

HAJNAL'S TRIANGLE-FREE GAME AND EXTREMAL GRAPH PROBLEMS

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Abstract: Starting with the empty graph on n points, two players alternately pick edges with the restriction that no player may complete a triangle. The score is the total number of edges drawn and the first player's objective is to obtain as high score as possible. It will be shown that the first player can achieve a score of $\Omega(n \log n)$.

The result will follow from a lower bound on the minimum number of edges in a maximal triangle-free graph containing a large matching. More generally, we determine the asymptotic behavior of the minimum number of edges in maximal triangle-free graphs containing a matching of size $\lfloor n/2 \rfloor$ and having each vertex valency $\leq D$.

1. The triangle-free game

András Hajnal proposed the following game. Starting with the empty graph on n points for some $n \geq 3$, two players, \mathcal{A} and \mathcal{B} , alternately draw edges. The only restriction is that they are not allowed to pick an edge which completes a triangle with two previously chosen edges and the loser is the player who cannot move. The problem is to determine the winner as a function of n and give a winning strategy. Note that the difference between this game and the usual Ramsey-type games is that we do not distinguish edges chosen by \mathcal{A} and \mathcal{B} .

The winning strategy is known only for small values of n ; namely, \mathcal{A} wins if $n = 6$ and \mathcal{B} wins in $3 \leq n \leq 5$ or $7 \leq n \leq 9$. However, in [S], a complete analysis is given for the *connected* version of the game. In this version, the players must obey the additional rule that, after each pick, the graph of chosen edges must consist of one connected component. In other words, except the very first pick of \mathcal{A} , the players are not allowed to choose edges connecting two isolated vertices.

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THEOREM 1.1. [S] \mathfrak{A} wins the connected version of the triangle-free game if and only if n is even.

2. The length of the game

The winning strategy for the connected version described in [S] allows the loser to play essentially as long as possible in a triangle-free graph; i.e., the loser cannot move only after $n^2/4 - O(n)$ steps. This motivated the investigation of the following variant. The *score* of the game is the total number of edges drawn. The aim of player \mathfrak{A} is to achieve a high score; on the other hand, player \mathfrak{B} tries to keep the score low. The main result of this note is the following. (All logarithms are of base 2.)

THEOREM 2.1. \mathfrak{A} can score $(n \log n)/2 - 2n \log \log n + O(n)$.

On the other hand, Paul Erdős [E] proved that \mathfrak{B} can always finish the game in $n^2/5$ steps.

It is easy to see (cf. Lemma 5.1) that \mathfrak{A} can achieve that the graph created by the players contains a matching of size $\lfloor n/2 \rfloor$. So Theorem 2.1 follows from Theorem 2.2:

THEOREM 2.2. If a maximal triangle-free graph $G(V, E)$ contains a matching of size ν , then G has at least $\nu \log \nu - 4\nu \log \log \nu + \Theta(n)$ edges.

3. A maximal triangle-free graph with a large matching

The following example shows that Theorem 2.2 is asymptotically best possible.

LEMMA 3.1. There exists a maximal triangle-free graph $G(V, E)$ containing a matching of size $\lfloor n/2 \rfloor$ and with $(n \log n)/2 + (n \log \log n)/4 + O(n)$ edges.

Proof. First, let us suppose that n is even. Let m be the smallest integer such that $4m + \binom{2m}{m} \geq n \geq 14$. Then $m = (\log n)/2 + (\log \log n)/4 + O(1)$. Let $V = X \cup Y \cup Z$, $Z = \{a_i, b_i : 1 \leq i \leq n/2 - 2m\}$, $|X| = |Y| = 2m$. We define the edge set of G as follows. X and Y are independent sets and the only edges in Z are the pairs $\{(a_i, b_i) : 1 \leq i \leq n/2 - 2m\}$. There are no edges between Y and Z and there is a complete bipartite graph between X and Y . Finally, each $z \in Z$ is connected to an m -element subset $N(z) \subset X$ such that $N(z_1) \neq N(z_2)$ for all $z_1 \neq z_2$, and $N(a_i) = X \setminus N(b_i)$ for all $1 \leq i \leq n/2 - 2m$. Clearly, G satisfies the conditions of the lemma.

In the case when n is odd, we perform the above construction on $n - 1$ vertices and, with the addition of $< n/2$ edges incident to the last point, we augment the graph to a maximal triangle-free one. ■

In case of $\nu < \lfloor n/2 \rfloor$, we perform the above example on $2\nu - 6$ vertices, and add the vertices a, b, x and the set W to the vertex set, where $|W| = n - 2\nu + 3$. Join

a to all a_i and Y , join b to all b_j and to Y , join x to the vertices of X , and join every vertex in W to a , b and x . The obtained graph has matching number ν and $2\nu \log \nu + (\nu \log \log \nu)/2 + O(n)$ edges.

4. Degree restrictions

If a triangle-free graph contains an almost perfect matching, then the maximal valency of points is at most $(n+1)/2$. Hence a lower bound on the number of edges in maximal triangle-free graphs with maximal valency $\leq (n+1)/2$ provides a lower bound for the score \mathfrak{A} can achieve. Partly, this argument served as motivation in [FS] for the study of the function $F(n, D)$, where $F(n, D)$ denotes the minimum number of edges in maximal triangle-free graphs with each vertex having valency $\leq D$. In the range $D = \Omega(n)$ the following result was proved.

THEOREM 4.1. [FS] *There exists a monotone decreasing, piecewise linear, right-continuous function $K(c)$ defined on the interval $(0, \infty)$ such that the points of discontinuities of $K(c)$ are all rational and are included in a sequence $c_1 > c_2 > \dots \rightarrow 0$ and, for all $c \neq c_i$,*

$$F(n, cn) = K(c)n + o(n).$$

Moreover, for each $c^* > 0$, the determination of $K(c)$ on (c^*, ∞) is a finite problem (by solving finitely many linear programming problems).

This result is related to a problem of Duffus and Hanson [DH]. They investigated the following more general problem: Determine $E(n, k, \delta)$, the minimum number of edges of a maximal K_k -free graph on n vertices with minimum degree δ .

Theorem 4.1 gives only a linear lower bound for the score \mathfrak{A} can achieve by creating a matching of size $\lfloor n/2 \rfloor$. (To be more specific, the value of $F(n, D)$ is known exactly for $D \geq (n-2)/2$; in particular, $F(n, (n+1)/2) = 3n - 15$.) However, it was proven that maximal triangle-free graphs with $F(n, D)$ edges must have an independent set of size $n - o(n)$ and, consequently, cannot contain a matching of size $\Omega(n)$. The presence of a large matching enables us to prove the nonlinear lower bound of Theorem 2.2.

Similarly to the above argument, we can define the function $M(n, D)$ as the minimum number of edges in a maximal triangle-free graph containing a matching of size $\lfloor n/2 \rfloor$ and having maximal valency $\leq D$. The second part of the paper is devoted to the asymptotic evaluation of $M(n, cn)$.

THEOREM 4.2. *For all $c > 0$, $(n \log n)/2 - 2n \log \log n + O(n) \leq M(n, cn) \leq (n \log n)/2 + (n \log \log n)/4 + O(n)$.*

We conjecture that for fixed $c > 0$, $|M(n, cn) - (n \log n)/2 - (n \log \log n)/4| = O(n)$.

5. Forcing a large matching

LEMMA 5.1. *Either player can achieve that the created graph contains a matching of size $\lfloor n/2 \rfloor$.*

Proof. We present a possible strategy for \mathfrak{A} ; the strategy for \mathfrak{B} is similar. Let X_i denote the set of non-isolated vertices after \mathfrak{A} 's i^{th} move. The strategy is to ensure that as long as $|X_i| < n$, $|X_i|$ is even and there is a perfect matching in X_i . This condition is satisfied after \mathfrak{A} 's first move (with $|X_1| = 2$). Suppose that the condition holds for X_i ; we shall give an appropriate response for each possible i^{th} move of \mathfrak{B} . If \mathfrak{B} 's i^{th} move connects two vertices in X_i or two isolated vertices then \mathfrak{A} connects two isolated vertices; if there are less than two isolated vertices left, \mathfrak{A} can pick any edge not completing a triangle. On the other hand, if \mathfrak{B} 's i^{th} move connects a vertex in X_i and an isolated vertex y then \mathfrak{A} connects y to an isolated vertex. Clearly, the strategy ensures that the resulting graph contains a matching of size $\lfloor n/2 \rfloor$. ■

6. Proof of the lower bound

Here we prove Theorem 2.2. Let $\{(a_i, b_i) : 1 \leq i \leq \nu\}$ be a matching in G and let $W = \{a_i, b_i : \deg(a_i) \geq d \vee \deg(b_i) \geq d\}$. We also add the unmatched points to W . We shall choose the value of the parameter d later. Also, we choose the notation such that $V \setminus W = \{a_i, b_i : 1 \leq i \leq s\}$ for some $s \leq \nu$. For $i \leq s$, let $A_i \subset W$ and $B_i \subset W$ the set of neighbours of a_i and b_i in W , respectively. We shall give a lower bound for

$$e = \sum_{i=1}^s (|A_i| + |B_i|);$$

clearly, this will be a lower bound for $|E|$ as well.

For each $i \leq s$, $A_i \cap B_i = \emptyset$ since G is triangle-free. Moreover, for fixed $i \leq s$, there are $2^{|W|-|A_i|-|B_i|}$ sets $X \subset W$ such that $A_i \subset X$ and $B_i \subset W \setminus X$. On the other hand, for fixed $X \subset W$, there are at most d^2 indices $i \leq s$ such that $A_i \subset X$ and $B_i \subset W \setminus X$. To see this, observe that if $A_i, A_j \subset X$ and $B_i, B_j \subset W \setminus X$ for some $i \neq j$ then, from the maximality of G , there is a path of length ≤ 2 from a_i to b_j in $V \setminus W$. However, since the valency of vertices in $V \setminus W$ is $< d$, the number of vertices reachable via a path of length ≤ 2 from a_i is $< d^2$.

Thus, counting the number of pairs (i, X) , $1 \leq i \leq s$, $A_i \subset X \subset W \setminus B_i$ two ways, we obtain

$$\sum_{i=1}^s 2^{|W|-|A_i|-|B_i|} < 2^{|W|} d^2.$$

Since the function $f(x) = 2^{-x}$ is convex, Jensen's inequality implies $s 2^{-(e/s)} < d^2$, i.e.

$$e > s \log s - 2s \log d. \quad (1)$$

Now let us choose $d = \log^2 \nu$. If $s \geq \nu - (2\nu/\log \nu)$ then (1) gives $|E| \geq e > \nu \log \nu - 4\nu \log \log \nu + O(\nu)$, and we are done. Otherwise, there are at least $2\nu/\log \nu$ vertices in G with valency $\geq \log^2 \nu$ and $|E| \geq \nu \log \nu$. ■

7. A maximal triangle-free graph with a large matching and small maximum degree

Theorem 2.2 provides the lower bound for $M(n, cn)$. The upper bound is proven in the next theorem.

THEOREM 7.1. *For $0 < c \leq 1/2$, there exists a function $f(c)$ such that $M(n, cn) \leq (n \log n)/2 + (n \log \log n)/4 + f(c)n + o(n)$.*

Proof. Suppose that n is even. (In the case when n is odd, we perform the construction on $n - 1$ vertices and connect the unmatched point to all vertices in X .) Given c , let q be a prime power such that $2qc > 1$. Let p be the smallest integer such that $\binom{2p}{p} \geq 2(q+1)$ and let l be the smallest integer such that $\binom{2l}{l} > \lceil n/(q^2 + q + 1) \rceil$. Let $P_i, P'_i, L_i, L'_i, Z_i, 1 \leq i \leq q^2 + q + 1$, be pairwise disjoint sets such that $|P_i| = |P'_i| = p$, $|L_i| = |L'_i| = l$ for all i . Moreover, each Z_i has an even number of elements, $0 \leq |Z_i| - |Z_j| \leq 2$ for all $1 \leq i < j \leq q^2 + q + 1$, and

$$V = \bigcup_{i=1}^{q^2+q+1} P_i \cup \bigcup_{i=1}^{q^2+q+1} P'_i \cup \bigcup_{i=1}^{q^2+q+1} L_i \cup \bigcup_{i=1}^{q^2+q+1} L'_i \cup \bigcup_{i=1}^{q^2+q+1} Z_i$$

is an n -element set. Heuristically, after defining P_i, P'_i, L_i, L'_i , we distribute the remaining points of V into the sets Z_i as evenly as possible. We shall define a maximal triangle-free graph G on the set V .

The sets $X = \bigcup_i P_i \cup \bigcup_i L_i$ and $Y = \bigcup_i P'_i \cup \bigcup_i L'_i$ are independent in G and the only edges in $Z = \bigcup_i Z_i$ are that of a perfect matching $\{(a_j, b_j) : 1 \leq j \leq |Z|/2\}$. Moreover, there is no edge between distinct Z_i 's. Each $z \in Z$ is connected to an l -element set $N_1(z)$ in one of the $(L_i \cup L'_i)$'s and to p -element sets in $q+1$ of the $(P_i \cup P'_i)$'s. We denote the union of these p -element sets by $N_2(z)$, $|N_2(z)| = p(q+1)$. $N_1(z)$, $N_2(z)$, and the edges between X and Y are defined by the following rules.

Let p_1, \dots, p_{q^2+q+1} and l_1, \dots, l_{q^2+q+1} be the points and lines of a projective plane of order q , respectively. If $z \in Z_i$ then $N_1(z) \subset (L_i \cup L'_i)$ such that $N_1(z)$ contains points from both L_i and L'_i . We also require that $N_1(z_1) \neq N_1(z_2)$ for all $z_1 \neq z_2$ and $N_1(a_j) = (L_i \cup L'_i) \setminus N_1(b_j)$ for $a_j, b_j \in Z_i, 1 \leq j \leq |Z|/2$. Moreover, $z \in Z_i$ is connected to p points exactly in those $(P_k \cup P'_k)$ for which $p_k \in l_i$ in the projective plane. We require that $N_2(z)$ contains points from both P_k and P'_k . Also, if $a_{j_1}, a_{j_2} \in Z_i$ then $N_2(a_{j_1}) = N_2(a_{j_2})$ and $N_2(a_j) \cap N_2(b_j) = \emptyset$ for all $1 \leq j \leq |Z|/2$. For z_1, z_2 in different Z_i 's, we require that they are not connected to the same p -element set

in any of the $(P_k \cup P'_k)$'s. After that, we add those edges of the complete bipartite graph $K(X, Y)$ to G which do not complete a triangle with some $z \in Z$. In particular, $x \in L_i$ is connected to all vertices in $\bigcup_{j \neq i} L'_j \cup \bigcup_{p_k \notin l_i} P'_k$ and $y \in L'_i$ is connected to all vertices in $\bigcup_{j \neq i} L_j \cup \bigcup_{p_k \notin l_i} P_k$.

First, we check that G is indeed a triangle-free graph. The restrictions of G to $X \cup Y$ and to Z are bipartite. There are no triangles with two vertices in Z since we ensured $N_1(a_j) \cap N_1(b_j) = \emptyset$ and $N_2(a_j) \cap N_2(b_j) = \emptyset$. Also, it is obvious that there are no triangles with one vertex in Z . Next, we prove that G is maximal. If $z_1, z_2 \in Z_i$ then either they are paired in the perfect matching in Z or they have a common neighbour in $(L_i \cup L'_i)$. If $z_1 \in Z_i, z_2 \in Z_j, (i \neq j)$ then they have a common neighbour in $(P_k \cup P'_k)$ where p_k is the intersection of the lines l_i, l_j in the projective plane. $z \in Z_i, x \in (L_i \cup L'_i)$ are either connected or x is connected to the pair of z in the perfect matching. The same argument works for $z \in Z_i, x \in (P_k \cup P'_k)$ if $p_k \in l_i$. For the remaining points x of $X \cup Y, z$ and x have a common neighbour in $(L_i \cup L'_i)$. Pairs of points in X (Y) have common neighbours in L'_j (L_j), respectively, for some appropriate j . Finally, the definition of G on $X \cup Y$ clearly implies that $x \in X, y \in Y$ are either connected or they have some common neighbour $z \in Z$. Also, it is easy to see that there is a perfect matching in the restriction of G to $X \cup Y$.

If we choose q in the interval $(1/2c, 1/c)$ then $p = O(1)$ and $l = (\log n)/2 + (\log \log n)/4 + O(1)$. Hence $|X| = |Y| = o(n)$ and the number of edges in G is $< n(l + p(q+1)) + o(n) = (n \log n)/2 + (n \log \log n)/4 + O(n)$, as required. The valencies of points in Z are of $o(n)$. Vertices in the $(L_i \cup L'_i)$'s are of degree $n/2(q^2 + q + 1) + o(n)$ and vertices in the $(P_i \cup P'_i)$'s have degree $(q+1)n/2(q^2 + q + 1) + o(n) < n/2q$ if n is large enough. Hence the maximal valency in G is $\leq cn$.

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