Discrete Mathematics 127 (1993) 187–207
North-Holland


# Intersecting designs from linear programming and graphs of diameter two*

Zoltán Füredi
*Mathematical Institute of the Hungarian Academy of Sciences, P.O.B. 127, Budapest 1364, Hungary*

Received 31 October 1990
Revised 29 May 1991

*Abstract*

We investigate 1-designs (regular intersecting families) and graphs of diameter 2. The optimal configurations are either projective planes or design-like structures closely related to finite geometries. The new results presented here are corollaries of a recent improvement about uniform hypergraphs with maximal fractional matchings. We propose several open problems.

## 1. Introduction

The purpose of this paper is to survey some extremal combinatorial problems where the solution naturally leads to a linear programming problem on an intersecting hypergraph. There are deep connections between combinatorial designs and different branches of algebra. Here we obtain designs as solutions of extremal problems in hypergraph theory, and the defining relations are linear inequalities (i.e. a linear program). In this way we usually have a more relaxed structure, and there is plenty of room for further research. We propose several problems and conjectures.

This paper is organized as follows. In the next section we recall some definitions and introduce notations. Then we investigate the maximum size of an $r$-uniform 1-design. In Section 4 we review recent results and problems concerning fractional matchings of intersecting hypergraphs. The second part of the paper is devoted to graphs of diameter two. We determine $e_2(n, D)$, the minimum number of edges of a graph of diameter 2 with $n$ vertices and with maximum degree at most $D$, for infinitely many small intervals. The proof is contained in Section 6.

*Correspondence to*: Zoltán Füredi, Department of Mathematics, University of Illinois at Urbana-Champaign, 1409 West Green Street, Urbana, IL 61801, USA.
* Research supported in part by the Hungarian National Science Foundation under grant 1812.


*SSDI* 0012-365X(92)00478-7

## 2. Definitions concerning hypergraphs

A *multihypergraph* $\boldsymbol{H}$ is a pair $(V, \mathscr{E})$ where $V$ is a (finite) set, the *vertex set*, and $\mathscr{E}$ is a collection of subsets of $V$, the *edge set*. If $\mathscr{E}$ does not contain multiple edges then $\boldsymbol{H}$ is called a *hypergraph*. For brevity we use the word 'hypergraph' instead of 'multihypergraph' if it does not cause ambiguity. A hypergraph is an *r-graph*, or an *r-uniform* hypergraph if all edges have $r$ elements. The *rank* of $\boldsymbol{H}$ is $r$ if $\max\{|E|: E\in\mathscr{E}(\boldsymbol{H})\}=r$. $\boldsymbol{G}$ is a *subhypergraph* of $\boldsymbol{H}$ if $V(\boldsymbol{G})\subset V(\boldsymbol{H})$ and $\mathscr{E}(\boldsymbol{G})\subset\mathscr{E}(\boldsymbol{H})$. The number of edges containing $v\in V$ is the *degree* of the vertex $v$ and it is denoted by $\deg_{\boldsymbol{H}}(v)$, or briefly by $\deg(v)$. The maximum of $\deg_{\boldsymbol{H}}(v)$ for $v\in V$ is denoted by $D(\boldsymbol{H})$. If every vertex has the same degree $D$, then $\boldsymbol{H}$ is called *D-regular*, or a *1-design*.

A hypergraph is *t-wise s-intersecting* if any $t$ edges have at least $s$ common elements. Instead of $t$-wise 1-intersecting we simply say $t$-wise intersecting, instead of pairwise $s$-intersecting we say $s$-intersecting and the case of pairwise 1-intersecting is abbreviated to *intersecting*. To distinguish these two notions easily, we will write $\hat{t}$-wise intersecting instead of a simple $t$. An $r$-graph $\boldsymbol{H}$ is *r-partite* if the vertex set has a partition $V(\boldsymbol{H})=X_1\cup\cdots\cup X_r$ such that $|X_i\cap E|=1$ holds for all $E\in\mathscr{E}(\boldsymbol{H})$, $1\leqslant i\leqslant r$. We use the notations $\lfloor x\rfloor$ and $\lceil x\rceil$ for the lower and upper integer part of $x$, respectively.

## 3. Intersecting 1-designs

An $r$-uniform hypergraph over $r^2-r+1$ vertices is called a *finite projective plane* of order $r-1$, denoted by $PG(2,r-1)$, if it is an $S(r^2-r+1,r,2)$ Steiner system. Such planes are known to exist if $r-1$ is a prime power or $r=1,2$. Every two edges intersect in exactly one element, so it is a regular, intersecting, $r$-uniform hypergraph (a 1-design). Bollobás [3] and Erdős [11] conjectured that such an intersecting family can have at most $r^2-r+1$ vertices; Lovász [33] proved this to be so. In [21] it was proved that the only extremal configuration is the finite plane. A new proof using association schemes was given by Calderbank [7]. The following two intersecting 1-designs have only $r^2-r$ vertices.

An $r$-graph is called a *truncated projective plane* of order $r-1$ if it is obtained from a $PG(2,r-1)$ by deleting a vertex $v$ and the $r$ edges through $v$.

The 1-design $\boldsymbol{G}$ is a *twisted* plane if $|V(\boldsymbol{G})|=|E(\boldsymbol{G})|=r^2-r$, it is $r$-uniform, every degree is $r$ and the edges cover all pairs. Such a hypergraph is known to exist only for $r\leqslant 4$ (see Fig. 1).

**Theorem 3.1.** *Suppose that $\boldsymbol{H}$ is a regular, intersecting hypergraph of rank $r$. Then either*

(i) *$\boldsymbol{H}$ is a $PG(2,r-1)$, and then $|V(\boldsymbol{H})|=r^2-r+1$, or*
(ii) *$\boldsymbol{H}$ is a truncated projective plane, and then $|V(\boldsymbol{H})|=r^2-r$, or*
(iii) *$\boldsymbol{H}$ is a twisted plane, and then again $|V(\boldsymbol{H})|=r^2-r$, or*

| | | | | | |
|---|---|---|---|---|---|
| 1 | 1 | 1 | · | · | · |
| 1 | 1 | · | 1 | · | · |
| · | · | 1 | 1 | 1 | · |
| · | · | 1 | 1 | · | 1 |
| 1 | · | · | · | 1 | 1 |
| · | 1 | · | · | 1 | 1 |

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | · | · | 1 | · | 1 | · | · | · | · | · |
| 1 | 1 | · | · | · | 1 | · | 1 | · | · | · | · |
| · | · | 1 | 1 | 1 | · | · | 1 | · | · | · | · |
| · | · | 1 | 1 | · | 1 | 1 | · | · | · | · | · |
| · | · | · | · | 1 | 1 | · | · | 1 | · | 1 | · |
| · | · | · | · | 1 | 1 | · | · | · | 1 | · | 1 |
| · | · | · | · | · | · | 1 | 1 | 1 | · | · | 1 |
| · | · | · | · | · | · | 1 | 1 | · | 1 | 1 | · |
| 1 | · | 1 | · | · | · | · | · | 1 | 1 | · | · |
| · | 1 | · | 1 | · | · | · | · | 1 | 1 | · | · |
| 1 | · | · | 1 | · | · | · | · | · | · | 1 | 1 |
| · | 1 | 1 | · | · | · | · | · | · | · | 1 | 1 |

Fig. 1. Incidence matrices of the 3- and the 4-uniform twisted planes.

| | | | | | |
|---|---|---|---|---|---|
| 1 | 1 | 1 | · | · | · |
| 1 | 1 | · | 1 | · | · |
| · | · | 1 | 1 | 1 | · |
| · | · | 1 | 1 | · | 1 |
| 1 | · | · | · | 1 | 1 |
| · | 1 | · | · | 1 | 1 |
| 1 | · | 1 | · | 1 | · |
| 1 | · | · | 1 | · | 1 |
| · | 1 | 1 | · | · | 1 |
| · | 1 | · | 1 | 1 | · |

Fig. 2. Incidence matrix of a 3-uniform intersecting 1-design with 6 vertices.

(iii/a) $r=3$, ***H*** *contains a twisted plane*, $\mathscr{E}(\boldsymbol{H})=\{123, 124, 345, 346, 156, 256, 135, 146, 236, 245\}$ (*see Fig.* 2), *and again* $V(\boldsymbol{H})=r^2-r$, *or*

(iv) $|V(\boldsymbol{H})|<r^2-r$.

The above theorem easily follows from a recent result on fractional matchings of hypergraphs. The proof is postponed to the next section. Here we mention some open problems concerning 1-designs.

**Problem 3.2.** Are there twisted planes for $r>4$?

It is easy to see that a twisted plane is a group divisible design, every pair of vertices is covered once except $(r^2-r)/2$ of them which form a perfect matching. As with other symmetric designs, their existence is not clear. Considering the determinant of the incidence matrix it follows that $r$ or $r-2$ is a square. Further constraints about the existence of twisted planes can be found in [32].

Let $h(r) := \max\{|V(\boldsymbol{H})|\colon \boldsymbol{H}$ is an $r$-uniform, intersecting 1-design$\}$. Replacing each edge by $(r+1)$-element sets containing it, we get $h(r) \leqslant h^+(r+1)$, where $h^+$ is defined as $h$ but multiple edges are allowed. This monotonicity is not obvious for the function $h(r)$. In general, let $\partial^k \boldsymbol{H}$ be the (multi)hypergraph defined by $\{K \subset V\colon |K| = k$, there exists an edge $E \in \mathscr{E}(\boldsymbol{H})$ with $E \subset K\}$. Theorem 3.1 and the example $\partial^r PG(2,q)$ (with $r > q \geqslant r/2$) give that both $h^+(r)$ and $h(r)$ are at least $q^2+q+1$ for $r > q$, and hence they are both equal to $r^2 + \mathrm{O}(r^{2-\varepsilon})$.

**Problem 3.3.** Find sharper bounds for $h(r)$. How large is $h(7)$, the first unsolved case?

We have $31 \leqslant h(7) \leqslant 41$ by the above arguments.

We can consider the number of edges instead of the vertices of a 1-design. Let $h(n,r) := \max\{|\mathscr{E}(\boldsymbol{H})|\colon \boldsymbol{H}$ is an $r$-uniform, intersecting 1-design on $n$ vertices$\}$ (with no repeated edges). Frankl [17] proved that $r^{r-o(r)} \leqslant \max_n h(n,r) < r^r e^r$ holds for all $r$. The example $\partial^r PG(2,q)$ with $q \sim (1-\varepsilon)r$ shows that $h(n,r)$ could be as large as $r^{r(1-\mathrm{O}(\varepsilon))}$ (for all $\varepsilon > 0$). The upper bound follows from the trivial inequality $h(n,r) \leqslant \binom{n}{r}$, (here equality holds for $n < 2r$), and from the fact $n \leqslant r^2 - r + 1$.

**Problem 3.4.** Estimate $h(n,r)$. Is it true that $h(n,r) < r^r$ for all $r$ and $n$?

**Problem 3.5.** Determine the maximum cardinality of a $\hat{t}$-wise $s$-intersecting regular hypergraph on $n$ vertices.

Let $R(n,\hat{t},s)$ be the quantity defined in the problem above. Answering a question of Daykin, Frankl [18] showed that $R(n,\hat{t},1) \geqslant 2^n/2^{2^{t+1}-t-1}$ (a positive fraction of $2^n$!). He conjectures that this lower bound is the exact value of $R(n,\hat{t},1)$. On the other hand he proved $R(n,\hat{t},1) < 2^{n-1} b^{-2^{t-3}}$ where $b = (\sqrt{5}-1)/2$. His results (and methods) in all probability can be applied for $\hat{t}$-wise $s$-intersecting families too.

**Problem 3.6.** Determine the maximum cardinality of a $\hat{t}$-wise $s$-intersecting hypergraph on $n$ vertices with a vertex-transitive automorphism group.

Let $T(n,\hat{t},s)$ be the quantity defined in the problem above. Of course, $T \leqslant R$. Frankl [18] proved that its order of magnitude indeed is much less, $T(n,\hat{t},1) = \mathrm{o}(2^n)$ as $n$ tends to infinity and $t \geqslant 4$ fixed. He also conjectures that $T(n,\hat{3},1)$ is only $\mathrm{o}(2^n)$. He obtained in [17] that $T(n,\hat{t},5)2^{-n} \leqslant \exp(-c\sqrt[3]{n})$ for some $c > 0$, and in general for $t \geqslant 6$.

$$(1+\mathrm{o}(1))2^{-n^{(t-1)/t}} < T(n,\hat{t},1)2^{-n} < \left(\frac{\sqrt{5}-1}{2}\right)^{-n^{(t-3)/t}}$$

Here the lower bound holds for all $t \geqslant 3$. There are some improved bounds in [8].

## 4. Fractional matchings of intersecting hypergraphs

A set $T\subset V(\boldsymbol{H})$ is a *transversal* of $\boldsymbol{H}$ if $T\cap E\neq\emptyset$ for each edge $E\in\mathscr{E}(\boldsymbol{H})$. The minimum cardinality of a transversal of $\boldsymbol{H}$ is $\tau(\boldsymbol{H})$, the *transversal number* of $\boldsymbol{H}$. A *fractional transversal* of $\boldsymbol{H}=(V,\mathscr{E})$ is a nonnegative function $t: V\to\boldsymbol{R}^+$ such that $t(E):=\sum_{x\in E}t(x)\geqslant 1$ for all $E\in\boldsymbol{H}$. The *value of t* is defined as

$$|t|=\sum_{x\in V}t(x).$$

The *fractional transversal number*, $\tau^*(\boldsymbol{H})$, is the infimum of $|t|$ over all fractional transversals.

A *matching* is a subfamily of pairwise disjoint edges, the *matching number* $\nu(\boldsymbol{H})$ is the maximum number of edges in a matching in $\boldsymbol{H}$. A *fractional matching* of $\boldsymbol{H}=(V,\mathscr{E})$ is a function $w:\mathscr{E}\to\boldsymbol{R}^+$ such that

$$\sum_{E\ni p}w(E)\leqslant 1 \quad \text{for all } p\in V.$$

The value of $w$ is defined as $|w|=\sum_{E\in\boldsymbol{H}}w(E)$. The *fractional matching number* $\nu^*(\boldsymbol{H})$ is the supremum of $|w|$ over all fractional matchings of $\boldsymbol{H}$.

The duality theorem of linear programming implies that there is an optimal fractional transversal $t$, and an optimal fractional matching $w$ with $|t|=|w|=\nu^*(\boldsymbol{H})$. Observe that $w(E)\equiv 1/D(\boldsymbol{H})$ is always a fractional matching of $\boldsymbol{H}$. Its value is $|\mathscr{E}(\boldsymbol{H})|/D(\boldsymbol{H})$; therefore, $\nu^*(\boldsymbol{H})\geqslant|\mathscr{E}(\boldsymbol{H})|/D(\boldsymbol{H})$, i.e.

$$D(\boldsymbol{H})\geqslant\frac{|\mathscr{E}(\boldsymbol{H})|}{\nu^*(\boldsymbol{H})}. \tag{4.1}$$

It is easy to see that $\nu^*(PG(2,r-1))=r-1+1/r$. Lovász [33] proved that for an intersecting $r$-graph $\boldsymbol{H}$ $\nu^*(\boldsymbol{H})\leqslant r-1+2/(r+1)$ and conjectured $\nu^*(\boldsymbol{H})\leqslant r-1+1/r$. In [18] this conjecture was settled, and recently it was sharpened as follows.

**Theorem 4.1** (Füredi [25]). *Suppose that $\boldsymbol{H}$ is an intersecting hypergraph of rank $r$. Then either*

(i) *$\boldsymbol{H}$ is a $PG(2,r-1)$, and then $\nu^*(\boldsymbol{H})=r-1+1/r$, or*
(ii) *$\mathscr{E}(\boldsymbol{H})$ contains a truncated projective plane, and then $\nu^*(\boldsymbol{H})=r-1$, or*
(iii) *$\boldsymbol{H}$ is a twisted plane, and then $\nu^*(\boldsymbol{H})=r-1$, or*
(iii/a) *$r=3$, $\boldsymbol{H}$ contains a twisted plane, and then $\nu^*(\boldsymbol{H})=r-1$, or*
(iv) *$\nu^*(\boldsymbol{H})\leqslant r-1-1/(r^2-r-1)$.*

**Proof of Theorem 3.1.** Regularity implies $|\mathscr{E}|r\geqslant D|V|$. Multiplying this with (4.1) we get $r^*(\boldsymbol{H})\geqslant|V|$. Then Theorem 3.1 follows from the upper bounds for the fractional matching number in Theorem 4.1.

**Conjecture 4.2.** Suppose that $\boldsymbol{H}$ is an interesecting hypergraph of rank $r\geqslant 4$ with $\nu^*(\boldsymbol{H})<r-1$. Then $\nu^*(\boldsymbol{H})\leqslant r-1-1/(2r-3)$.

For $r=3$ we have that $\max\{\nu^*(H)$: $H$ is 3-uniform, intersecting with $\nu^*<2\}=9/5$ (see [9]). Conjecture 4.2 is probably not too difficult for $r=4$. Delete three nonconcurrent lines of a $PG(2,r-1)$. The obtained hypergraph shows that (if it is true) the above conjecture is the best possible.

**Problem 4.3.** Determine $\nu^*(r,\hat{t},s):=\sup\{\nu^*(H)$: $H$ is $r$-uniform, $\hat{t}$-wise $s$-intersecting$\}$.

It easily follows [23, p. 165] that in the above definition the supremum can be replaced by the maximum. This value is known for $s>r-\sqrt{r(t-1)}$ [19], and in the case $s=1$ if $r<3t/2$ [22]. Using the notation $q^{[a]}=q^a+q^{a-1}+\cdots+q+1$, $q^{[0]}=1$ we have [19] $\nu^*(q^{[t+s-1]},\hat{t},q^{[s]})=q^{[t+s]}/q^{[t+s-1]}$. Here equality holds for $PG(t+s,q)$.

**Conjecture 4.4.** Suppose that $H$ is a $\hat{t}$-wise $q^{[s]}$-intersecting family of rank $q^{[t+s-1]}$ other than the hyperplanes of $PG(t+s,q)$. Then $\nu^*(H)\leqslant q$.

The most general result here (proved in [19]), which implies the above mentioned results, is as follows. If $H$ is $s$-intersecting of rank $r$, then either $H$ is a symmetric $(r,s)$-design (an $S_s((r^2-r+s)/s,r,2)$ block design), and then $\nu^*=(r-1)/s+1/r$, or $\nu^*\leqslant(r-1)/s+1/r-(r-s)/r(r-1)s$.

**Conjecture 4.5.** If $H$ is $s$-intersecting of rank $r$ other than a symmetric $(r,s)$-design, then $\nu^*(H)\leqslant(r-1)/s$.

For $r$-partite hypergraphs Conjecture 4.2 holds [24]. If $H$ is an $r$-partite, intersecting hypergraph, then either $\nu^*(H)\leqslant r-1-1/(r-1)$, or $H$ is a truncated projective plane of order $r-1$ (and then $\nu^*(H)=r-1$). Deleting a line of a truncated projective plane, we obtain an $r$-partite hypergraph with $\nu^*=r-1-1/(r-1)$.

**Problem 4.6.** Find $\max\nu^*(H)$ for intersecting 7-partite hypergraphs.

For $r$-partite hypergraphs Conjectures 4.4 and 4.5 were proved in [26]. (Note that a symmetric $(r,s)$-design, including projective spaces, is not $r$-partite.)

**Problem 4.7.** Determine $\nu_r^*(\hat{t},s):=\sup\{\nu^*(H)$: $H$ is $r$-partite, $\hat{t}$-wise $s$-intersecting$\}$.

It seems interesting to determine the maximum of $\nu^*$ for other classes of hypergraphs. For example the following.

**Problem 4.8.** Determine $\mu(\hat{t},s,p)=\max\{\nu^*(H)$: $H$ is $\hat{t}$-wise $s$-intersecting, and $|V(H)|\leqslant p\}$.

Denote $\mu(\hat{2}, 1, p)$ by $\mu(p)$. It is easy to see that $\mu(q^2+q+1) \leqslant q+1/(q+1)$, and here equality holds if a $PG(2,q)$ exists [1, 36]. As a corollary of Theorem 4.1 we have the following: if $\boldsymbol{H}$ is an intersecting hypergraph over $q^2+q+1$ elements, then either $\boldsymbol{H}$ contains a $PG(2,q)$ as a subhypergraph, and then $v^*(\boldsymbol{H}) = q+1/(q+1)$, or

$$v^*(\boldsymbol{H}) \leqslant q + \frac{q-1}{q^2+q-1}. \tag{4.2}$$

If we replace a line $L$ of a $PG(2,q)$ by a superset $L \cup \{x\}$, where $x \in V(PG(2,q)) - L$, then for the intersecting hypergraph obtained equality holds in (4.2). So the upper bound in (4.2) could not be improved in general, but seems interesting to find, for example, the value $\mu(7)$.

Obviously, $\mu(q^2+q) \leqslant q$. Theorem 4.1 implies the following improvement [25]. Let $\boldsymbol{H}$ be an intersecting hypergraph over $q^2+q$ elements; then either $\boldsymbol{H}$ contains a truncated plane, or it contains a twisted plane, or $v^*(\boldsymbol{H}) \leqslant q - [1/3(q+1)^3]$. Mills [34] determined the value of $\mu(r)$ for $r \leqslant 13$ (also see [40] for $r \leqslant 7$). It seems hopeful to determine $\mu(q^2+q+1+a)$ if $|a|$ is small and a $PG(2,q)$ exists.

**Conjecture 4.9.** $\mu(q^2+q+2) \leqslant q+2/(2q+1)$, and here equality holds if a $PG(2,q)$ exists.

We can consider larger classes of hypergraphs. In [21] the following theorem was proved: if the (multi)hypergraph $\boldsymbol{H}$ of rank $r$ (where $r \geqslant 3$) does not contain $p+1$ (pointwise) disjoint copies of $PG(2, r-1)$, then

$$v^*(\boldsymbol{H}) \leqslant v(r-1) + p/r. \tag{4.3}$$

This is a slight improvement on the trivial inequality $v^* \leqslant \tau \leqslant rv$. For $r$-partite hypergraphs (4.3) was proved by Gyárfás [29]. Let $\tau^*(r, v) = \sup\{v^*(\boldsymbol{H}) \colon r(\boldsymbol{H}) \leqslant r$, with matching number $v(\boldsymbol{H}) \leqslant v\}$. By the above result we have that $\tau^*(r, v) = (r-1+1/r)v$ if and only if a $PG(2, r-1)$ exists. Otherwise $\tau^*(r, v) \leqslant (r-1)v$.

**Conjecture 4.10.** $\tau^*(r, v) = v\tau^*(r, 1)$ for all $r$.

In the same way that Theorem 3.1 implies Theorem 3.1 via the inequality (4.1), all of the above results have a consequence for the maximum degree of the corresponding class of regular intersecting families.

The most general conjecture concerning fractional matchings can be found in [27], and is as follows.

**Conjecture 4.11.** For a hypergraph $\boldsymbol{H}$, for a fractional matching $w : \mathscr{E}(\boldsymbol{H}) \to \boldsymbol{R}^+$ and for an arbitrary function $b : \mathscr{E}(\boldsymbol{H}) \to \boldsymbol{R}^+$, one can find a matching $\mathscr{M} \subset \mathscr{E}$ with

$$\sum_{A \in \mathscr{E}} (|A| - 1 + 1/|A|) b(A) \geqslant \sum_{E \in \mathscr{E}} w(E) b(E).$$

For uniform $H$ and $b$ constant this is the weak version of (4.3). In [27] the conjecture is proved if $H$ is either uniform, or intersecting, or if $b$ is constant. A consequence of these results is the following inequality. For any $r$-uniform intersecting hypergraph with $\bigcap \mathscr{E}(H)=\emptyset$

$$\sum_{E\in\mathscr{E}}\sum_{F\in\mathscr{E}}|E\cap F|\geqslant\frac{r^2}{r^2-r+1}|\mathscr{E}(H)|^2.$$

Conjecture 4.11 is related to the ratio of the matching and fractional matching polytopes. In [27] we formulate an even stronger version of it which includes a number of other conjetures, e.g. a possible generalization of Shannon's theorem [39] for $r$-graphs proposed by Faber and Lovász [16].

## 5. Graphs of diameter 2 with a given maximum degree

The graph $G$ has diameter two if the distance between any two vertices is at most two. Let $e_2(n,D)$ denote the minimum number of edges in a (simple) graph of diameter 2 with $n$ vertices and maximum degree at most $D$. Erdős and Rényi [14] proposed the problem of determining $e_2(n,D)$. An excellent survey can be found in Bollobás' book [4, Ch. 4]. The smallest graph of diameter 2 is the star, it has $n-1$ edges and its maximum degree is $n-1$. In [14] it was proved that for any other graph (i.e. for any graph of diameter 2 with $D(G)<n-1$) we have $|\mathscr{E}(G)|\geqslant 2n-5$. For example, a graph obtained from the five cycle $C_5$ by replacing a vertex by an independent set of size $n-4$ has $2n-5$ edges and maximum degree $n-3$. Erdős et al. [15] determined the exact value of $e_2(n,D)$ for $D>n/2$. Some of their statements, especially those without proofs, were corrected by Vrto and Znám [41]. The following construction shows that

$$e_2(n,D)=2n-4 \quad \text{for } \tfrac{2}{3}n-1<D\leqslant n-5.$$

For simplicity we define $G$ only in the case $n/3$ is an integer. Let $V(G)=\{x_1,x_2,x_3\}\cup V_1\cup V_2\cup V_3$ be a disjoint union of these four sets with $|V_i|=(n/3)-1$. Let $E_i:=\{x_1,x_2,x_3\}\setminus\{x_i\}$. To obtain $\mathscr{E}(G)$, join all vertices of $V_i$ to both vertices of $E_i$ and finally join $x_1$ to $x_2$ and $x_3$. Then $D(G)=2n/3$.

Bollobás [2] proved that

$$\frac{1}{c}n<e_2(n,cn)<\left(\frac{1}{c}+\left(\frac{1}{c}\right)^{5/8}\right)n,$$

i.e., $nc^{-1}$ is in fact the correct order of magnitude of $e_2(n,cn)$. The construction giving $(q+1)n+O(1)$ edges for $(q+1)/(q^2+q+1)<c<1/q$ (and $n>n_0(c)$) is as follows. Let $A\subset V(G)$ be a $(q^2+q+1)$-element set, and let $\mathscr{L}$ consist of the $q^2+q+1$ lines of a finite projective plane of order $q$ on the set $A$. We divide the remaining vertices of $G$ into $q^2+q+1$ approximately equal classes and we join each vertex of a class to all vertices belonging to a corresponding line $L\in\mathscr{L}$. Finally, the set $A$ will span a complete subgraph in $G$. Pach and Surányi [36] proved that, indeed, in this range ($c$ is fixed $n>n_0(c)$) if there exists a finite plane of order $q$, then $e_2(n,cn)=(q+1)n+O(1)$. They also proved [35] that there exists a sequence $1=c_1>c_2>\cdots$ tending to zero such that for $c\notin\{c_k\}$

$$a(c):=\lim_{n\to\infty} e_2(n,cn)/n$$

exists for every $0<c<1$. Moreover, the function $a(c)$ is linear in the intervals $(c_i, c_{i-1})$ but may jump at the exceptional points $c_i$. With this terminology the above-mentioned results imply that

$$a(c)=\begin{cases} 2 & \text{for } 1>c>2/3,\\ 3-c & \text{for } 2/3>c>3/5,\\ 5-4c & \text{for } 3/5>c>5/9,\\ 4-2c & \text{for } 5/9>c>1/2,\\ 3 & \text{for } 1/2>c>3/7. \end{cases}$$

The last case was proved in [36]. This was improved by Znám [43] as follows. For $(3/7)n\leqslant D\leqslant n/2-\sqrt{21n}$ we have $e_2(n,D)=3n-12$.

To obtain $a(c)$ Pach and Surányi [35] developed the following method. For any hypergraph $H$ with $\mathscr{E}(H)=\{E_1,E_2,\ldots,E_m\}$ and positive real $c$ define $a(\mathbf{H},c)$ as the minimum of $\sum|E_i|y_i$, where each $y_i$ is a nonnegative weight under the following restrictions:

(1) the sum of weights of the edges through every point is at most $c$, and

(2) the total sum of the weights is equal to 1.

Then $a(c):=\inf a(H,c)$ over all intersecting hypergraphs. The determination of $a(c)$ (theoretically) is a finite process for any given $c$, as in the above infimum we can consider only intersecting hypergraphs with at most $3/c^2$ edges and vertices, i.e.

$$a(c)=\min\{a(H,c)\colon H \text{ intersecting}, |V|,|\mathscr{E}|\leqslant 3/c^2\}. \tag{5.1}$$

An intersecting hypergraph $H$ is called *$a(c)$-extremal* if $a(H,c)=a(c)$. Reformulating the earlier results we have that for $(q+1)/(q^2+q+1)<c<1/q$, the only $a(c)$-extremal hypergraph is a $PG(2,q)$ (if it exists). The only $a(c)$-extremal hypergraphs for $3/7<c<1$ are shown in Fig. 3.

If $\mathbf{G}$ is an extremal graph (i.e. $|\mathscr{E}(G)|=e_2(n,cn)$ with $D(G)\leqslant cn$), $n$ sufficiently large, $n>n_0(c)$, and $c$ not an exceptional value, then there exists an $a(c)$-extremal hypergraph $H=\{E_1,\ldots,E_m\}$ with $V(H)\subset V(G)$ of size $m$,

$$|V(H)|=o(n) \tag{5.2}$$

and a partition $V_1,\ldots,V_m$ of the remaining vertices $V(G)\setminus V(H)$ such that for all $i$ and $x\in E_i$, $y\in V_i$ the edge $\{x,y\}$ is in $\mathscr{E}(G)$. So the determination of $e_2(n,cn)$ is more or less equivalent to the search for $a(c)$-extremal hypergraphs.

It is obvious that an $a(c)$-extremal $H$ is *v-critical.* (This means that it has no multiple edges, and substituting any edge $E\in\mathscr{E}$ by a smaller nonempty edge $E'\subset E$ the obtained family $(\mathscr{E}\setminus\{E\})\cup\{E'\}$ is not intersecting anymore.) Other properties are given in Section 6.

$$\begin{matrix}1/3\\1/3\\1/3\end{matrix}\begin{pmatrix}1&1&\cdot\\1&\cdot&1\\\cdot&1&1\end{pmatrix} \qquad 1>c>\frac{2}{3} \qquad a(c)=2$$

$$\begin{matrix}(1-c)/2\\(1-c)/2\\2c-1\\2c-1\\2-3c\end{matrix}\begin{pmatrix}1&1&1&\cdot&\cdot\\1&1&\cdot&1&\cdot\\1&\cdot&\cdot&\cdot&1\\\cdot&1&\cdot&\cdot&1\\\cdot&\cdot&1&1&1\end{pmatrix} \qquad \frac{3}{5}>c>\frac{5}{9} \qquad a(c)=5-4c$$

$$\begin{matrix}1/7\\1/7\\1/7\\1/7\\1/7\\1/7\\1/7\end{matrix}\begin{pmatrix}1&1&1&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&1&1&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&1&1\\\cdot&1&\cdot&1&\cdot&1&\cdot\\\cdot&1&\cdot&\cdot&1&\cdot&1\\\cdot&\cdot&1&1&\cdot&\cdot&1\\\cdot&\cdot&1&\cdot&1&1&\cdot\end{pmatrix} \qquad \frac{1}{2}>c>\frac{3}{7} \qquad a(c)=3$$

$$\begin{matrix}c/3\\c/3\\c/3\\1-c\end{matrix}\begin{pmatrix}1&1&\cdot&\cdot\\1&\cdot&1&\cdot\\1&\cdot&\cdot&1\\\cdot&1&1&1\end{pmatrix} \qquad \frac{2}{3}>c>\frac{3}{5} \qquad a(c)=3-c$$

$$\begin{matrix}2c-1\\(1-c)/2\\(1-c)/2\\(1-c)/2\\(1-c)/2\end{matrix}\begin{pmatrix}1&1&\cdot&\cdot&\cdot&\cdot\\1&\cdot&1&\cdot&1&\cdot\\1&\cdot&\cdot&1&\cdot&1\\\cdot&1&1&\cdot&\cdot&1\\\cdot&1&\cdot&1&1&\cdot\end{pmatrix} \qquad \frac{5}{9}>c>\frac{1}{2} \qquad a(c)=4-2c$$

Fig. 3. The $a(c)$-extremal minimal designs for $c>3/7$.

Apparently, the $a(c)$-extermal designs are finite projective planes, or small intersecting structures obtained from these planes, at least for $c>3/7$. Now we are ready to state our main result of this paper, which makes precise the previous impression at least in infinitely many short intervals.

**Theorem 5.1.** *Suppsoe that there exists a finite projective plane $PG(2,q)$, and let $(1\backslash q)<c<(1/q)+1/(2q^4+2q^3)$. Then $a(c)=[q^2(q-1)]-[q/(q-1)]c$, and an $a(c)$-extremal design is an extended punctured plane of order $q$, $EPP(q)$.*

The proof makes use of Theorem 4.1 and it is postponed to the next section. A *punctured plane* of order $q$, denoted by $PP(q)$, is obtained from a $PG(2,q)$ by deleting a vertex $x$ and the $q+1$ edges through $x$, and adding a new edge $E_0:=L\backslash\{p\}$, where $p\in L\in\mathscr{E}(PG(2,q))$. (See Fig. 4 for $q=3$.) The $q$-element edge $E_0$ is called the *special* edge of $PP(q)$. The *extended punctured plane* $EPP(q)$ is obtained from a $PP(q)$ by adding new edges of size at most $q+1$ so that they must not contain each other, but, of course, keep the intersection property. It follows (see after (6.11)) that the only two ways to do this extension are as follows. All new edges will be containd in the original $V(PP(q))$ and have $q+1$ elements. Let us denote the traces of the deleted lines of

$$\begin{array}{c} (qc-1)/(q-1) \\ (1-c)/(q^2-q) \\ (1-c)/(q^2-q) \\ (1-c)/(q^2-q) \\ (1-c)/(q^2-q) \\ (1-c)/(q^2-q) \\ (1-c)/(q^2-q) \\ (1-c)/(q^2-q) \\ (1-c)/(q^2-q) \\ (1-c)/(q^2-q) \end{array} \begin{pmatrix} 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & 1 & \cdot \end{pmatrix}$$

$$\frac{1}{3}+\frac{1}{216} > c > \frac{1}{3}$$
$$a(c) = \frac{9}{2} - \frac{3}{2}c$$

Fig. 4. The $a(c)$-extremal minimal design obtained from $PG(2,q)$ (here $q=3$).

$PG(2,q)$ on $V(PP(q))$ by $L_0, L_1, \ldots, L_q$. Then, $EPP(q)$ is obtained either by joining some edges of the form

$$L_i \cup \{x\}, \tag{5.3}$$

where $x \in L_0$ is fixed and $1 \leqslant i \leqslant q$, or by joining some edges of the same form with $L_i$ fixed and $x$ allowed to vary.

To compute the value $a(EPP(q), c)$, define the weight function $y : \mathscr{E}(EPP(q)) \to \boldsymbol{R}^+$ as follows. Let $y(E_0) := (qc-1)/(q-1)$, $y(E) := (1-c)/q(q-1)$ for all other edges of $PP(q)$ and 0 for the edges from $\mathscr{E}(EPP(q)) \setminus \mathscr{E}(PP(q))$. We obtain that $a(EPP(q), c)$ is at most $q^2/(q-1) - cq/(q-1)$. On the other hand, a solution, $t : V(PP(q)) \cup \{*\} \to \boldsymbol{R}^+$, of the dual linear program defined by $t(*) := q^2/(q-1)$, $t(x) := 1/(q-1)$ for $x \in E_0$, and $t(x) := 0$ for $x \in V \setminus E_0$ shows that $a(EPP(q), c)$ is indeed equal to the claimed value (in the range $1/q < c < 1$). ($t(*)$ is the variable corresponding to the constraint $\sum -y(E) \leqslant -1$.)

Returning to the original problem about graphs of diameter 2, we sharpen the basic theorem of [36] as follows. We can replace the upper bounds in (5.2) by an absolute constant depending only on $i$ ($c_{i-1} > c > c_i$). As a consequence of this we get a sharper bound for $e_2(n, D)$.

**Theorem 5.2.** *There exists a sequence* $1 = c_0 > c_1 > \cdots$ *(tending to* 0*), and constants* $M_i$*, such that for* $c_i n - M_i > D > c_{i-1} n + M_i$ *we have*

$$|e_2(n, D) - a(c)n| < M_i$$

**Proof.** (*Sketch.*) It is rather technical, and copies an argument dealing with a similar problem in [20], so we give only a sketch. First we prove that $a(c)$ is linear in the segment $(c_i, c_{i-1})$, so it has the Lipschitz property. Using this and an argument similar to Lemma 15 in [20], we sharpen the main lemma (Lemma 2.4) from [36]. Applying this to the sets $X, Y, Z, U$, defined in the course of the proof in [36] we show that the size of each of them is bounded. The main difference from [36] is that we separate the degrees larger than $O(1/c)$ instead of splitting at $O(\log\log n)$. We repeatedly have to use the trivial inequality $e_2(n, cn) \leqslant a(c)n + O(1/c^2)$. □

Build a graph $G$ of diameter 2 using the core $EPP(q)$ as follows. Suppose that $n \geqslant 2q^2+2q$ and $n-(2q^2-q) > D \geqslant (n/q)+q-2$. Let $E_0$ be the special edge of $PP(q)$, $E_0 := \{x_1, \ldots, x_q\}$. Denote the edges of $PP(q)$ through $x_i$ by $E_{(i-1)q+1}, \ldots, E_{iq}$. Let $V(EPP(G)) := L_0 \cup L_1 \cup \cdots \cup L_q$, where the $q$-element $L_i$ is the trace of a deleted line of the $PG(2,q)$, which the punctured plane derived from, $L_0 = E_0$. By (5.3), $EPP(q)$ can have at most $q$ additional edges of the form $L_\alpha \cup \{x_\beta\}$; denote them by $E_{q^2+1}, \ldots, E_{q^2+k}$ $(0 \leqslant k \leqslant q)$. Let $V(G)$ be an $n$-element set containing $V(PP(G))$ such that the remaining vertices partition into $1+q^2+k$ sets $V_0, V_1, \ldots$ with cardinalities $|V_0| = \lfloor (qD-n+q)/(q-1) \rfloor$, and $\sum\{|V_j|: x_i \in E_j, j>0\} = D - |V_0| - q$ for $x_i \in E_0$. (This quantity equals $\lceil (n-D-q^2)/(q-1) \rceil$.) Also suppose that the sets $V_j$ are nonempty for $j \leqslant q^2$. Finally, suppose that

$$\sum_{x \in E_j} |V_j| \leqslant D-q$$

holds for every $x \in V(PP(q))$. (For the points of $E_0$ equality hold.) There are several ways to partite $V(G)$ in this way, for example, whenever all the $|V_j|$'s are almost equal for $1 \leqslant j \leqslant q^2$ and $V_j = \emptyset$ for $j > q^2$.

Define the edge set of $G$ as follows. Put a complete graph for each $(q+1)$-element set of the form $L_i \cup \{x_i\}$ $(1 \leqslant i \leqslant q)$. Join each $x \in E_j$ to each $y \in V_j$ for all $j$. Denote the class of graphs obtained in this way by $\mathscr{G}_q(n, D)$. Then each graph $G \in \mathscr{G}_q(n, D)$ of this type has maximum degree $D$, has diameter 2 and

$$|\mathscr{E}(G)| = \frac{q^2}{q-1}n - \frac{q}{q-1}D - \frac{q^2(q+5)}{2} + q^2\left\{\frac{qD-n}{q-1}\right\}. \tag{5.4}$$

Here $\{x\}$ stands for the fractional part of $x \in \mathbf{R}$, i.e. $\{x\} := x - \lfloor x \rfloor$.

**Theorem 5.3.** *Suppose that there exists a finite plane $PG(2,q)$. There exists a constant $M_q$ such that if $nq^{-1} + M_q < D < n(q^{-1} + (2q^4+2q^3)^{-1}) - M_q$ and the graph $G$ with $n$ vertices and maximum degree at most $D$ has diameter 2, then the right-hand side of (5.4) is a lower bound for $|\mathscr{E}(G)|$. Moreover, equality holds only for the members of $\mathscr{G}_q(n, D)$.*

**Problem 5.4.** Describe the $a(c)$-extremal hypergraphs.

The range 'close' to a $PG(2,q)$ looks especially promising, for example, when $(q+1)/(q^2+q+1)-\varepsilon(q)<c<(q+1)/(q^2+q+1)$. To fill the first gap (between 3/7 and 1/3) Znám has the following conjecture.

**Conjecture 5.5** (Znám [42]).

$$g(c)=\begin{cases} 5-4c & \text{for } \frac{3}{7}>c>\frac{5}{12}, \\ 8-11c & \text{for } \frac{5}{12}>c>\frac{2}{5}, \\ 6-6c & \text{for } \frac{2}{5}>c>\frac{3}{8}, \\ \frac{11}{2}-\frac{9}{2}c & \text{for } \frac{3}{8}>c>\frac{11}{29}, \\ 5-3c & \text{for } \frac{11}{29}>c>\frac{5}{14}, \\ \frac{9}{2}-\frac{3}{2}c & \text{for } \frac{5}{14}>c>\frac{1}{3}, \end{cases}$$

Theorem 5.1 established the range $1/3+1/216>c>1/3$. If Conjecture 4.2 is true, then our proof works without any change for the range $(1/q)+(1/5q^3)>c>1/q$ as well.

*5.1. Further problems concerning graphs of diameter 2*

**Problem 5.6.** Determine $e_2(n,D,d)$, where this denotes the minimum number of edges in a (simple) graph of diameter 2 with $n$ vertices, maximum degree at most $D$ and minimum degree at least $d$.

The investigation of $e_2(n,n-1,d)$, i.e. when only a lower restriction is put on the valencies, was started by Bondy and Murty [6]. Their result was generalized by Bollobás and Harary [5], who showed that $e_2(n,n-1,d)=\lceil (n-1)(d+1)/2\rceil$ for $d<\sqrt{n/3}$. Pach and Surányi [36] extended most of the above results for $e_2(n,cn,d)$, where $c$ and $d$ are fixed.

Concerning minimum degrees the following result is due to Duffus and Hanson [10]: If $G$ is a maximal triangle-free graph on $n$ vertices with minimum degree 3, then $|E(G)|\geqslant 3n-15$. (Note that such a graph has diameter two.) They investigated the following more general problem.

**Problem 5.7.** Determine $E(n,k,\delta)$, the minimum number of edges of a maximal $K_k$-free graph on $n$ vertices with minimum degree $\delta$.

Hajnal (see [38]) proposed the following problem.

**Problem 5.8.** At least how many edges must a maximal triangle-free graph have if the maximal degree of vertices is at most $D$ for some $D<n-1$?

Denote the minimal number of edges of a maximal triangle-free graph with maximal valency at most $D$ by $F(n, D)$. Clearly, $\boldsymbol{G}$ is a maximal triangle-free graph if and only if it is triangle-free and has diameter 2. Hence $e_2(n, D) \leqslant F(n, D)$. For $D \geqslant n/2$, the complete bipartite graph $\boldsymbol{K}_{D,n-D}$ provides an example of a maximal triangle-free graph with maximal valency $\leqslant D$. However, there are maximal triangle-free graphs with much less edges.

Let $(n-2)/2 < D \leqslant n-3$ and let $V = V_1 \cup V_2 \cup \{x_3\} \cup V_4 \cup \{x_5\}$ be a partition of a set $V$ of cardinality $n$ into parts of size $|V_1| = |V_2| = n-2-D$ and $|V_4| = 2D-(n-2)$. Let the graph $\boldsymbol{G}(C_5)$ have the following set of edges. $x_3$ is connected to each vertex in $V_2 \cup V_4$; $x_5$ is connected to each vertex in $V_4 \cup V_1$; finally, each $z \in V_1$ is connected to each $w \in V_2$. Then $\boldsymbol{G}(C_5)$ is a maximal triangle-free graph with maximal valency $D$ and $2n-5+(n-3-D)^2$ edges.

Another example for a triangle-free graph of diameter 2 can be obtained from the Petersen graph. Let $V_0 = \{x_1, x_2, \ldots, x_{10}\}$ be the vertex set of the Petersen graph $\boldsymbol{P}$ such that $x_1, x_2, x_3, x_4$ are pairwise nonadjacent. Note that the Petersen graph itself is a maximal triangle-free graph with $3 \times 10 - 15$ edges. Let $n \geqslant 10$ be given. Let $V_1$, $V_2$, $V_3$, $V_4$ be pairwise disjoint sets, also disjoint from $V_0$, of size $|V_i| = \lfloor (n-6+(i-1))/4 \rfloor$. Then $\sum_{i=1}^{4} |V_i| = n-6$. For $1 \leqslant i \leqslant 4$, replace $x_i$ by the independent set $V_i$ in $\boldsymbol{P}$, connecting the vertices in $V_i$ to the original neighbours of $x_i$ in $\boldsymbol{P}$. The resulting graph $\boldsymbol{G}(\boldsymbol{P})$ has $n$ vertices, $3n-15$ edges, and maximal valency is $D = n/2 - \mathrm{O}(1)$. The vertex-duplication procedure described above maintains the maximal triangle-free property so $\boldsymbol{G}(\boldsymbol{P})$ is maximal triangle-free.

In [28] it was proved that for $n > 2^{2^{28}}$

$$F(n, D) = \begin{cases} 2n-5 & \text{for } D = n-2, \\ 2n-5+(n-3-D)^2 & \text{for } n-3-\sqrt{n-10} \leqslant D \leqslant n-3, \\ 3n-15 & \text{for } (n-2)/2 \leqslant D < n-3-\sqrt{n-10}. \end{cases}$$

The main tool of the proof is the result of Duffus and Hanson [10] mentioned above, and a theorem analogous to the results of Pach and Surányi. A general example is the following.

Let $PG(2, q)$ be a projective plane on $W_1 = \{x_1, \ldots, x_{q^2+q+1}\}$ with line set $\{L_1, \ldots, L_{q^2+q+1}\}$. We can suppose that the lines containing $x_{q^2+q+1}$ are $L_i$ for $q^2+1 \leqslant i \leqslant q^2+q+1$. Let $W_2 = \{y_1, \ldots, y_{q^2+q}\}$ be a set disjoint from $W_1$. First, we define a set system $\boldsymbol{H}$ and a graph $\boldsymbol{G}$ on the $2(q^2+q)$ vertices $V = \{x_i, y_i \colon 1 \leqslant i \leqslant q^2+q\}$. $\boldsymbol{H}$ consists of $q^2$ sets of size $2q$; namely, let $H_i = L_i \cup \{y_j \colon x_j \in L_i\}$ $(1 \leqslant i \leqslant q^2\}$. $\boldsymbol{G}$ is a $(q-1)$-regular bipartite graph defined as follows. The sets $W_1 \setminus \{x_{q^2+q+1}\}$ and $W_2$ are independent in $\boldsymbol{G}$. $x_i$ and $y_j$ are connected if and only if $i \neq j$ and $\{x_i, x_j\} \subset L_k$ for some $q^2+1 \leqslant k \leqslant q^2+q+1$. Based on $\boldsymbol{H}$ and $\boldsymbol{G}$, we can build a maximal triangle-free graph $\boldsymbol{G}^q(n)$. Let $n \geqslant 3q^2+2q$. For $1 \leqslant i \leqslant q^2$, we choose sets $V_i$ disjoint from each other and from $V$ such that $|V_i| = \lfloor (n-2(q^2+q)+(i-1))/q^2 \rfloor$ for all $i$. Then the sets $V_i$ are nonempty and $|V| + \sum_{i=1}^{q^2} |V_i| = n$. We define $\boldsymbol{G}^q(n)$ on the vertex set

$V \cup V_1 \cup \cdots \cup V_{q^2}$. $x, y \in V$ are adjacent in $G^q(n)$ if and only if they are adjacent in $G$. The set $V_1 \cup \cdots \cup V_{q^2}$ is independent in $G^q(n)$. Finally, $x \in V$ and $y \in V_i$ are connected if and only if $x \in H_i$.

This example implies that the upper bound in the following inequality for $D \geq 5\sqrt{n}$ (the lower bound is trivial).

$$\frac{n^2}{2D} - n < F(n, D) < \frac{4n^2}{D} + 2n.$$

Hence, the order of the magnitude of the function $F(n, cn)$ is linear in $n$ for a fixed $c$. The theorem analogous to the results of Pach and Surányi states that there exists a sequence $1 = c_1 > c_2 > \cdots$ tending to zero such that for $c \notin \{c_k\}$

$$A(c) := \lim_{n \to \infty} F(n, cn + B(c))/n$$

exists for every $0 < c < 1$. Here $B(c)$ is a constant depending only on $c$.

To obtain $A(c)$ in [28] the following method was developed. Certain hypergraph–graph pairs are intimately related to maximal triangle-free graphs. Let $\boldsymbol{H} = (V, E(\boldsymbol{H}))$ be hypergraph and $\boldsymbol{G} = (V, \mathscr{E})$ be a graph on some set $V$. The pair $\boldsymbol{H}, \boldsymbol{G}$ is a *core* if it satisfies the following properties:

(1) $\boldsymbol{H}$ is intersecting,

(2) $\boldsymbol{G}$ is triangle-free,

(3) for all $e \in \mathscr{E}$ and $H \in E(\boldsymbol{H})$, $e \not\subset H$,

(4) for all $x \in V$ and $H \in E(\boldsymbol{H})$, $x \notin H$, there exists $y \in H$ such that $\{x, y\} \in \mathscr{E}$;

(5) for all $x, y \in V$, if $\{x, y\} \not\subset H$ for any $H \in E(\boldsymbol{H})$ then either $\{x, y\} \in \mathscr{E}$ or there exists $z \in V$ with $\{x, z\} \in \mathscr{E}$ and $\{z, y\} \in \mathscr{E}$.

Finally, the function $A(c)$ is defined as $A(c) = \inf\{a(\boldsymbol{H}, c)\}$ where the infimum is taken over all hypergraphs $\boldsymbol{H}$ which occur in a core with an appropriate graph $\boldsymbol{G}$ and $c \geq 1/v^*(\boldsymbol{H})$.

**Problem 5.9.** Describe the $A(c)$-extremal hypergraphs.

We do not have such a general result for infinitely many intervals as for $e_2(n, cn)$. Although it seems certain that the minimal size of an $A(c)$-extremal hypergraph $\boldsymbol{H}$ is relatively small (we can prove $|\mathscr{E}(\boldsymbol{H})| < 5/c^2$), we have only the following bound for

$$|V(\boldsymbol{H})| \leq B(c).$$

For $c > 0$, we define a function $B(c)$ the following way. If $c > 1$ then $B(c) := 1$. For $0 < c \leq 1$, let

$$B_0(c) := 2^{(2/c^2) + (2/c) + 1},$$

$$B_{k+1}(c) := 2^{(2/c^2) + (2/c) + 1 + \sum_{i=0}^{k} B_i(c)},$$

$$B(c) := B_{\lfloor (2/c^2) + (2/c) \rfloor}(c).$$

**Conjecture 5.10.** For $(q+1)/(q^2+q+1)<c<1/q$ we have $A(c)=q+2$ if a $PG(2,q)$ exists.

A construction can be given as follows. Define a core on a set $V=V_1\cup V_2$ of cardinality $2(q^2+q+1)$. Let $E_1,\dots,E_{q^2+q+1}$ be the line set of the projective plane on a set $V_1, |V_1|=q^2+q+1$. Let $V_2=\{x_1,\dots,x_{q^2+q+1}\}$. We define $E(\boldsymbol{H})=\{E_i\cup\{x_i\}: 1\leqslant i\leqslant q^2+q+1\}$. The graph $\boldsymbol{G}$ is bipartite with classes $V_1$ and $V_2$; we connect $x_i$ to all points in $V_1\setminus E_i$. It is clear that the pair $(\boldsymbol{H},\boldsymbol{G})$ is a core. Also, the weight function $y(H_i):=1/(q^2+q+1)$ gives a feasible solution of the required linear program and $\sum y(H_i)|H_i|=q+2$.

**Problem 5.11.** Determine the minimum $D=D_2(n)$ such that there exists a triangle-free graph of diameter 2 over $n$ vertices and maximum degree $D$.

This problem was proposed by Erdős and Fajtlowicz [12]. They pointed out that the random method gives only $D_2(n)\leqslant O(\sqrt{n}\log n)$. This upper bound was lowered by an example due to Hanson and Seyffarth [30] showing that for some circular graphs $D_2(n)\leqslant(2+o(1))\sqrt{n}$. Other circular graphs were found by Hanson and Strayer [31]. The example $\boldsymbol{G}^q(3q^2+2q)$ of the previous section indicates that their upper bound in fact can be improved to $D_2(n)\leqslant(2/\sqrt{3}+o(1))\sqrt{n}$ (for all $n$).

Further generalizations were investigated by Erdős and Pach [13], who considered graphs *with property* $I_k$, i.e. graphs in which every independent set of size $k$ has a common neighbour.

## 6. Proof of Theorem 5.1

Let $c$ be fixed and let $\boldsymbol{H}$ be an $a(c)$-extremal hypergraph with optimal weight function $y:\mathscr{E}\to\boldsymbol{R}^+$ (this means that $\sum y(E)|E|=a(c)$). Suppose that $\boldsymbol{H}$ has minimal number of edges (among the $a(c)$-extremal designs contained in $\mathscr{E}(\boldsymbol{H})$). Call a vertex $x$ *saturated* if $\sum\{y(E): x\in E\in\mathscr{E}\}=c$. The set of saturated vertices is $S$. Of course $|S|$ is not larger than $a(c)/c$. Suppose that $y$ has maximal number of saturated vertices (among the optimal weights of $\boldsymbol{H}$ saturated at $S$). Then

$$|\mathscr{E}(\boldsymbol{H})|\leqslant|S|+1\leqslant\lfloor a(c)/c\rfloor+1. \tag{6.1}$$

This follows from the fact that $a(\boldsymbol{H},c)$ is a solution of a linear program with $|\mathscr{E}|$ variables and $|V(\boldsymbol{H})|$ constraints corresponding to the vertices with one additional constraint $\sum y(E)=1$. The complementary slackness theorem of linear programming implies that the minimal number of nonzero variables in an optimal solution is not more than the maximal number of constraints fulfilled with equality. Applying this to our case, the minimality of $\mathscr{E}(\boldsymbol{H})$ implies that all edges have nonzero weights, and hence their number is not more than $|S|+1$.

The function $y/c$ is a fractional matching of $\boldsymbol{H}$; hence

$$\nu^*(\boldsymbol{H}) \geqslant \sum_E y(E)/c = 1/c. \tag{6.2}$$

As $|E| \geqslant \nu^*$ for all edges, we obtain

$$|E| \geqslant 1/c. \tag{6.3}$$

From now on, we suppose that $c = (1/q) + \delta$ with $0 < \delta < 1/(2q^4 + 2q^3)$, and $\bar{\boldsymbol{H}}$ is an $a(c)$-extremal design. The existence of a punctured plane, $PP(q)$, gives

$$a(\bar{\boldsymbol{H}}, c) \leqslant \frac{q^2}{q-1} - \frac{q}{q-1}c. \tag{6.4}$$

So we have to give a proof only for the lower bound for $a(c)$. Let $\boldsymbol{H}$ be an $a(c)$-extremal subfamily of $\bar{\boldsymbol{H}}$ with minimal number of edges, and let $y$ be an optimal weight function with maximal number of saturated vertices. Then (6.1) can be applied, and, of course, (6.2) and (6.3), too. We get

$$|\mathscr{E}| \leqslant q^2 + q, \tag{6.5}$$

$$|\mathscr{E}| \geqslant q \quad \text{for all } E \in \mathscr{E}. \tag{6.6}$$

Split $\mathscr{E}(\boldsymbol{H})$ into three parts, $\mathscr{E} = \mathscr{E}_q \cup \mathscr{E}_{q+1} \cup \mathscr{E}_{>q+1}$, where the index indicates edge sizes, $\mathscr{E}_x := \{E \in \mathscr{E} : |E| = x\}$. Then (6.4) gives $a(c) < q+1$, implying $\mathscr{E}_q \neq \emptyset$. Consider any edge $E_0 \in \mathscr{E}$. We obtain

$$\begin{aligned} c|E_0| &\geqslant \sum_{x \in E_0} \left( \sum_{E \ni x} y(E) \right) = \sum |E_0 \cap E| y(E) \\ &= \sum_E y(E) + \sum_E (|E \cap E_0| - 1) y(E) \\ &\geqslant 1 + (|E_0| - 1) y(E_0). \end{aligned}$$

The comparison of the extreme sides of this inequality gives

$$\frac{q}{q-1}\delta \geqslant y(E) \quad \text{for } E \in \mathscr{E}_q, \tag{6.7}$$

$$\frac{1}{q^2} + \frac{q+1}{q}\delta \geqslant y(E) \quad \text{for } E \in \mathscr{E}_{q+1}. \tag{6.8}$$

Denote the sum of $y(E)$ over $\mathscr{E}_x$ by $Y_x$, for example, $Y_q := \sum \{y(E) : E \in \mathscr{E}_q\}$. We have $Y_q + Y_{q+1} + Y_{>q+1} = 1$. Equation (6.4) implies that

$$q + 1 - \frac{q}{q-1}\delta \geqslant a(\boldsymbol{H}, c) \geqslant q Y_q + (q+1) Y_{q+1} + (q+2) Y_{>q+1}.$$

We obtain

$$Y_q \geqslant \frac{q}{q-1}\delta + Y_{>q+1}, \tag{6.9}$$

$$Y_{q+1} \geqslant 1 + \frac{q}{q-1}\delta - 2Y_q. \tag{6.10}$$

The proof of Theorem 5.2 consists of two parts. First, we consider the case when the fractional matching number satisfies the following condition.

(i) $\quad \nu^*(\mathscr{E}_{\leqslant q+1}) > q - 1/(q^2+q-1)$.

Then Theorem 4.1 implies that $\mathscr{E}_{q+1}$ is either a projective plane or it contains a twisted plane or a truncated plane. The transversal number of a projective plane is $q+1$, $\tau(PG(2,q)) = q+1$. Even more, if $T$ is a $q+1$-element transversal then

$$T \in \mathscr{E}(PG(2,q)). \tag{6.11}$$

So the $\mathscr{E}_{\leqslant q+1}$ cannot contain both a $PG(2,q)$ and a $q$-element set. The transversal number of a $q+1$-uniform twisted plane is $q+1$ (see, e.g. [25, p. 259]), so the above argument implies that the only possibility is that $\mathscr{E}_{\leqslant q+1}$ contains a truncated plane.

Denote this truncated plane by $\boldsymbol{P}$, i.e. $\mathscr{E}(\boldsymbol{P}) \subset \mathscr{E}$. Let $V(\boldsymbol{P}) = L_1 \cup \cdots \cup L_{q+1}$, where $L_i$ is the trace of a deleted line of the $PG(2,q)$, the $\boldsymbol{P}$ obtained from. Then (6.11) implies that the sets $L_i$ are the only $q$-element transversals of $\boldsymbol{P}$. Only one of them, say $L_1$, can be a member of $\mathscr{E}(\boldsymbol{H})$. Hence, a $PP(q) := \boldsymbol{P} \cup \{L_1\}$ is a subfamily of $\boldsymbol{H}$. It follows from (6.7) and (6.9) that the weight of $L_1$ is exactly $\delta q/(q-1)$, and then the weights of all large edges are 0, implying $\mathscr{E}_{>q+1} = \emptyset$. Equality holds in (6.9), and therefore in (6.4) too. This implies that $a(\boldsymbol{H}, c) = a(PP(q), c)$. Then, the minimality of the edge set of $\boldsymbol{H}$ implies $\boldsymbol{P} \cup \{L_1\} = \boldsymbol{H}$.

Considering $\bar{\boldsymbol{H}}$, we claim that it is an $EPP(q)$. First, it is easy to see that $L_1$ is the only $q$-element member of $\mathscr{E}(\bar{\boldsymbol{H}})$. It follows that $\bar{y}(L_1) = \delta q/(q-1)$ in any $a(c)$-optimal weight function $\bar{y}$ over $\mathscr{E}(\bar{\boldsymbol{H}})$. Then, the weights of all large edges are 0, implying $\mathscr{E}_{>q+1}(\bar{\boldsymbol{H}}) = \emptyset$.

As $PP(q)$ is a subfamily of $\bar{\boldsymbol{H}}$, every additional edge $F \in \mathscr{E}(\bar{\boldsymbol{H}}) \setminus \mathscr{E}(\boldsymbol{H})$ is a transversal of $PP(q)$. Moreover, $F$ has exactly $q+1$ elements. Then, for $q \geqslant 3$, we finish the proof by using the following sharpening of (6.11), due to Pelikán [37]. If $T$ is a transversal of $PG(q,2)$ and it does not contain any line, then for $q \geqslant 3$ its size $|T| \geqslant q+2$. This implies, as $F$ does not contain an edge of $PP(q)$, that it has the form $L_i \cup \{x\}$ $(i > 1)$, as desired. The case $q = 2$ can be finished easily by hand. Secondly, we consider the case when the fractional matching number satisfies the following condition.

(ii) $\quad \nu^*(\mathscr{E}_{\leqslant q+1}) \leqslant q - \dfrac{1}{q^2+q-1}$.

As the function $y/c$ is a fractional matching of $\boldsymbol{H}_{\leqslant q+1}$ we get that

$$Y_q + Y_{q+1} \leqslant c\nu^*(\boldsymbol{H}_{\leqslant q+1}) \leqslant c\left(q - \frac{1}{q^2+q-1}\right).$$

Comparing this with (6.10), we obtain

$$Y_q \geqslant 1 + \frac{q}{q-1}\delta - \left(\frac{1}{q} + \delta\right)\left(q - \frac{1}{q^2+q-1}\right)$$

$$= \frac{1}{q(q^2+q-1)} - \delta\left(q - \frac{1}{q^2+q-1} - \frac{q}{q-1}\right). \tag{6.12}$$

The right-hand side of (ii) is less than $1/c$ (for $\delta < 1/(q^4+q^3-q^2-q)$), so (6.2) implies that $\mathscr{E}_{>q+1} = \emptyset$. Then (6.5) gives

$$|\mathscr{E}_q| + |\mathscr{E}_{q+1}| = |\mathscr{E}(H)| - |\mathscr{E}_{>q+1}| \leqslant q^2 + q - 1. \tag{6.13}$$

Now apply (6.7) and (6.8) to get a lower bound for $|\mathscr{E}_q|$ and $|\mathscr{E}_{q+1}|$, respectively.

$$|\mathscr{E}_q| + |\mathscr{E}_{q+1}| \geqslant \frac{Y_q}{\delta \dfrac{q}{q-1}} + \frac{Y_{q+1}}{\dfrac{1}{q^2} + \delta \dfrac{q+1}{q}}.$$

Apply, the lower bound from (6.10) to $Y_{q+1}$. We get

$$|\mathscr{E}_q| + |\mathscr{E}_{q+1}| \geqslant \frac{Y_q}{\delta \dfrac{q}{q-1}} + \frac{1 + \dfrac{q}{q-1}\delta - 2Y_q}{\dfrac{1}{q^2} + \delta \dfrac{q+1}{q}}$$

$$= \frac{q^2 + \dfrac{q^3}{q-1}\delta}{1 + \delta q(q+1)} + Y_q\left(\frac{1}{\delta \dfrac{q}{q-1}} - \frac{2}{\dfrac{1}{q^2} + \delta \dfrac{q+1}{q}}\right).$$

Here the coefficient of $Y_q$ is positive (for $0 < \delta < (q-1)/(q^3+q)$). We can apply the lower bound of (6.12) for $Y_q$. The lower bound obtained for $|\mathscr{E}_q| + |\mathscr{E}_{q+1}|$ contradicts (6.13) if $\delta < 1/(2q^4 + 2q^3)$. This completes the proof for the case (ii). □

**Note added in proof.** Erdős and Holzman [44] recently solved Problem 5.9 for $2/5 < c < 1/2$, and thus disproved Conjecture 5.10 in case $q = 2$.

## Acknowledgement

The author is grateful to the organizers of the Second Japan Conference on Graph Theory and Combinatorics, 18–22 August 1990, at Hakone, Japan, where this paper was presented.

## References

[1] H.L. Abbott, M. Katchalski and A.C. Liu, An extremal problem in graph theory II, J. Austral. Math. Soc. Ser. A 29 (1980) 417–424.

[2] B. Bollobás, Graphs with a given diameter and maximal valency and with a minimal number of edges, in: D.J.A. Welsh, ed., Combinatorial Mathematics and its Applications (Academic Press, London, 1971) 25–37.

[3] B. Bollobás, Disjoint triples in a 3-graph with given maximum degree, Quart. J. Math. Oxford (2) 28 (1977) 81–85.

[4] B. Bollobás, Extremal Graph Theory (Academic Press, London, 1978).

[5] B. Bollobás and F. Harary, Extremal graphs with given diameter and connectivity, Ars Combin. 1 (1976) 281–296.

[6] J.A. Bondy and U.S.R. Murty, Extremal graphs of diameter 2 with prescribed minimum degree, Studia Sci. Math. Hungar. 7 (1972) 239–241.

[7] A.R. Calderbank, Symmetric designs as the solution of an extremal problem in combinatorial set theory, European J. Combin. 9 (1988) 171–173.

[8] P.J. Cameron, P. Frankl and W.M. Kantor, Intersecting families of finite sets and fixed-point-free 2-elements, European J. Combin. 10 (1989) 149–159.

[9] F.R.K. Chung, Z. Füredi, M.R. Garey and R.L. Graham, On the fractional covering number of hypergraphs, SIAM J. Discrete Math. 1 (1988) 45–49.

[10] D.A. Duffus and D. Hanson, Minimal $k$-saturated and color critical graphs of prescribed minimum degree, J. Graph Theory 10 (1986) 55–67.

[11] P. Erdős, Problems and results in graph theory and combinatorial analysis, in: C. St. J.A. Nash-Williams et al., eds., Proc. 5th British comb. conf., Aberdeen, 1975 (Utilitas Math., Winnipeg); Congr. Numer. 15 (1976) 169–172.

[12] P. Erdős and S. Fajtlowicz, Domination in graphs of diameter 2, unpublished, see P. Erdős and J. Pach, Remarks on stars and independent sets, in: Aspects of Topology, pp. 307–313: London Math. Soc. Lecture Note Ser. Vol. 93 (Cambridge Univ. Press, Cambridge, 1985).

[13] P. Erdős And J. Pach, Remarks on stars and independent sets, in: I.M. James and E.H. Kronheimer, eds., Aspects of Topology, London Math. Soc. Lecture Notes Series, 93 (1985) 307–313.

[14] P. Erdős and A. Rényi, On a problem in the theory of graphs, Publ. Math. Inst. Hungar. Acad. Sci. 7A (1962) 623–641, in Hungarian.

[15] P. Erdős, A. Rényi and V.T. Sós, On a problem of graph theory, Studia Sci. Math. Hungar. 1 (1966) 215–235.

[16] V. Faber and L. Lovász, Problem 18, in: C. Berge and D.K. Ray-Chaudhuri, eds., Hypergraph Seminar, Ohio State Univ., 1972, Lecture Notes in Mathematics 411 (Springer, Berlin, 1974) 284.

[17] P. Frankl, Intersecting hypergraphs with transitive groups, in: L. Lovász and V.T. Sós, eds., Algebraic Methods in Graph Theory, Szeged, Hungary, 1978, Proc. Coll. Math. Soc. János Bolyai, Vol. 25 (North-Holland, Amsterdam, 1981) 171–176.

[18] P. Frankl, Regularity conditions and intersecting hypergraphs, Proc. Amer. Math. Soc. 82 (1981) 309–311.

[19] P. Frankl and Z. Füredi, Finite projective spaces and intersecting hypergraphs, Combinatorica 6 (1986) 335–354.

[20] Z. Füredi, Erdős-Ko-Rado type theorems with upper bounds on the maximum degree, in: L. Lovász, ed., Algebraic Methods in Graph Theory, Proc. Colloq. Math. Soc. J. Bolyai 25 (Szeged, Hungary, 1978) (North-Holland, Amsterdam, 1981) 177–207.

[21] Z. Füredi, Maximum degree and fractional matchings in uniform hypergraphs, Combinatorica 1 (1981) 155–162.

[22] Z. Füredi, $t$-expansive and $t$-wise intersecting hypergraphs, Graphs Combin. 2 (1986) 67–80.

[23] Z. Füredi, Matchings and covers in hypergraphs, Graphs Combin. 4 (1988) 115–206.

[24] Z. Füredi, Covering the complete graph by partitions, Discrete Math. 75 (1989) 217–226. (Proceedings of the 1988 Cambridge Coll.)

[25] Z. Füredi, Covering pairs by $q^2+q+1$ sets, J. Combin. Theory Ser. A 54 (1990) 248–271.

[26] Z. Füredi and A. Gyárfás, Covering $t$-element sets by partitions, European J. Combin., 12 (1991) 483–489.

[27] Z. Füredi, J. Kahn and P. Seymour, On the fractional matching polytope of a hypergraph, Combinatorica, 13 (1993) 167–180.
[28] Z. Füredi and Á. Seress, Maximal triangle-free graphs with restrictions on the degrees, J. Graph Theory, to appear.
[29] A. Gyárfás, Partition covers and blocking sets in hypergraphs, Ph.D. Thesis; in Hungarian, MTA SzTAKI Tanulmányok, Vol. 71, Budapest, 1977.
[30] D. Hanson and K. Seyffarth, $k$-saturated graphs of prescribed maximum degree, in: Proc. 13th Manitoba Conf. on Numerical Math. and Computing, Winnipeg, Man., 1983, Congr. Numer. 42 (1984) 169–182.
[31] D. Hanson and H.J. Strayer, 3-saturated circulant graphs, Congr. Numer. 71 (1990); Proc. 20th Southeast Conf., Boca Raton, 1989.
[32] E.R. Lamken, R.C. Mullin and S.A. Vanstone, Some non-existence results on twisted planes related to minimum covers, Congr. Numer. 48 (1985) 265–275.
[33] L. Lovász, On minimax theorems of combinatorics, Doctoral Thesis, in Hungarian; Mat. Lapok 26 (1975) 209–264.
[34] W.H. Mills, Covering designs I: covering by a small number of subsets, Ars Combin. 8 (1979) 199–315.
[35] J. Pach and L. Surányi, Graphs of diameter 2 and linear programming, in: L. Lovász et al. eds., Algebraic Methods in Graph Theory, Szeged, Hungary, 1978, Proc. Coll. Math. Soc. J. Bolyai, Vol. 25 (North-Holland, Amsterdam, 1981) 599–629.
[36] J. Pach and L. Surányi, On graphs of diameter 2, Ars Combin. 11 (1981) 61–78.
[37] J. Pelikán, Properties of balanced incomplete block designs, in: Combinatorial Theory and its Appl., Balatonfüred, Hungary, 1969, Proc. Coll. Math. Soc. J. Bolyai, Vol. 4 (North-Holland, Amsterdam, 1970) 869–889.
[38] Á. Seress, On Hajnal's triangle-free game, Graphs and Combin. 8 (1992) 75–80.
[39] C. Shannon, A theorem on coloring the lines of a network, J. Math. Phys. 28 (1949) 148–151.
[40] Y. Shiloach, U. Vishkin and S. Zaks, Golde ratios in a pairs covering problem, Discrete Math. 41 (1982) 57–65.
[41] I. Vrto and Š. Znám, Minimal graphs of diameter two and given maximal degree, Studia Sci. Math. Hungar. 17 (1982) 283–290.
[42] Š. Znám, private communication, Eger, Hungary, 1981.
[43] Š. Znám, Minimal graphs of diameter two, Studia Sci. Math. Hungar. 19 (1987) 187–191.
[44] P. Erdős and R. Holzman, On maximal triangle-free graphs, J. Graph Theory, to appear.