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The grid revisited*

Paul Erdős, Zoltán Füredi, János Pach and Imre Z. Ruzsa

Mathematical Institute of the Hungarian Academy of Sciences, P.O.B. 127, Budapest 1364 Hungary

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Abstract

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We construct a set of n points (i) on the unit sphere S^{d-1} ($d \geqslant 4$) so that they determine $o(n)$ distinct distances and (ii) in the plane, in general position, so that they determine $o(n^{1+\varepsilon})$ distinct distances for any $\varepsilon > 0$. We also prove that if P is a set of n points in a disk of radius n such that the minimum distance between them is 1, and $|P|/n \to \infty$, then the set of angles determined by these points is everywhere dense in $[0, 2\pi]$.

1. Introduction

There are many extremal problems in discrete and combinatorial geometry whose optimal solution is provided by a *grid*, i.e., by a piece of

$$\Lambda = \{n_1 v_1 + n_2 v_2 + \cdots + n_d v_d \colon n_1, n_2, \ldots, n_d \in \mathbb{Z}\},$$

where $v_1, v_2, \ldots, v_d$ is a system of d linearly independent vectors in Euclidean d-space.

Perhaps the most famous example is the problem of densest packing of disks. According to an old theorem of Thue [21, 22], the density of any packing of nonoverlapping unit disks in the plane is at most $\pi/\sqrt{12}$, and this bound is attained for a system of disks whose centers form a hexagonal lattice, i.e., a grid Λ with $v_1 = (1, 0)$ and $v_2 = (1/2, \sqrt{3}/2)$. This result was generalized by Rogers [18] and Fejes Tóth [8, 9] to packings of translates of any centrally symmetric convex set in the plane, and the optimum is always attained for a suitable lattice packing. The corresponding problems in higher dimensions are under persistent attack. (See, e.g., [5, 7, 16] for recent surveys on the subject, and Wu-Yi Hsiang [14] for a new attempt.) We recall another related problem of Fejes Tóth [8, 9], for which the optimal configuration is conjectured to be a piece of the hexagonal lattice: $p_1, p_2, \ldots, p_n$ be n points in the plane whose

Correspondence to: János Pach, Courant Institute, NYU, New York, NY 10012, USA.

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minimum distance is 1. Minimize

$$\sum_{1 \leqslant i < j \leqslant n} d(p_i, p_j),$$

where $d(p_i, p_j)$ denotes the distance between p_i and p_j.

Some 45 years ago, Erdős [3] initiated the investigation of a branch of problems of a completely different kind. In particular, he asked:

(i) What is the maximum number of times that the same distance can occur among n points in a fixed metric space M?

(ii) What is the minimum number of distinct distances determined by n points of M? In other words, determine or estimate

$$f^M(n) = \max \max_{\alpha > 0} |\{\{p_i, p_j\}: d(p_i, p_j) = \alpha\}|, \tag{1}$$

$$g^M(n) = \min |\{d(p_i, p_j): 1 \leqslant i < j \leqslant n\}|, \tag{2}$$

where the max and min are taken over all n-element point sets $\{p_1, p_2, \dots, p_n\} \subseteq M$. If $M = \mathbb{R}^d$, the Euclidean d-space, then we shall write $f_d(n)$ and $g_d(n)$ for $f^M(n)$ and $g^M(n)$, respectively. Spencer et al. [20] and (with a more elegant proof and with a better constant $C > 0$) Clarkson et al. [2] showed that $f_2(n) \leqslant Cn^{4/3}$. However, according to Erdős' conjecture, the maximum is attained for some piece of a grid; hence, $f_2(n) \leqslant n^{1 + C/\log\log n}$. In the same spirit, Erdős conjectured that the extremal configuration $\{p_1, p_2, \dots, p_n\} \subseteq \mathbb{R}^2$ realizing the least number of distinct distances has grid-like structure and $g_2(n) \geqslant cn/\sqrt{\log n}$ for some $c > 0$. The best result in this direction is due to Chung et al. [1]. A similar phenomenon appears to occur in higher dimensions. For instance, in 3-dimensional space the conjectured bounds are $f_3(n) \leqslant C'n^{4/3} \log\log n$ and $g_3(n) \geqslant c'n^{2/3}$.

A related result of Pach and Sharir [17] states that the maximum number of right-angle triangles spanned by n points in the plane is $O(n^2 \log n)$, and this bound is asymptotically tight, as is shown by the example of a $\sqrt{n}$ by $\sqrt{n}$ piece of the integer lattice.

In the following two sections we are going to discuss two very similar extremal problems. In both cases we shall present a construction which is probably close to being optimal. The first construction can be obtained from a grid by choosing a suitable small subset of its vertices, while the second one can be derived from such a subset by orthogonal projection into a plane in general position. In the last section we shall consider a somewhat different question whose solution also requires a grid. The theorem in the last section has also been proved independently by Noga Alon.

2. Distinct distances among points on a sphere

Moser [15] conjectured that there exists a constant c such that, given any set of n points on the unit sphere S^2 in Euclidean 3-space, no distance can occur among them more than cn times. This has been disproved by Erdős et al. [6]. However, it seems plausible that the following weaker version of Moser's conjecture is still true. There is a positive constant c' such that any set of n points on S^2 determines at least $c'n$ distinct distances. Our next theorem shows that the same result cannot hold in higher dimensions.

Let S^{d-1} denote the unit sphere in Euclidean d-space. For any finite subset $P \subseteq S^{d-1}$, let $g(P)$ be the number of distinct distances determined by P, i.e., the number of those reals $\alpha > 0$ for which one can find two elements $p_1, p_2 \in P$ at a distance α from each other.

Theorem 2.1. Let $d \geqslant 4$. *Then there exists a constant c_d with the property that, for any $n > 2$, one can find an n-element point set $P \subseteq S^{d-1}$ determining*

$$g(P) \leqslant \begin{cases} c_4 \dfrac{n}{\log \log n} & \text{if } d=4, \\ c_d n^{2/(d-2)} & \text{if } d>4 \end{cases}$$

distinct distances.

Proof. Let $d \geqslant 4$, $n = m^{d-2}/d$, and consider the set L of all lattice points $(x_1, \ldots, x_d)$, with integer coordinates $0 \leqslant x_i < m$. The number of distinct distances determined by L is at most dm^2, because there are at most that many numbers of the form $(\sum_{i=1}^{d}(x_i - x_i')^2)^{1/2}$. In particular, there is a sphere around the origin which contains at least $|L|/dm^2 = m^{d-2}/d = n$ elements of L. Letting P denote the set of these points,

$$g(P) \leqslant dm^2 = d^{d/(d-2)} n^{2/(d-2)},$$

which proves the theorem for $d > 4$.

If $d = 4$, then we have to use a little more delicate number-theoretic argument. A well known theorem of Fermat and Lagrange states that any natural number k can be expressed as the sum of four squares, i.e.,

$$k = x_1^2 + x_2^2 + x_3^2 + x_4^2,$$

where every x_i is an integer. Jacobi obtained a far-reaching generalization of this result (see, e.g., [13]). He proved that there exists a constant $c > 0$ such that the number of different solutions of Eq. (1) is at least $c\sigma(k)$, where

$$\sigma(k) = \sum_{d/k} d,$$

i.e., the sum of all positive divisors of k. On the other hand, it is well known that there is a constant $c' > 0$ such that $\sigma(k) \geqslant c' k \log \log k$ for infinitely many integers k.

Let k be such an integer, and set $m = \lceil \sqrt{k} \rceil$. Then there are at least $c\sigma(k) \geqslant cc' k \log \log k \geqslant c'' m^2 \log \log m$ points of L on the sphere of radius $\sqrt{k}$ around the origin. Let P denote the set of these lattice points, $|P|=n=\lceil c''m^2 \log\log m \rceil$. Now we obtain, as before, that

$$g(P) \leqslant 4m^2 \leqslant c_4 \frac{n}{\log\log n}$$

for a suitable $c_4 > 0$.

Note that, strictly speaking, we have proved only that our theorem is true for infinitely many values n, but our argument easily extends to the general case. □

3. Distinct distances among points in the plane in general position

We say that a set P of n points in the plane is *in general position*, if no 3 points are on a line and no 4 on a circle. Obviously, this implies that $g(P)$, the number of distinct distances determined by P, is at least $(n-1)/3$. Erdős [4] posed the following question: Is it true that

$$\lim_{n\to\infty} \min_{|P|=n} \frac{g(P)}{n} = \infty, \tag{3}$$

where the min is taken over all n-element point sets in the plane in general position? This problem is still open. On the other hand, a construction of Erdős et al. [6] shows that

$$\min_{|P|=n} g(P) \leqslant cn^{\log 3/\log 2}.$$

Our next theorem improves this upper bound to be $O(n^{1+\varepsilon})$ for any $\varepsilon > 0$.

Theorem 3.1. *There exists a constant c such that, for any natural number n, one can find an n-element point set P in the plane in general position such that the number of distinct distances determined by P,*

$$g(P) \leqslant n 2^{c\sqrt{\log n}}.$$

Proof. Assume for the sake of simplicity that $n = \lfloor 2^{d(d-2)}/d \rfloor$ for some natural number $d \geqslant 4$. Consider the set L of all lattice points $(x_1, \ldots, x_d)$ with integer coordinates $0 \leqslant x_i < m$, where $m = 2^d$. Then we can show in exactly the same way as in the proof of Theorem 2.1 that there is a sphere in $\mathbb{R}^d$ around the origin, which contains at least $|L|/dm^2 = m^{d-2}/d \geqslant n$ elements of L. Let Q denote the set of these points.

Let $Q-Q=\{p_1-p_2 \colon p_1, p_2 \in Q\}$, i.e., the set of all *vectors determined by* Q. Similarly, $Q+Q=\{p_1+p_2 \colon p_1, p_2 \in Q\}$, the *Minkowski sum* of Q with itself. Observe that every element of $Q-Q$ is a vector $(x_1, \ldots, x_d)$ with integer coordinates $-m < x_i < m$; hence,

$$|Q-Q| < (2m)^d = 2^{(d+1)d} < n 2^{4d} \leqslant n 2^{8\sqrt{\log n}}.$$

Fix a 2-dimensional plane Π in $\mathbb{R}^d$, and, for any $p \in Q$, let p' denote the orthogonal projection of p into Π. Evidently, we can choose Π so as to satisfy the following two conditions:

(i) $p'_1 = p'_2$ if and only if $p_1 = p_2$;

(ii) the point set $P = \{p' \colon p \in Q\} \subseteq \Pi$ is in general position.

Furthermore, in view of the fact that $p_1 - p_2 = p_3 - p_4$ implies $d(p'_1, p'_2) = d(p'_3, p'_4)$, we have

$$g(P) \leqslant |Q - Q| \leqslant n 2^{8\sqrt{\log n}},$$

as required. Dropping the assumption that n is of some special form affects only the constant factor in the exponent. □

Distinct distances in the 3-dimensional space. If we project the above example Q to a 3-dimensional subspace in general position, then the obtained set P_3 has n points, $g(P) \leqslant n \exp(8\sqrt{\log n})$, no three on a line, and no more than $\exp(\log n / \log\log n)$ on a plane. Starting with this construction, and selecting elements randomly, independently, and with uniform distribution, we obtain the following. There exists a $C > 0$ such that for all n and $t < \sqrt{\log n}$, one can find an n-element point set $P_3(n, t) \subset \boldsymbol{R}^3$, with no three on a line, no t on a plane and $g(P_3(n, t)) < n^{1 + C/t}$.

There are other obvious extensions to higher dimensions, and for other independence properties (e.g., no four on a circle), but we were unable to prove the following.

Conjecture. There exists a $t_0 > 0$ such that, for every $\varepsilon > 0$, there exists an n-element point set $P \subset \boldsymbol{R}^3$, with no three on a line, no t_0 on a plane, and $g(P) = O(n^{1+\varepsilon})$.

Distinct vectors in the plane. Given a set P of n points in the plane, let $g^*(P) = |P - P|$, i.e., the number of vectors determined by P. Clearly, $g^*(P) \geqslant g(P)$. In fact, we have proved above a slightly stronger result than Theorem 3.1. We have shown the existence of an n-element point set in the plane in general position with $g^*(P) \leqslant n 2^{c\sqrt{\log n}}$.

However, replacing g by g^*, the validity of (3) follows from some deep results in additive set theory.

Theorem 3.2. *For any $C > 0$, there exists an integer $n_0 = n_0(C)$ such that every n-element point set P in the plane in general position determines at least Cn vectors, provided that $n \geqslant n_0$.*

Proof. Assume that $g^*(P) = |P - P| \leqslant Cn$ for some P. Then, by a result of Ruzsa [19], we have $|P + P| \leqslant C'n$, with a suitable constant C' depending only on C. A well-known result of Freiman [10–12] states that, for any C', there exists a constant C'' with the property that any n-element set P in the plane with $|P+P| \leqslant C'n$ can be covered by the projection of a grid of dimension C' and size $C''n$. That is,

$$P \subseteq \{v_0 + n_1 v_1 + n_2 v_2 + \cdots + n_{C'} v_{C'}:\ 1 \leqslant n_i \leqslant k_i\}$$

for suitable vectors $v_i \in \mathbb{R}^2$ and natural numbers k_i satisfying $\prod_{i=1}^{C'} k_i \leqslant C''n$.

Suppose, without loss of generality, that $k_1 \geqslant n^{1/C'}$. Obviously, we can fix some values $\bar{n}_2, \ldots, \bar{n}_{C'}$ so that

$$v_0 + n_1 v_1 + \bar{n}_2 v_2 + \cdots + \bar{n}_{C'} v_{C'} \in P$$

for at least $n/(k_2 \cdots k_{C'}) \geqslant k_1/C'' \geqslant n^{1/C'}/C''$ different integers n_1. However, the corresponding points of P are all on a line, contradicting our assumption that P is in general position. $\square$

4. Distinct angles determined by a point set

Let P be a set of points in a disk of radius n so that the minimum distance between them is at least 1. If all elements of P are on a line, then every triple determines angle 0 or π. Clearly, $|P| > 2n+1$ implies that some other angles must also occur. Our following result shows that, if $|P|/n \to \infty$, then the set of angles determined by triples of P is everywhere dense in $[0, 2\pi]$.

Theorem 3.3. *For any $\varepsilon > 0$, there exists $C = C(\varepsilon)$ with the following property. Let P be any set of at least Cn points in a disk of radius n such that no two points are closer than 1. Then, for any $0 \leqslant \alpha \leqslant 2\pi$, we can find three elements of P so that the angle determined by them differs from α by at most ε.*

Proof. Suppose that, for some $0 \leqslant \alpha \leqslant 2\pi$, no triple of P determines an angle which differs from α by at most ε.

Set $k = \lceil 2\pi/(\varepsilon/2) \rceil$, and let $v_1^1, v_1^2, \ldots, v_1^k$ be vectors of length 1/2 such that v_1^{i+1} can be obtained from v_1^i by counterclockwise rotation through angle $2\pi/k$. Further, let v_2^i be a vector of length 1/2 orthogonal to v_1^i. Consider the square grids

$$\Lambda^i = \{n_1 v_1^i + n_2 v_2^i:\ n_1, n_2 \in \mathbb{Z}\}, \quad 1 \leqslant i \leqslant k.$$

Joining two vertices of Λ^i by a straight-line segment if their distance is 1/2, we obtain a decomposition of the plane into squares (*cells*). Suppose, without loss of generality, that no point of P is on the boundary of any cell of Λ^i $(1 \leqslant i \leqslant k)$. A cell of Λ^i is called nontrivial if it contains an element of P. (Obviously, no cell can contain more than one point of P.)

Given any $p \in P$, $1 \leqslant i \leqslant k$, let $r^i(p)$ denote the ray starting from p and pointing in the direction of v_1^i. We shall say that p is of *type* i if $r^i(p)$ intersects at most $10/\varepsilon$ nontrivial cells of Λ^i. Obviously, p can have more than one type.

Choose any two (not necessarily distinct) indices i and j so that the angle between v_1^i and v_1^j differs from α by at most $\varepsilon/2$. We claim that any $p\in P$ is either of type i or of type j. Suppose the contrary. Then, letting $p^i, p^j \in P$ be chosen from the last nontrivial cells of Λ^i and Λ^j which are crossed by $r^i(p)$ and $r^j(p)$, respectively, we would obtain by an easy calculation that

$$|\alpha - \sphericalangle p^i p p^j| < \varepsilon,$$

a contradiction.

Clearly, the total number of points in P having type i is at most $2n(10/\varepsilon)$ (because Λ^i consists of $2n$ 'rows' of cells, and in each 'row' there are at most $10/\varepsilon$ such points). On the other hand, every point of P has at least $k/2$ different types (because there are no two consecutive elements missing from the set of types of $p\in P$ in the circular sequence $i+t(j-i)$, $t=0,1,2,\ldots$). Thus,

$$|P|\frac{k}{2} \leqslant k2n\frac{10}{\varepsilon},$$

and $|P| \leqslant (40/\varepsilon)n$. Hence, the theorem is true with $C(\varepsilon)=40/\varepsilon$. □

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