length-minimizing network: three edges must meet at 120° angles. Hwang [11] surveys results on the Steiner problem, and Alfaro et al.

[1] and Levy [14] obtain results on two-dimensional singularities for general norms.

Determining the maximal complexity of possible singularities leads to the following extremal problems for unit vectors in Minkowski space.

(1) **EQUILATERAL SET PROBLEM.** What is the maximum cardinality of a Φ -equilateral set $\{x_i\} \subset \mathbb{R}^n$, i.e., a set such that $\Phi(x_i - x_j) = 1$ whenever $i \neq j$?

(2) **SUMS OF UNIT VECTORS PROBLEM.** What is the maximal number m in a set $S = \{x_i\}$ of Φ -unit vectors $\Phi(x_i) = 1$ such that $\Phi(x_i + x_j) \leq 1$ whenever $i \neq j$ and $\sum_{i=1}^m x_i = 0$.

Here, (2) has weakened condition (1.1) of Theorem II to requiring a bound only on sums of *pairs* of vectors in the set S ; we will see, presently, that this does not affect the answer. In the context of Theorem II, we are especially interested in the case that Φ is strictly convex.

These are natural problems, and versions of them have been raised repeatedly as pure questions in combinatorial geometry. For example, Kusner [12] lists a number of such problems concerning equilateral sets. Variants of these problems include putting extra restrictions on the norm (e.g., smoothness) and, in the case of (2), by adding or removing the balancing condition $\sum_{i=1}^m x_i = 0$.

The purpose of this paper is to give new results and new proofs of old results on these problems. We also describe known results and state some open problems. It turns out that there are upper bounds exponential in n for both problems (Theorems 2.1 and 3.5), and exponential lower bounds for most variants of these problems (Theorems 2.4 and 3.6); however, the balancing condition in (2) is shown to imply sharp upper and lower bounds linear in n (Theorems 3.1 and 3.4).

These extremal problems turn out to have interesting connections to several other problems in combinatorial geometry. These include the problem of characterizing convex bodies that can be perfectly packed with identical smaller copies of themselves, and Hadwiger's problem of finding the maximal number of translates of a body \mathcal{B} that all intersect \mathcal{B} and have disjoint interiors. These also include problems of finding large sets of points $\{x_i\}$ and $\{y_j\}$ in \mathbb{R}^n whose inner products $\langle x_i, y_j \rangle$ are constrained in various ways, such as the Inner Product Problem discussed in §3, in which all angles $x_i \cdot y_j$ are required to be at least $\frac{\pi}{2}$.

2. Equilateral sets

The Equilateral Set Problem for a general norm was settled by Petty [20].

THEOREM 2.1 [20]. Any set $S = \{x_i\}$ of points in (\mathbb{R}^n, Φ) such that

$$\Phi(x_i - x_j) = 1 \quad \text{for } i \neq j$$

has cardinality $|S| \leq 2^n$. Equality is attained only when the unit ball of Φ is affinely equivalent to the n -cube.

Petty [20] deduced this theorem from results of Danzer and Grünbaum [3] on sets of pairwise antipodal points. We give an alternate proof.

PROOF OF THEOREM 2.1. The upper bound 2^n is a simple consequence of the

ISODIAMETRIC INEQUALITY. *Let Φ be a norm on \mathbb{R}^n with unit ball \mathcal{B} and let \mathcal{K} be a closed body of Φ -diameter ≤ 2 . Then*

$$(2.1) \quad \text{Vol}(\mathcal{K}) \leq \text{Vol}(\mathcal{B}),$$

with equality if and only if $\mathcal{K} = \mathcal{B}$.

This inequality is a generalization of the so-called Bieberbach inequality. In the case of the Euclidean norm, Bieberbach proved it for \mathbb{R}^2 and Urysohn [26] for \mathbb{R}^n . Mel'nikov [15] proved it for general finite-dimensional normed spaces. Burago and Zalgaller [2, 11.2.1, p. 93] give a short proof of the isodiametric inequality using the Brunn-Minkowski theorem.

To deduce Theorem 2.1, observe that if $\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$ is a set of Φ -equidistant points, then the interiors of the Φ -balls $\frac{1}{2}\mathcal{B} + \mathbf{x}_i$ are disjoint, and $\mathcal{W} = \bigcup_{i=1}^k (\frac{1}{2}\mathcal{B} + \mathbf{x}_i)$ has diameter ≤ 2 . Hence,

$$(2.2) \quad \text{vol}(\mathcal{B}) \geq \text{vol}(\mathcal{W}) = k2^{-n} \text{vol}(\mathcal{B}).$$

Thus, $k \leq 2^n$.

For the case of equality $k = 2^n$, one must have equality in (2.2), and the isodiametric inequality requires that

$$\mathcal{B} = \bigcup_{i=1}^k (\frac{1}{2}\mathcal{B} + \mathbf{x}_i),$$

i.e., \mathcal{B} is perfectly packed with translates of $\frac{1}{2}\mathcal{B}$. We use the following result of Groemer [9, Hilfssatz 2], which says that for arbitrary full-dimensional convex bodies \mathcal{K} , this last property is already sufficient to force \mathcal{K} to be affinely equivalent to an n -cube. Here, \mathcal{K} is not assumed to be centrally symmetric.

THEOREM 2.2 [9]. *Let \mathcal{K} be a bounded convex body in \mathbb{R}^n that is the closure of its interior, such that for some finite $t > 1$, the body $t\mathcal{K}$ can be perfectly packed by translates of \mathcal{K} . Then \mathcal{K} is affinely equivalent to an n -cube, and t is an integer. The packing is unique and extends to a lattice packing of \mathbb{R}^n .*

The remaining part of Theorem 2.1 follows immediately. Since the packing is unique, any equilateral set S of cardinality 2^n must be affinely equivalent to the set of vertices of the n -cube, under the same affine equivalence taking \mathcal{B} to an n -cube. \square

We remark that Gritzmann [7, Theorem 3.4] has proved a stronger variant of Theorem 2.2; see also [8].

THEOREM 2.3 [8]. *Let \mathcal{L} be a bounded convex body in \mathbb{R}^n that is the closure of its interior and that can be perfectly packed by translates of a convex body \mathcal{K} . Then there are complementary subspaces of dimension k and $n-k$ and convex bodies \mathcal{K}_1 and \mathcal{K}_2 in these subspaces such that \mathcal{K}_1 is affinely equivalent to a k -cube, with $\mathcal{K} = \mathcal{K}_1 + \mathcal{K}_2$ and $\mathcal{L} = t\mathcal{K}_1 + \mathcal{K}_2$, where t is a positive integer and $+$ denotes Minkowski sum.*

Now we consider the Equilateral Set Problem for strictly convex norms. For $n = 2$, a sharp bound is easily seen to be 3. For $n = 3$, Petty [20] observes that results of Grünbaum [10] imply that a strictly convex norm in \mathbb{R}^3 has at most five vectors at unit distances from each other. Lawlor and Morgan [13, §3.5] give a smooth and strictly convex norm where five vectors occur. The exact answer for strictly convex norms is not known for any $n \geq 4$.

For n dimensions, there is an exponential lower bound for strictly convex norms.

THEOREM 2.4. *There is a strictly convex norm Φ in \mathbb{R}^n and a set $S = \{\mathbf{x}_i\}$ of Φ -unit vectors such that*

$$\Phi(\mathbf{x}_i - \mathbf{x}_j) = 1 \quad \text{if } i \neq j,$$

which has cardinality $|S| \geq (1.02)^n$, for all $n \geq n_0$.

To prove this theorem, we will construct a suitable norm. The fundamental fact used to do this is that for the Euclidean norm there exist exponentially large sets of “nearly orthogonal” unit vectors.

LEMMA 2.5. *For any fixed $\delta > 0$, there is a constant $f(\delta) > 0$ such that for all sufficiently large n there exists a set $Y = \{\mathbf{y}_i\}$ in \mathbb{R}^n such that*

$$(2.3a) \quad \langle \mathbf{y}_i, \mathbf{y}_i \rangle = 1 \quad \text{for all } i,$$

$$(2.3b) \quad |\langle \mathbf{y}_i, \mathbf{y}_j \rangle| \leq \delta \quad \text{whenever } i \neq j,$$

which has cardinality $|Y| \geq (1 + f(\delta))^n$. In particular for $\delta = \frac{1}{6}$, one may take $f(\delta) > .02$.

PROOF. Without loss of generality, $\delta \leq \frac{1}{2}$. We use a simple random construction. Draw m independent uniform samples $\{\mathbf{z}_i\}$ from the set of all ± 1 vectors in \mathbb{R}^n . Any such sample having the property that

$$|\langle \mathbf{z}_i, \mathbf{z}_j \rangle| \leq \delta n \quad \text{whenever } i \neq j,$$

gives a set $Y = \{\frac{1}{\sqrt{n}}\mathbf{z}_i\}$ satisfying (2.3). Now for any pair of samples $(\mathbf{z}_i, \mathbf{z}_j)$, one has

$$\text{Prob}\{|\langle \mathbf{z}_i, \mathbf{z}_j \rangle| > \delta n\} = 2^{-n+1} \sum_{0 \leq j \leq (1/2-\delta)n} \binom{n}{j} \leq 2^{1+(H(1/2-\delta)-1)n},$$

where $H(t) = -t \log_2 t - (1-t) \log_2 (1-t)$ is the binary entropy function; [27, Theorem 1.4.5]. Since there are $\binom{m}{2}$ such pairs, one has

$$\text{Prob}\{\text{all } |\langle \mathbf{z}_i, \mathbf{z}_j \rangle| \leq \delta n\} \geq 1 - \binom{m}{2} 2^{1+(H(1/2-\delta)-1)n}.$$

This probability is nonzero if we choose $m = |Y| = 2^{1/2(1-H(1/2-\delta))n}$. Since $H(\frac{1}{2}-\delta) < 1$, the desired bound follows.

For $\delta = \frac{1}{6}$, $H(\frac{1}{2}-\delta) \leq .918$, and one may take $f(\frac{1}{6}) > .02$. \square

PROOF OF THEOREM 2.4. By Lemma 2.5, for all large enough n there exists a set $Y = \{\mathbf{w}_0, \dots, \mathbf{w}_m\}$ of $m+1$ Euclidean unit vectors in \mathbb{R}^n having

$$(2.4) \quad |\langle \mathbf{w}_i, \mathbf{w}_j \rangle| \leq \frac{1}{6} \quad \text{whenever } i \neq j,$$

with $m \geq (1.02)^n$.

Consider the centrally symmetric polytope \mathcal{B}_1 which is the convex hull of all $\mathbf{w}_i - \mathbf{w}_j$ with $i \neq j$. We show that all points $\mathbf{w}_i - \mathbf{w}_j$ lie on the boundary of \mathcal{B}_1 ; and, furthermore, that each point can serve as its own dual vector in the norm determined by \mathcal{B}_1 .

CLAIM 1. For $i \neq j$ and small enough $\eta > 0$, the hyperplane

$$H_{ij}(\eta) = \{\mathbf{x} : \langle \mathbf{x}, \mathbf{w}_i - \mathbf{w}_j \rangle = \|\mathbf{w}_i - \mathbf{w}_j\|^2 - \eta\}$$

separates $\mathbf{w}_i - \mathbf{w}_j$ from $\mathbf{0}$, and from all $\mathbf{w}_k - \mathbf{w}_l$ with $(k, l) \neq (i, j)$.

The proof of Claim 1 starts with the estimate

$$\langle \mathbf{w}_i - \mathbf{w}_j, \mathbf{w}_i - \mathbf{w}_j \rangle = 2 - 2\langle \mathbf{w}_i, \mathbf{w}_j \rangle \geq \frac{10}{6}.$$

Separation of $\mathbf{w}_i - \mathbf{w}_j$ from $\mathbf{0}$ and $\mathbf{w}_j - \mathbf{w}_i$ is clear. For i, j, k, l , all distinct

$$\langle \mathbf{w}_k - \mathbf{w}_l, \mathbf{w}_i - \mathbf{w}_j \rangle \leq \frac{4}{6},$$

while if exactly one of k, l equals i or j , then

$$\langle \mathbf{w}_k - \mathbf{w}_l, \mathbf{w}_i - \mathbf{w}_j \rangle \geq \frac{9}{6},$$

proving Claim 1.

Now let \mathcal{B}_2 be the intersection of the half-spaces containing $\mathbf{0}$ cut out by all the hyperplanes $H_{ij}(0)$. Then $\mathcal{B}_1 \subseteq \mathcal{B}_2$, and Claim 1 implies, on letting $\eta \rightarrow 0$, that the boundaries of \mathcal{B}_1 and \mathcal{B}_2 intersect exactly in the points $\mathbf{w}_i - \mathbf{w}_j$ for $i \neq j$ and that each of these points lies in the relative interior of a facet of \mathcal{B}_2 .

It suffices to show that there exists a strictly convex, centrally symmetric body \mathcal{B} with $\mathcal{B}_1 \subset \mathcal{B} \subset \mathcal{B}_2$. If so, then the strictly convex norm Φ determined by \mathcal{B} has

$$\Phi(\mathbf{w}_i - \mathbf{w}_j) = 1 \quad \text{whenever } i \neq j,$$

since they are on the boundaries of both \mathcal{B}_1 and \mathcal{B}_2 . If we then set $\mathbf{x}_i = \mathbf{w}_i - \mathbf{w}_0$ for $1 \leq i \leq m$, then

$$(2.5) \quad \Phi(\mathbf{x}_i) = \Phi(\mathbf{x}_i - \mathbf{x}_j) = 1 \quad \text{whenever } i \neq j,$$

which proves the theorem.

Thus, it remains to show

CLAIM 2. Let \mathcal{B}_1 and \mathcal{B}_2 be convex polytopes with $\mathcal{B}_1 \subseteq \mathcal{B}_2$, whose boundaries intersect in a finite number of points, all in the relative interior of facets of \mathcal{B}_2 . Then there exists a strictly convex body \mathcal{B} with $\mathcal{B}_1 \subset \mathcal{B} \subset \mathcal{B}_2$. If \mathcal{B}_1 and \mathcal{B}_2 are centrally symmetric about $\mathbf{0}$, then such a \mathcal{B} exists that is centrally symmetric about $\mathbf{0}$.

To prove Claim 2, let \mathcal{C} be a closed halfspace with the bounding hyperplane H , and suppose first that \mathcal{B}_1 is a convex polytope which is contained in \mathcal{C} and which touches H only at a single point \mathbf{q} . Then for all sufficiently large radii r , the spherical ball of radius r contained in \mathcal{C} that is tangent to H at \mathbf{q} contains $\mathcal{B}_1 - \{\mathbf{q}\}$ in its interior. If \mathcal{B}_1 is strictly inside \mathcal{C} , then for an arbitrary point \mathbf{q} on H , one can find such a ball in \mathcal{C} containing \mathcal{B}_1 . Next, given $\mathcal{B}_1 \subseteq \mathcal{B}_2$ satisfying the given hypotheses, one constructs such balls associated to pairs $(\mathcal{C}, \mathbf{q})$ for each hyperplane \mathcal{C} containing a facet of \mathcal{B}_2 . Finally, take \mathcal{B} to be the intersection of the balls associated to the various pairs $(\mathcal{C}, \mathbf{q})$; it is strictly convex. In the centrally symmetric case, these balls can be chosen in symmetric pairs for opposite facets $(\mathcal{C}, \mathbf{q})$ and $(-\mathcal{C}, -\mathbf{q})$ so that \mathcal{B} is centrally symmetric. \square

The proof of Claim 2 actually constructs a body \mathcal{B} that is uniformly convex, a property that is stronger than strict convexity. A norm Φ is *uniformly convex* if there exists $\beta > 0$ such that $\Phi(\mathbf{x}) - \beta\|\mathbf{x}\|$ is also a norm. Claim 2 probably remains true with its conclusion further strengthened to require that \mathcal{B} be both of class C^∞ and uniformly convex. Any strengthenings of the conditions of the norm constructed in Claim 2 automatically carry over to corresponding strengthenings of Theorems 2.4 and 3.6.

The following problem is open.

CONJECTURE 2.6. There is a constant $\gamma > 0$ such that, for any strictly convex norm Φ in \mathbb{R}^n , any equilateral set S for Φ has

$$|S| \leq (2 - \gamma)^n.$$

This is analogous to a conjecture of Erdős and Füredi [6], which states: If S is a finite set of points in \mathbb{R}^n such that any angle determined by three of its points is less than $\frac{\pi}{2}$, then its cardinality $|S| \leq (2 - \gamma)^n$.

3. Sums of unit vectors

Given $S = \{\mathbf{x}_i : 1 \leq i \leq m\}$ with $\Phi(\mathbf{x}_i) = 1$, we begin by relating the bounded sums condition

$$(3.1) \quad \Phi(\mathbf{x}_i + \mathbf{x}_j) \leq 1$$

to a condition involving only the Euclidean norm. Consider the dual norm Φ^* , specified by the unit ball

$$(3.2) \quad \mathcal{B}^* = \{\mathbf{x}^* : \langle \mathbf{x}^*, \mathbf{y} \rangle \leq 1 \text{ for all } \mathbf{y} \in \mathcal{B}\},$$

where $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^n x_i y_i$ is the Euclidean inner product on \mathbb{R}^n . For each vector \mathbf{x} with $\Phi(\mathbf{x}) = 1$, there exists at least one dual vector \mathbf{x}^* with $\Phi^*(\mathbf{x}^*) = 1$ and

$$(3.3) \quad \langle \mathbf{x}, \mathbf{x}^* \rangle = 1,$$

e.g., \mathbf{x}^* points in a direction normal to a tangent hyperplane to the unit ball \mathcal{B} of Φ at its boundary point \mathbf{x} . Choose for each $\mathbf{x}_i \in S$ a corresponding $\mathbf{x}_i^* \in \mathcal{B}^*$ satisfying (3.3), and set $S^* = \{\mathbf{x}_i^* : 1 \leq i \leq m\}$. We claim that if S satisfies (3.1), then

$$(3.4) \quad \langle \mathbf{x}_i, \mathbf{x}_j^* \rangle \leq 0 \quad \text{whenever } i \neq j.$$

Indeed by (3.1) and $\mathbf{x}_j^* \in \mathcal{B}^*$,

$$\langle \mathbf{x}_i, \mathbf{x}_j^* \rangle + 1 = \langle \mathbf{x}_i + \mathbf{x}_j, \mathbf{x}_j^* \rangle \leq \Phi(\mathbf{x}_i + \mathbf{x}_j) \leq 1,$$

which proves (3.4).

This leads us to consider the auxiliary problem

INNER PRODUCT PROBLEM. Let X and Y be sets in \mathbb{R}^n with $|X| = |Y| = m$, such that

$$(3.5a) \quad \langle \mathbf{x}_i, \mathbf{y}_i \rangle > 0 \quad \text{for } 1 \leq i \leq m.$$

$$(3.5b) \quad \langle \mathbf{x}_i, \mathbf{y}_j \rangle \leq 0 \quad \text{whenever } i \neq j.$$

Bound m , under given side conditions on X and Y .

The condition (3.5b) is a condition on angles: all angles $\mathbf{x}_i \mathbf{0} \mathbf{y}_j$ are at least $\frac{\pi}{2}$.

Upper bounds for the Inner Product Problem (with side conditions) will imply the same upper bounds for the Sums of Unit Vectors Problem (with side conditions); the converse is not necessarily true. We consider, in particular, the following side conditions:

(i) **EUCLIDEAN NORM CASE:** $X = Y$.

(ii) **WEAK BALANCING CONDITION:** $\mathbf{0}$ is in the relative interior of the convex hull of $X = \{\mathbf{x}_i : 1 \leq i \leq m\}$.

Note that the balancing condition

$$\sum_{i=1}^m \mathbf{x}_i = \mathbf{0}$$

implies that the weak balancing condition (ii) holds.

We first consider the Sums of Unit Vectors Problem, assuming the weak balancing condition.

THEOREM 3.1. *Let Φ be a norm on \mathbb{R}^n . If $S = \{\mathbf{x}_i\}$ is a set of Φ -unit vectors in \mathbb{R}^n such that*

$$(3.6) \quad \Phi(\mathbf{x}_i + \mathbf{x}_j) \leq 1 \quad \text{for } i \neq j,$$

which satisfies the weak balancing condition, then it has cardinality $|S| \leq 2n$. Equality can be attained only when the set S is linearly equivalent to the set $\{\pm \mathbf{e}_i : 1 \leq i \leq n\}$, where \mathbf{e}_i are the unit vectors in each coordinate direction.

In fact the equality $|S| = 2n$ is still attained under the stronger hypotheses

$$\Phi\left(\sum_{\mathbf{x}_i \in J} \mathbf{x}_i\right) \leq 1 \quad \text{for all } J \subseteq S, \quad \sum_{i=1}^n \mathbf{x}_i = \mathbf{0},$$

if Φ has the n -cube as unit ball and $S = \{\pm \mathbf{e}_i : 1 \leq i \leq n\}$. Under these stronger hypotheses, it can be proved that the case of equality $|S| = 2n$ holds only when the unit ball of Φ is affinely equivalent to the n -cube.

Theorem 3.1 is an immediate corollary of a similar upper bound for the Inner Product Problem.

THEOREM 3.2. (a) *Given sets X and Y in \mathbb{R}^n with cardinalities $|X| = |Y| = m$ such that*

$$(3.7a) \quad \langle \mathbf{x}_i, \mathbf{y}_i \rangle > 0 \quad \text{for } 1 \leq i \leq m,$$

$$(3.7b) \quad \langle \mathbf{x}_i, \mathbf{y}_j \rangle \leq 0 \quad \text{whenever } i \neq j,$$

where X satisfies the condition that $\mathbf{0}$ is in the relative interior of the convex hull of X . Then $m \leq 2n$, and equality can hold only if the \mathbf{x}_i can be renumbered so that $\mathbf{x}_{n+i} = -\lambda_i \mathbf{x}_i$ with $\lambda_i > 0$ for $1 \leq i \leq n$.

(b) *If in addition*

$$\langle \mathbf{x}_i, \mathbf{y}_j \rangle < 0 \quad \text{whenever } i \neq j,$$

then $m \leq n + 1$.

Our proof of Theorem 3.2 is based on an observation of I. Bárány, improving on the original proof of the authors'. It uses

STEINITZ'S THEOREM. (1) *Let S be a finite set in \mathbb{R}^n whose convex hull is a body \mathcal{K} of dimension r , such that $\mathbf{0}$ lies in the relative interior of \mathcal{K} . Then there is a subset T of S of cardinality $t \leq 2r$, say $T = \{\mathbf{x}_i : 1 \leq i \leq t\}$, whose convex hull is of dimension r , such that $\mathbf{0}$ is a strict convex combination of T , i.e.,*

$$\mathbf{0} = \sum_{i=1}^t \lambda_i \mathbf{x}_i,$$

with all $\lambda_i > 0$, $\sum_{i=1}^t \lambda_i = 1$.

(2) *If there exists no subset T with $t \leq 2r - 1$ having the above property, then necessarily T consists of exactly $2r$ vectors, which are collinear in pairs.*

The first part of Steinitz's theorem is due to Steinitz and is discussed at length in Danzer, Grünbaum, and Klee [4, §3], and the second part is due to Robinson [22, Lemma 2a].

PROOF OF THEOREM 3.2. The weak balancing condition implies by Steinitz's theorem that there is a subset T of $t \leq 2n$ vectors of X , say $T = \{\mathbf{x}_i : 1 \leq i \leq t\}$, with

$$(3.8) \quad \mathbf{0} = \sum_{i=1}^t \lambda_i \mathbf{x}_i \quad \text{all } \lambda_i > 0,$$

and $\dim(T) = \dim(X) = r$. If $|X| \geq t+1$ then

$$\mathbf{x}_{t+1} = \sum_{i=1}^t \gamma_i \mathbf{x}_i,$$

since \mathbf{x}_{t+1} is in the subspace spanned by T . By adding a large enough multiple of (3.8) to this equation, we obtain

$$\mathbf{x}_{t+1} = \sum_{i=1}^t \beta_i \mathbf{x}_i$$

with all $\beta_i > 0$. But $\langle \mathbf{x}_{t+1}, \mathbf{y}_{t+1} \rangle > 0$ by hypothesis, while

$$\langle \mathbf{x}_{t+1}, \mathbf{y}_{t+1} \rangle = \sum_{i=1}^t \beta_i \langle \mathbf{x}_i, \mathbf{y}_{t+1} \rangle \leq 0$$

by (3.7b), a contradiction. Hence, $T = X$ and $|X| \leq t \leq 2n$.

If $|X| = 2n$, then the case (2) of equality in Steinitz's theorem requires that the vectors in X be collinear in pairs. This proves the second assertion in (a).

For part (b) we use Caratheodory's theorem in place of Steinitz's theorem: Every vector in the convex hull of X is a convex combination of at most $n+1$ vectors in X . Thus, the weak balancing condition implies that

$$\mathbf{0} = \sum_{i=1}^{n+1} \alpha_i \mathbf{x}_i,$$

with all $\alpha_i \geq 0$, $\sum_{i=1}^{n+1} \alpha_i = 1$, after renumbering X if necessary. If $|X| \geq n+2$, then

$$0 = \left\langle \sum_{i=1}^{n+1} \alpha_i \mathbf{x}_i, \mathbf{y}_{n+2} \right\rangle = \sum_{i=1}^{n+1} \alpha_i \langle \mathbf{x}_i, \mathbf{y}_{n+2} \rangle < 0,$$

a contradiction proving part (b). \square

A similar easy upper bound $2n$ holds for the Inner Product Problem assuming the Euclidean norm side condition $X = Y$, without any balancing condition.

THEOREM 3.3. *Given a set X of cardinality $|X| = m$ in \mathbb{R}^n such that*

$$(3.9a) \quad \langle \mathbf{x}_i, \mathbf{x}_i \rangle > 0 \quad \text{for } 1 \leq i \leq m,$$

$$(3.9b) \quad \langle \mathbf{x}_i, \mathbf{x}_j \rangle \leq 0 \quad \text{whenever } i \neq j,$$

then $m \leq 2n$.

PROOF. This is proved by induction on the dimension n , the case $n = 1$ being obvious. Given X in \mathbb{R}^n , let \mathbf{w}_i denote the projection of \mathbf{x}_i onto the $(n-1)$ -dimensional subspace perpendicular to \mathbf{x}_1 . Then $\mathbf{w}_1 = \mathbf{0}$, and at most one other $\mathbf{w}_i = \mathbf{0}$, which occurs only if there is some $\mathbf{x}_i = -\lambda \mathbf{x}_1$ with $\lambda > 0$. Now for $i \geq 2$, (3.9b) implies that $\mathbf{x}_i = \mathbf{w}_i - \lambda_i \mathbf{x}_1$ with $\lambda_i \geq 0$, hence for $i \neq j$

$$\langle \mathbf{x}_i, \mathbf{x}_j \rangle = \langle \mathbf{w}_i, \mathbf{w}_j \rangle + \lambda_i \lambda_j \|\mathbf{x}_1\|^2 \leq 0.$$

Thus, the set of nonzero \mathbf{w}_i satisfies the hypotheses (3.9) in \mathbb{R}^{n-1} , so there are at most $2n-2$ of them, and the induction step follows. \square

Note that in (3.9a) we can rescale \mathbf{x}_i to require $\langle \mathbf{x}_i, \mathbf{x}_i \rangle = 1$ without changing (3.9b). Then (3.9) asserts that all open spherical caps of angular measure $\frac{\pi}{4}$ about each \mathbf{x}_i are disjoint. In this reformulation the inequality $|X| \leq 2n$ is derived in Rankin [21], exactly as above.

For a strictly convex norm the weak balancing condition gives the stronger bound $n+1$ in place of $2n$.

THEOREM 3.4. *Let Φ be a strictly convex norm in \mathbb{R}^n . Let $S = \{\mathbf{x}_i\}$ be a set of Φ -unit vectors in \mathbb{R}^n such that*

$$\Phi(\mathbf{x}_i + \mathbf{x}_j) \leq 1 \quad \text{for } i \neq j,$$

which satisfies the weak balancing condition. Then S has cardinality $|S| \leq n+1$.

PROOF. The condition of strict convexity of Φ sharpens the condition (3.4) to

$$(3.10) \quad \langle \mathbf{x}_i, \mathbf{x}_j^* \rangle < 0 \quad \text{whenever } i \neq j.$$

Now the result follows from Theorem 3.2(b). \square

Next, we consider the Sums of Unit Vectors Problem without any balancing condition and give an exponential upper bound.

THEOREM 3.5. *Consider any norm Φ in \mathbb{R}^n and any set $S = \{\mathbf{x}_i\}$ of Φ -unit vectors satisfying*

$$(3.11) \quad \Phi(\mathbf{x}_i + \mathbf{x}_j) \leq 1 \quad \text{whenever } i \neq j.$$

Then S has cardinality $|S| \leq 3^n - 1$.

PROOF. By the triangle inequality

$$\Phi(\mathbf{x}_i - \mathbf{x}_j) \geq \Phi(2\mathbf{x}_i) - \Phi(\mathbf{x}_i + \mathbf{x}_j) \geq 1,$$

whenever $i \neq j$, and all $\Phi(\mathbf{x}_i) \geq 1$ also. Hence, the interiors of all the sets $\mathbf{x}_i + \frac{1}{2}\mathcal{B}$ and of $\frac{1}{2}\mathcal{B}$ are pairwise disjoint. Since these sets all lie in $\frac{3}{2}\mathcal{B}$, volume considerations yield $|S| + 1 \leq 3^n$. \square

The proof of Theorem 3.5 shows that the sets $\{2\mathbf{x}_i + \mathcal{B}\}$ satisfy the conditions of *Hadwiger's problem*, which is that of bounding the maximum number of translates of a closed convex body \mathcal{B} that intersect \mathcal{B} , but which have pairwise disjoint interiors. The bound 3^n is sharp for Hadwiger's problem on taking \mathcal{B} to be affinely equivalent to the n -cube, and Theorem 2.2 shows that this is the only case of equality. Since (3.11) does not hold for the n -cube, improvement is possible in the upper bound of Theorem 3.5. See Danzer, Grünbaum, and Klee [4, p. 149] for history and results on Hadwiger's problem.

In fact, exponential size sets can occur in the Sums of Unit Vectors Problem when no balancing condition is present, even with strict convexity imposed.

THEOREM 3.6. *There exists a strictly convex norm Φ in \mathbb{R}^n and a set $S = \{\mathbf{x}_i\}$ of Φ -unit vectors satisfying*

$$(3.12) \quad \Phi(\mathbf{x}_i + \mathbf{x}_j) < 1 \quad \text{whenever } i \neq j,$$

which has cardinality $|S| \geq (1.02)^n$.

PROOF. The proof uses similar ideas to Theorem 2.4. By Lemma 2.5, for all large enough n , there exists a set of $Y = \{\mathbf{w}_i\}$ in \mathbb{R}^{n-1} having $|Y| \geq (1.02)^n$ and

$$(3.13) \quad |\langle \mathbf{w}_i, \mathbf{w}_j \rangle| \leq \frac{1}{6} \quad \text{whenever } i \neq j.$$

View \mathbb{R}^{n-1} embedded in \mathbb{R}^n as the first $n-1$ coordinates, take the unit vector $\mathbf{e} = (0, 0, \dots, 0, 1)$ orthogonal to all \mathbf{w}_i , and set

$$\mathbf{x}_i = \mathbf{w}_i + \lambda \mathbf{e}, \quad 1 \leq i \leq m,$$

where $\lambda > 0$ is arbitrary. Now let \mathcal{B}_1 be the convex hull of all the vectors $\pm \mathbf{x}_i$ and $\pm 1.01(\mathbf{x}_i + \mathbf{x}_j)$ where $i \neq j$. Clearly each $\mathbf{x}_i + \mathbf{x}_j \in \text{Int}(\mathcal{B}_1)$ whenever $i \neq j$, and we assert that

$$(*) \quad \text{All } \mathbf{x}_i \text{ are on the boundary of } \mathcal{B}_1.$$

To show this we use the dual vectors

$$\mathbf{y}_i = \mathbf{w}_i - \frac{1}{5\lambda} \mathbf{e}, \quad 1 \leq i \leq m.$$

One has

$$\langle \mathbf{x}_i, \mathbf{y}_i \rangle = \langle \mathbf{w}_i + \lambda \mathbf{e}, \mathbf{w}_i - \frac{1}{5\lambda} \mathbf{e} \rangle = 1 - \frac{1}{5} \langle \mathbf{e}, \mathbf{e} \rangle = \frac{4}{5}.$$

Then $(*)$ is an immediate consequence of the following:

CLAIM. The hyperplane $H_i = \{\mathbf{x} : \langle \mathbf{x}, \mathbf{y}_i \rangle = \frac{4}{5} - \frac{1}{40}\}$ separates \mathbf{x}_i from $\mathbf{0}$, from $-\mathbf{x}_i$, from all other $\pm \mathbf{x}_j$, and from all $\pm 1.01(\mathbf{x}_k + \mathbf{x}_l)$ having $k \neq l$.

The claim is obvious for $\mathbf{0}$ and $-\mathbf{x}_i$, while for $j \neq i$,

$$\langle \mathbf{x}_j, \mathbf{y}_i \rangle = \langle \mathbf{w}_j + \lambda \mathbf{e}, \mathbf{w}_i - \frac{1}{5\lambda} \mathbf{e} \rangle = \langle \mathbf{w}_j, \mathbf{w}_i \rangle - \frac{1}{5} \leq -\frac{1}{30}$$

and

$$\langle -\mathbf{x}_j, \mathbf{y}_i \rangle = \langle -\mathbf{w}_j - \lambda \mathbf{e}, \mathbf{w}_i - \frac{1}{5\lambda} \mathbf{e} \rangle = \langle -\mathbf{w}_j, \mathbf{w}_i \rangle + \frac{1}{5} \leq \frac{1}{6} + \frac{1}{5}.$$

Also for $j \neq i$,

$$\langle \mathbf{x}_i + \mathbf{x}_j, \mathbf{y}_i \rangle = \frac{4}{5} + \langle \mathbf{x}_j, \mathbf{y}_i \rangle \leq \frac{4}{5} - \frac{1}{30},$$

$$\langle -(\mathbf{x}_i + \mathbf{x}_j), \mathbf{y}_i \rangle = -\frac{4}{5} + \langle -\mathbf{x}_j, \mathbf{y}_i \rangle \leq -\frac{2}{5},$$

and, finally, for i, k, l all distinct,

$$\langle \mathbf{x}_k + \mathbf{x}_l, \mathbf{y}_i \rangle = \langle \mathbf{x}_k, \mathbf{y}_i \rangle + \langle \mathbf{x}_l, \mathbf{y}_i \rangle \leq 0$$

$$\langle -(\mathbf{x}_k + \mathbf{x}_l), \mathbf{y}_i \rangle = \langle -\mathbf{x}_k, \mathbf{y}_i \rangle + \langle -\mathbf{x}_l, \mathbf{y}_i \rangle \leq 2\left(\frac{1}{6} + \frac{1}{5}\right) \leq \frac{4}{5} - \frac{2}{30}.$$

Multiplying all these inequalities by 1.01 keeps all inner products $\leq \frac{4}{5} - \frac{1}{40}$; hence, the claim is proved.

Thus, the norm Φ_1 determined by \mathcal{B}_1 has the desired properties of the theorem, except that it is not strictly convex. To finish the proof, take \mathcal{B}_2 to be the intersection of the closed half-spaces $Q_i = \{\mathbf{x} : \langle \mathbf{x}, \mathbf{y}_i \rangle \leq \frac{4}{5}\}$. Now $\mathcal{B}_1 \subseteq \mathcal{B}_2$, and their common boundary points are exactly $\{\pm \mathbf{x}_i : 1 \leq i \leq m\}$, so Claim 2 of Theorem 2.4 applies to give a body \mathcal{B} that is strictly convex and symmetric about $\mathbf{0}$, with $\mathcal{B}_1 \subseteq \mathcal{B} \subseteq \mathcal{B}_2$. The norm Φ determined by \mathcal{B} has the required property. \square

For the Inner Product Problem with no side conditions, there is *no upper bound at all on m* when $n \geq 3$. To see this, write $\mathbb{R}^3 = \{(x, y, z)\}$ and take for any fixed m the set $X = \{\mathbf{x}_i : 1 \leq i \leq m\}$ where all \mathbf{x}_i lie in the plane $z = 1$ and form an equilateral m -gon centered at $(0, 0, 1)$, say $\mathbf{x}_i = (\cos \frac{2\pi j}{m}, \sin \frac{2\pi j}{m}, 1)$ for $1 \leq j \leq m$. Now we can find m lines $\{l_i : 1 \leq i \leq m\}$ lying in the plane $z = 1$, which separate each \mathbf{x}_i from all the other \mathbf{x}_i , e.g.,

$$l_i : x(\cos \frac{2\pi j}{m}) + y(\sin \frac{2\pi j}{m}) = 1 - \varepsilon,$$

for small enough positive ε ; see Figure 3.1.

Now let $Y = \{\mathbf{y}_i\}$, where \mathbf{y}_i is a unit vector perpendicular to the plane H_i determined by the line l_i and the point $(0, 0, 0)$, and lying on the same side of H_i as \mathbf{x}_i does. Then (3.7a)–(3.7b) clearly hold since H_i separates \mathbf{x}_i from all $\{\mathbf{x}_j : j \neq i\}$. (In this example, the hyperplane $z = \frac{1}{2}$ separates all points in X from $\mathbf{0}$.)

Finally, we pose the problem of whether the stronger condition (1.1) on sums of unit vectors implies a polynomial upper bound on their number, when no balancing condition is present.

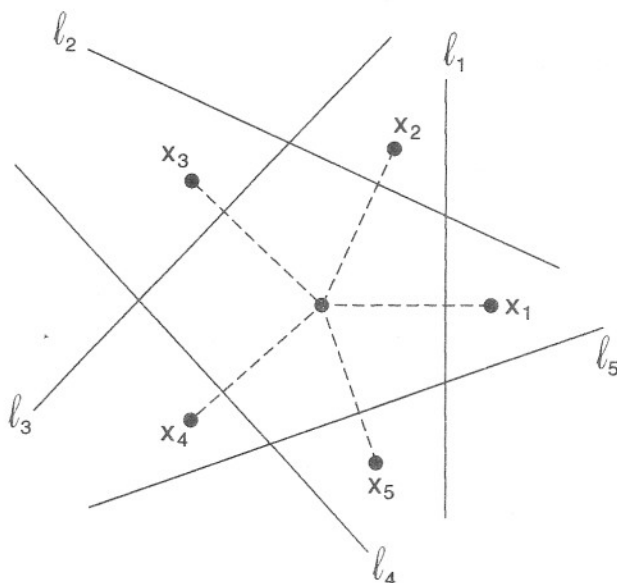


FIGURE 3.1. Lines separating each vertex of a regular m -gon from the other vertices are used in showing the necessity of same side condition in the Inner Product Problem.

PROBLEM 3.7. For a general norm (\mathbb{R}^n, Φ) and a set S of Φ -unit vectors, does the condition

$$\Phi \left(\sum_{j \in J} \mathbf{x}_j \right) \leq 1$$

for all $J \subseteq S$ imply a polynomial bound $p(n)$ on $|S|$?

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REFERENCES

1. M. Alfaro, M. Conger, K. Hodges, A. Levy, R. Kochar, L. Kuklinski, Z. Mahmood, and K. von Haam, *The structure of singularities in Φ -minimizing networks in \mathbb{R}^n* , Pacific J. Math. **149** (1991), 201–210.
2. Y. D. Burago and V. A. Zalgaller, *Geometric inequalities*, Springer-Verlag, New York, 1988.
3. L. Danzer and B. Grünbaum, *Über zwei Probleme bezüglich konvexen Körper von P. Erdős und V. L. Klee*, Math. Z. **79**, (1962), 95–99.
4. L. Danzer, B. Grünbaum, and V. Klee, *Helly's theorem and its relatives*, Convexity, Proc. Sympos. Pure Math, vol. VII, Amer. Math. Soc., Providence, RI, 1963, pp. 101–181.
5. M. M. Day, *Normed linear spaces*, Springer-Verlag, Berlin, 1958.

6. P. Erdős and Z. Füredi, *The greatest angle among n points in the d -dimensional Euclidean space*, Ann. Discrete Math. **17** (1983), 275–283.
7. P. Gritzmann, *Finite packungen und Überdeckungen*, Habilitationsschrift, Universität Siegen, 1984.
8. P. Gritzmann and J. Wills, *Finite packing and covering*, Studia Sci. Math. Hungar. **21** (1986), 149–162.
9. H. Groemer, *Abschätzungen für die Anzahl der Konvexen Körper, die einen konvexen Körper berühren*, Monatsh. Math. **65** (1961), 74–81.
10. B. Grünbaum, *Strictly antipodal sets*, Israel J. Math. **1** (1963), 5–10.
11. F. K. Hwang, *A primer of the Euclidean Steiner Problem*, Proc. NATO Advanced Research Workshop on Topological Network Design, 1991, (to appear).
12. Robert B. Kusner, R. K. Guy, *An olla-podrida of open problems, often oddly posed*, Amer. Math. Monthly **90** (1983), 196–199.
13. G. Lawlor and F. Morgan, *Paired calibrations applied to soap films, immiscible fluids, and surfaces and networks minimizing other norms*, preprint, 1991.
14. A. Levy, *Energy-minimizing networks meet only in threes*, J. Undergrad. Math. **22** (1990), 53–59.
15. M. S. Mel'nikov, *Dependence of volume and diameter of sets in n -dimensional Banach space*, Uspekhi Math. Nauk. **18** (1963), 165–170. (Russian)
16. F. Morgan, *The cone over the Clifford torus in \mathbb{R}^4 is Φ -minimizing*, Math. Ann. **289** (1991), 341–354.
17. —, *Minimal surfaces, crystals and norms on \mathbb{R}^n* , Proc. 7th Ann. Sympos. on Computational Geometry, ACM Press, 1991, pp. 204–213.
18. —, *Minimal surfaces, crystals, shortest networks and undergraduate research*, Math. Intelligencer, 1992 (to appear).
19. —, *Riemannian geometry: A beginner's guide*, Jones & Bartlett (to appear).
20. C. M. Petty, *Equilateral sets in Minkowski spaces*, Proc. Amer. Math. Soc. **29** (1971), 369–374.
21. R. A. Rankin, *The closest packing of spherical caps in n dimensions*, Proc. Glasgow Math. Assoc. **2** (1955), 139–144.
22. C. V. Robinson, *Spherical theorems of Helly type and congruence indices of spherical caps*, Amer. J. Math. **64** (1942), 260–272.
23. B. Simon, *Fifteen problems in mathematical physics*, Perspectives in Mathematics, Birkhauser-Verlag, Basel, 1984, pp. 423–454.
24. J. Taylor, *The structure of singularities in soap-bubble-like and soap-film-like minimal surfaces*, Ann. of Math. (2) **103** (1976), 489–539.
25. —, *Crystalline variational problems*, Bull. Amer. Math. Soc. **84** (1978), 568–588.
26. P. Urysohn, *Mittlere Breite und Volumen der konvexen Körper in m -dimensionalen Räume*, Mat. Sb. SSSR **31** (1924), 477–486.
27. J. H. van Lint, *Introduction to coding theory*, Springer-Verlag, New York, 1982.

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