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THE PRISON YARD PROBLEM

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Given a polygon Π with n vertices whose sides are *walls*. Guards, located at vertices can see all directions, but cannot see beyond walls. We prove that at most $\lceil n/2 \rceil$ guards suffice to see everywhere the whole plane. If Π is not convex, then $\lfloor n/2 \rfloor$ suffice.

1. Introduction

The *prison yard problem* is one of a family of guard problems, where one places guards at various points in or on a simple polygon (representing the walls of an enclosure) with the aim of covering (seeing) every point of the interior or exterior regions with at least one guard. Here *covering* (or *seeing*) a point means having an unobstructed line of sight from some guard to the point. The most celebrated versions are the art gallery and the rectilinear art gallery problems, raised by Klee [4] in 1973, and solved in [2] and in [3]. Aggarwal's thesis [1] and a monograph by O'Rourke [7] discuss these matters mainly from the computational geometric point of view. In the prison yard version only vertex guards are allowed (i.e. guards placed on vertices of the polygon) and they required to cover both the interior and the exterior of the polygon. It is easy to see that a (strictly) convex polygon with n vertices requires at most $\lceil n/2 \rceil$ guards and indeed needs that many.

Theorem 1. *$\lceil n/2 \rceil$ vertex guards suffice to cover both the interior and exterior of a simple polygon Π of n vertices. In the non-convex case $\lfloor n/2 \rfloor$ guards suffice.*

The prison yard problem was independently posed by D. Wood and J. Malkelvitch [7]. The value $\lceil n/2 \rceil$ was conjectured by O'Rourke, [7, Conjecture 6.1] who gave an upper bound $\lfloor 2n/3 \rfloor$. This upper bound was improved to $\lceil 7n/12 \rceil$

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by Kooshesh, Moret and Székely [5]. A recent excellent survey is [8] by Shermer; further partial results can be found in the paper of M. Watanabe [9].

We cannot resist to quote O'Rourke's book [7, p. 156.]: "Proving or disproving this conjecture is one of the most interesting open problems in this fields." O'Rourke also devotes a short section to negative results, two natural approaches of the prison yard problem. Constructing counterexamples he showed that neither one leads a solution. (One of the methods is dominating a triangulation by combinatorial vertex guards; the other one is, called Chazelle's method, to form a convex partitioning of the plane from the n-vertex polygon.)

O'Rourke also considered the orthogonal case (when the sides of Π are parallel to the axis) and proved an upper bound $\lfloor 7n/16 \rfloor + 5$. The determination of the exact bound remains open.

2. Prison yards and the Four Color Theorem

Using the n vertices of Π and adding the point infinity one can obtain a triangulation of the whole plane, i.e. a planar graph G with $n+1$ vertices. The Four Color Theorem implies, that there exists a subset C of the vertices of G (namely, the two smallest of the four color-classes) of size at most $(n+1)/2$ such that C meets all triangles, so C covers both the exterior and the interior of Π. The only problem is that C might contain the point infinity. So the essence of the proof is to show that for the graph G with $n+1$ vertices, and for any vertex w there exists a subset C (of size at most $(n+1)/2$) avoiding w and meeting all triangles.

But the real difficulty now is, that the above statement is not true for all planar graphs G and for all $w \in V(G)$. In the following example, G^ℓ, on $5\ell+1$ vertices ($\ell \geq 2$), one needs at least 3ℓ vertices to meet all triangles if one cannot use the vertex w. The vertex w is joined to all the others, the vertices $p_1, \dots, p_{5\ell}$ form a long cycle (in this order), (say, $p_1, \dots, p_{5\ell}$ is a convex 5ℓ-gon on the plain with w lying outside), moreover, for every i $(0 \leq i < \ell)$ $\mathscr{E}(G^\ell)$ contains the triangle formed by the edges $p_{5i+1}p_{5i+3}$, $p_{5i+3}p_{5i+5}$, $p_{5i+1}p_{5i+5}$; finally, the rest of the edges can be drawn arbitrarily (by maintaining planarity). Any cover avoiding w must contain at least 3 vertices from $p_{5i+1}, \dots, p_{5i+5}$ to meet all the triangles containing w and the triangle $p_{5i+1}p_{5i+3}p_{5i+5}$. This example is an extension of the one given by O'Rourke [7, p. 159.] on 10+1 vertices. But the statement is true and will be proved by a tedious induction, (defining not one but immediately two covers A and B, see the rest of the paper) if $G \setminus w$ has no chord:

Theorem 2. *Let $\mathcal{M}$ be a triangulation of the sphere with vertex set V, $w \in V$ an arbitrary vertex. Each triangle of $\mathcal{M}$ can be dominated by a subset of V of size $\lceil |V|/2 \rceil$ avoiding w if $\mathcal{M} \setminus w$ has no chord.*

Note that Theorems 1 and 2 are not equivalent, although the connection between them are ensured by Lemma 5, that the convex hull of a simple polygon Π can be triangulated without using any chord (except in the case Π itself is convex, but this case is trivial). It is very likely, that in Theorem 2 $\lceil |V|/2 \rceil$ can be replaced by $\lfloor |V|/2 \rfloor$.

Our proofs for Theorem 1 and 2 are constructive and give rise to low-order polynomial (in some sense, linear) time procedures for constructing a covering obeying this bound. In general, finding the minimum number of guards is known to be NP-hard (Lee and Lin [6]).

3. Definitions

The convex hull of a set S is denoted by $\operatorname{conv} S$. $[x,y]$ stands for the segment with endpoints x and y. The boundary of a region R is denoted by ∂R.

A planar *simplicial complex* $\mathcal{S}$ is a set of three kinds of members, $\mathcal{S}=V\cup\mathcal{E}\cup\mathcal{T}$, where $V\subset\mathbf{R}^2$ is a finite set, $\mathcal{E}$ is a collection of closed segments with endpoints from V, and $\mathcal{T}$ is a set of closed triangles with sides from $\mathcal{E}$. Moreover, the members of $\mathcal{S}$ are joined regularly, i.e. the set V and the interior points of the segments from $\mathcal{E}$ and the interior points of the triangles from $\mathcal{T}$ are pairwise disjoint. We use the notation $V(\mathcal{S})$, $\mathcal{E}(\mathcal{S})$, and $\mathcal{T}(\mathcal{S})$ to refer for the set of vertices, edges, and triangles of $\mathcal{S}$, respectively. We frequently identify $\mathcal{S}$ by its *abstract* simplicial complex, so $\{x,y\}\in\mathcal{S}$ and $\{a,b,c\}\in\mathcal{S}$ really mean that $[x,y]\in\mathcal{E}(\mathcal{S})$, and $\operatorname{conv}(a,b,c)\in\mathcal{T}(\mathcal{S})$. If $R=\cup\mathcal{S}$, then $\mathcal{S}$ is called a *triangulation* of R.

A simplicial complex $\mathcal{C}$ is called a *subcomplex* of $\mathcal{S}$ if $\mathcal{C}\subset\mathcal{S}$. For $W\subset V$ we define the *induced subcomplex* or *restriction* $\mathcal{S}|W$ as $\{X\in\mathcal{S} : X\subset W\}$, and let $\mathcal{S}\setminus W=\mathcal{S}|(V\setminus W)$. The members of $\mathcal{S}$ lying completely on the boundary of the region $\cup\mathcal{S}$ form again a simplicial complex, $\partial\mathcal{S}$. A *chord* $[x,y]$ is an edge from $\mathcal{E}(\mathcal{S})$, such that $[x,y]\notin\partial\mathcal{S}$ but both $x,y\in\partial\mathcal{S}$.

A *planar triangulated figure*, $\mathcal{R}$, is a simplicial complex with no holes, (i.e. $\mathbf{R}^2\setminus(\cup\mathcal{R})$ is connected), and it might equipped with a *special* vertex $q=q(\mathcal{R})$ chosen from the boundary, $q\in V(\mathcal{R})\cap\partial\mathcal{R}$. We call a vertex on the boundary of $\mathcal{R}$ *ordinary* if it is neither a *cut vertex* of $\mathcal{R}$ nor the special vertex q. The vertices/edges of $\partial\mathcal{R}$ are also called *external* vertices/edges.

$\mathcal{R}$ is 2-*connected* if it is connected and does not have any cut vertex. Then $\partial\mathcal{R}$ is a *cycle*. A general planar triangulated figure $\mathcal{R}$ may have (maximal) 2-connected subcomplexes that are linked together at cut vertices and perhaps at cut edges or *isthmuses*. A 2-connected subcomplex $\mathcal{C}$ of $\mathcal{R}$ may have chords, which can partition $\mathcal{C}$ into a tree like structure of blocks that are chordless.

We call a 2-connected subcomplex $\mathcal{C}$ of $\mathcal{R}$ *simple* if it either contains no chords (in $\mathcal{C}$), or $|V(\mathcal{C})|=4$, or if each of its chords cuts off from the rest of $\mathcal{C}$ exactly one triangle T whose third vertex is ordinary. Any such triangle T will be called a *petal* of $\mathcal{C}$, and the third vertex is a *petal vertex* of $\mathcal{C}$. A chord not cutting off a petal is called non-trivial.

4. Compatible covers

Let $\mathcal{S}$ be a planar triangulation. A *cover* of the triangles is a subset of the vertices $C\subset V(\mathcal{S})$ meeting all members of $\mathcal{T}(\mathcal{S})$. A cover C is *compatible* with the subset of edges $\mathcal{A}\subset\mathcal{E}(\mathcal{S})$, if it meets every member of $\mathcal{A}$, too.

A quasi 2-*coloring*, χ, of the edges of $\mathcal{E}$ with colors α and β is a mapping χ: $\mathcal{E}\to\{\emptyset,\alpha,\beta,\{\alpha,\beta\}\}$, i.e. it is a partial multicoloring, every edge can have both, one, or none of the colors α and β. We say that the covers A and B are compatible with a given quasi 2-coloring χ, if A is compatible with the set of edges having color α, and B is compatible with the set of edges having color β, i.e. with the set $\{E\in\mathcal{E}: \beta\in\chi(E)\}$.

Our main results, Theorem 1 and 2, are implied by the following.

Theorem 4. *Let $\mathcal{R}$ be a planar triangulated figure with n vertices, each of whose external edges is labeled with (at most) one label chosen from the set $\{\alpha,\beta\}$. Isthmuses, which are external on both sides, can have both labels. Let q be any vertex on the boundary of $\mathcal{R}$, (called the special vertex). Then there are subsets A and B of the vertex set of $\mathcal{R}$ such that:*
— *each triangle of $\mathcal{R}$ intersects set A and set B; vertex q lies in set B; and*
— *every external edge of $\mathcal{R}$ intersects the set denominated by its label(s); and*
— *the cardinalities of A and B sum to at most n.*

The proof of Theorem 4 goes by induction on the number of vertices, it consists of several reduction steps. We will describe a sequence of "reductions" that allow us to obtain sets A and B obeying our conditions given similar sets on a smaller figure. There may be many generalizations to non-polygonal regions that can be resolved by the same argument.

5. Triangulations with no chords

In this section before starting the coloring process, we find an appropriate triangulation of $\operatorname{conv}\Pi$. The following lemma was also proved (independently) by Kooshesh, Moret and Székely (see [5, Lemma 4]).

Lemma 5. *Given a non-convex, simple polygon Π with vertex set V. Then one can triangulate* $\operatorname{conv}\Pi$ *obtaining $\mathcal{R}$ so that*
(5.1) *$\mathcal{R}$ uses only vertices of Π, i.e. $V(\mathcal{R})=V$,*
(5.2) *$\mathcal{R}$ uses all edges of Π, and*
(5.3) *$\mathcal{R}$ does not use any chord of* $\operatorname{conv}\Pi$.

Proof of Lemma 5. First, note that no side of Π is a chord of $\operatorname{conv}\Pi$. Hence, starting with the sides of Π and $\operatorname{conv}\Pi$, and adding edges joining vertices of V as far as it is possible one gets a triangulation satisfying (5.1) and (5.2). Let $\mathcal{S}$ be a triangulation satisfying (5.1) and (5.2), and having minimum number of chords of $D:=V(\operatorname{conv}\Pi)$. We claim that $\mathcal{S}$ satisfies property (5.3), as well.

Suppose, on the contrary, that $\{x,y\}\in\mathcal{E}(\mathcal{S})$ with $x,y\in D$ is a chord. There are two triangles containing this inner edge $\{x,y,a\}$, $\{x,y,b\}\in\mathcal{T}(\mathcal{S})$. As x and y are on the boundary of $\operatorname{conv}\Pi$, $\{x,a,y,b\}$ form a convex quadrilateral. If $\{a,b\}\not\subset D$, i.e. either a or $b\in\operatorname{int}\operatorname{conv}\Pi$, then replacing $\{x,y\}$ by $\{a,b\}$ (and also $\{x,y,a\}$ and $\{x,y,b\}$ by $\{a,b,x\}$ and $\{a,b,y\}$) one gets a triangulation with fewer chords.

Otherwise, $a,b\in D$. Let $K\subset D$ be a set containing $\{a,x,b,y\}$ with maximal cardinality with the property $\cup(\mathcal{S}|K)=\operatorname{conv}K$. (This means, that $\operatorname{conv}K$ is triangulated only by the sides and chords of D, like $\{a,x,b,y\}$.) There are two neighboring vertices of $\partial\operatorname{conv}K$, say u,v, such that $[u,v]\notin\Pi$. Then there is a vertex $z\in\operatorname{int}\operatorname{conv}\Pi\setminus\operatorname{conv}K$ with $\{u,v,z\}\in\mathcal{S}$. The set $K\cup\{z\}$ forms a vertex set of a convex polygon. Delete $\mathcal{S}|(K\cup\{z\})$ from $\mathcal{S}$ and add a new triangulation obtained by the chords $\{\{z,k\}:k\in K\}$. The obtained new triangulation of $\operatorname{conv}\Pi$ has fewer chords of D than $\mathcal{S}$. This contradiction completes the proof. ∎

6. Proof of the Prison Yard Theorem (and Theorem 2) via Theorem 4

Proof of Theorem 1. As the convex case is easy, let Π be a non-convex, simple polygon with vertex set V, $|V|=n$. Denote by D the vertex set of $\operatorname{conv}\Pi$. These vertices are naturally ordered along $\partial\operatorname{conv}\Pi$, $D=\{x_1,x_2,\dots,x_d\}$. We may suppose that $[x_d x_1]\notin\Pi$. Apply Lemma 5 to get a triangulation $\mathcal{S}$ of $\operatorname{conv}\Pi$ with no chords. Put the vertices of D into two sets, A_0 and B_0, alternately, i.e. $A_0:=\{x_1,x_3,\dots,x_{2i+1},\dots\}$, and $B_0:=\{x_{2i}:2\le 2i\le d\}$.

Let $\mathcal{R}$ denote the triangulation $\mathcal{S}\setminus D$. It is a (connected) planar triangulated figure. Let q, the special vertex of $\mathcal{R}$, be defined as the third vertex of the triangle of $\mathcal{S}$ containing $[x_1,x_d]$. Label the edge $E\in\partial\mathcal{R}$ by the color α (or β) if a triangle $T\in\mathcal{S}$, $E\subset T\in\mathcal{S}$ meets B_0 (or A_0, respectively). Isthmuses can have both labels. Label α means that if we want to extend the set A_0 to a cover of the triangles of $\mathcal{S}$ using only vertices from $V\setminus D$, then one or both endpoints of that edge must be placed in A. Theorem 4 supplies two compatible covers, A and B, of $\mathcal{R}$ with $q\in B$, and with $|A|+|B|\le n-|D|$.

It is easy to check that both of the sets $A_0\cup A$ and $B_0\cup B$ meet all triangles of $\mathcal{S}$. Indeed, the triangles of $\mathcal{R}$ are covered by both A and B. The triangles of $\mathcal{S}$ with an exterior edge $[x_i x_{i+1}]$ meets both A_0 and B_0 (with possible exception $\{x_1,x_d,q\}$, but then $x_1\in A$, $q\in B_0$). All the other triangles meets D in exactly one point, so they are covered by the compatibility of α and A, and β and B.

These covers $A_0\cup A$ and $B_0\cup B$ form vertex guard sets for the exterior of $\operatorname{conv}D$, too, hence for the whole plane. Indeed, A_0 covers this exterior, and B_0 covers almost all the exterior with the possible exception is the three-sided region bounded by the segment $[x_1,x_d]$, and by a ray starting at x_1 opposite to x_2 and by another ray starting at x_d and opposite to x_{d-1}. However, this region is guarded by the vertex $q\in B$. Finally, the smaller of these sets has cardinality at most $n/2$. ∎

Proof of Theorem 2. Like the above proof. However, $B_0\cup B$ does not necessarily cover the triangle $\{x_1,x_d,w\}$, so we add to it one more element from $\{x_1,x_d\}$. We get $|A_0\cup A|+|B_0\cup B|\le|V|+1$. Take the smaller one. ∎

7. Starting the induction

To prove Theorem 4 we apply induction on n. If $\mathcal{R}$ is not connected, than we can construct the appropriate compatible covers for each component separately. If $\mathcal{I}(\mathcal{R})=\emptyset$ (and $\mathcal{R}$ is connected), than $\mathcal{E}(R)$ is a *tree*. It is a bipartite graph, so the color class not containing q can be A, the other one can be B. From now on, we may suppose, that $\mathcal{R}$ is a connected triangulated figure, and $n\ge 3$.

If the special vertex, q, is a cut vertex, then there exist $V_1,V_2\subset V(\mathcal{R})$, $V_1\cap V_2=\{q\}$, $V_1\cup V_2=V(\mathcal{R})$ such that the subcomplexes $\mathcal{R}_1=\mathcal{R}|V_1$ and $\mathcal{R}_2=\mathcal{R}|V_2$ are connected only at the vertex q, $\mathcal{R}=\mathcal{R}_1\cup\mathcal{R}_2$. Apply the induction hypothesis to $\mathcal{R}_i$ $(i=1,2)$ to obtain the compatible covers A_i, B_i with $|A_i|+|B_i|\le|V_i|$. Define $A=A_1\cup A_2$, $B=B_1\cup B_2$. As q is the special vertex we have $q\in B_1\cap B_2$ hence we get that $|B|\le|B_1|+|B_2|-1$. This implies $|A|+|B|\le|V_1|+|V_2|-1=n$.

If any vertex v on the boundary of $\mathcal{R}$ has no label β on any of its incident boundary edges, we may place v in set A, remove v from $\mathcal{R}$, and place labels β

on all the boundary edges of $\mathcal{R}\setminus v$ that were internal in $\mathcal{R}$ and labels α and β on any isthmus in $\mathcal{R}\setminus v$ that is not an isthmus in $\mathcal{R}$. Finding sets A and B in $\mathcal{R}$ can therefore be accomplished by finding corresponding sets in $\mathcal{R}\setminus v$ so labeled (using our induction hypothesis); and appending v to $\mathcal{R}\setminus v$, and to set A. The same statement holds with A and B interchanged except for vertex q. We may therefore restrict our attention to $\mathcal{R}$ such that every ordinary vertex has labels α and β on its two incident boundary edges.

If $\mathcal{R}$ contains a pendant vertex t, connected only by an isthmus, $[t,u]$, to the rest of $\mathcal{R}$, we may place u in both A and B, and t in neither. With u in both A and B, both sets intersect every triangle and edge containing t or u. If we find sets A and B in $\mathcal{R}\setminus\{t,u\}$ obeying our conditions with arbitrary labels on any edges external in $\mathcal{R}\setminus\{t,u\}$ but not in $\mathcal{R}$; by adding u to each set we will have such sets in $\mathcal{R}$. The above argument must be modified in the case $t=q$; then put t into B, and u into A, and call u to the special vertex of $\mathcal{R}\setminus t$. The role of α, A and β, B should be exchanged.

Lemma 7. *Every planar triangulated figure, $\mathcal{R}$, with $|V(\mathcal{R})|\geq 3$ and with the distinguished vertex q, contains one of*
(7.1) *a pendant vertex other than q; or*
(7.2) *a simple 2-connected region $\mathcal{C}$ connected to the rest of $\mathcal{R}$ at a single vertex v_1, with q not in $\mathcal{C}\setminus v_1$; or*
(7.3) *a simple 2-connected region $\mathcal{C}$ connected to the rest of $\mathcal{R}$ through two adjacent vertices $[v_1,v_2]\in\partial\mathcal{C}$, with q not in $\mathcal{C}\setminus\{v_1,v_2\}$ and with $|V(\mathcal{C})|\geq 4$; or*
(7.4) *$\mathcal{R}$ is itself simple.*

Proof. Recall that for $|V(\mathcal{C})|\geq 5$ simple triangulation means that it is 2-connected, no non-trivial chord, but it might have petals. We may further suppose in Case (7.2) that $v_1\neq q$ because we already supplied reduction for the case q being a cut vertex. Proof of Lemma 7 can be obtained by standard argument left to the reader. ∎

8. The main idea of the proof of Theorem 4

Consider a simple 2-connected part of $\mathcal{R}$, denote it by $\mathcal{C}$ (this is usually given by (7.2)-(7.4)). $\mathcal{C}$ is a planar triangulated figure, its boundary edges inherit the quasi 2-coloring of $\partial\mathcal{R}$ (of course, only those edges appearing simultaneously in $\partial\mathcal{C}$ and in $\partial\mathcal{R}$). Let $C:=V(\mathcal{C})$ denote its vertex set, D the vertex set of the boundary, $D=\{v_1,v_2,\dots,v_d\}$ in this order, $d\geq 3$, (i.e. $D=V(\partial\mathcal{C})$, $[v_iv_{i+1}]\in\mathcal{E}(\mathcal{C})$ for all $1\leq i\leq d$, $v_{d+1}=v_1$). We may suppose that the colors α and β alternate in the edges $\partial\mathcal{C}$, except in at most one vertex in case of d is odd. But, in general, we do not require that all the boundary edges are colored, neither the presence of a special vertex in D.

As $\mathcal{C}$ is simple, for any triangle $T\in\mathcal{T}(\mathcal{C})$ one of the following holds
(8.1) $T\cap D=\emptyset$, T is completely internal;
(8.2) $|T\cap D|=1$, T has exactly one vertex on the boundary;
(8.3) $T\cap D=E$ is a boundary edge from $\partial\mathcal{C}$, the third vertex is internal;
(8.4) $|T\cap D|=2$, T contains (exactly) one chord. This chord cuts off a petal;

(8.5) T is a petal, it contains three consecutive vertices of D;
(8.6) T has exactly two chords, then $\mathscr{C}$ is a pentagon with these two chords;
(8.7) T has exactly three chords, then $\mathscr{C}$ is a hexagon with these three chords;
(8.8) $T = \mathcal{I}(\mathscr{C})$.

Let X and Y be a partition of D, $X \cup Y = D$, $X \cap Y = \emptyset$. (In most cases we simply put the boundary vertices alternately in two sets, $X = \{v_1, v_3, v_5, \ldots\}$). Using the induction hypothesis, we are going to construct two subsets of $C \setminus D$, X_{in} and Y_{in}, such that $|X_{\text{in}}| + |Y_{\text{in}}| \le |C \setminus D|$, and the sets $X^* = X \cup X_{\text{in}}$ and $Y^* = Y \cup Y_{\text{in}}$ both cover all the triangles $T \in \mathcal{I}(\mathscr{C})$ whenever T has at most one vertex on the boundary D. We call X^* and Y^* the *extensions* of X and Y in $\mathscr{C}$.

These extensions are constructed exactly in the same way as in Section 6 the sets A and B. Consider the triangulation $\mathscr{C} \setminus D$. Label the edge E in its boundary by the color ξ (or v) if a triangle T, with $E \subset T \in \mathscr{C}$ meets Y (or X, respectively). (Isthmuses can have both labels.) Theorem 4 (applied to $\mathscr{C} \setminus D$ with this boundary coloration) yields, X_{in}, a cover of $\mathscr{C} \setminus D$ compatible with ξ and Y_{in}, a cover compatible with v, such that the sum of sizes of these sets does not exceed $|C| - |D|$. It is easy to check that both of the sets X_{in} and Y_{in} meet all triangles of $\mathscr{C}$ having no chord. Let us note, that we can further designate one vertex from $\partial(\mathscr{C} \setminus D)$ into X_{in} or Y_{in} (as a special vertex of that figure).

If we would like to use X as (a part of) a cover of $\mathscr{C}$ compatible to α, then a petal vertex $v_i \in X \cap D$ has no significant role in covering the triangles of $\mathcal{I}(\mathscr{C})$; it can be replaced by any of its neighbors v_{i-1} or by v_{i+1}. Even more, these vertices meet more triangles. v_i's only function is to meet those external edges $[v_{i-1}, v_i]$ and $[v_i, v_{i+1}]$ of $\mathscr{C}$ which are labeled α. If one of these edges, say $[v_{i-1}, v_i]$, is not labeled by α, then we take v_i out of X and instead put v_{i+1} in. (If none of the two external edges have label α then we replace v_i arbitrarily by one of its neighbors in D.) Starting with the set X and doing the above exchange operations simultaneously for all eligible petal vertices we obtain the set X_α. The definitions of X_β, Y_α and Y_β are similar. We do not modify the sets X_{in} and Y_{in}. We set $X^*_\alpha = X_\alpha \cup X_{\text{in}}$ and $X^*_\beta = X_\beta \cup X_{\text{in}}$, $Y^*_\alpha = Y_\alpha \cup Y_{\text{in}}$, $Y^*_\beta = Y_\beta \cup Y_{\text{in}}$. We call this process the *petal modification* of X, (of X^*, etc.). The modified set X_α (X^*_α, etc.) meets at least as many triangles and edges labeled α as the original X (X^*, etc.) does.

Summarizing, let $\mathscr{C}$ be a simple figure, and suppose that there is no triangle $T \in \mathcal{I}(\mathscr{C})$ of types (8.6)–(8.8). Take a partition $X \cup Y = D$ extend them in the interior of $\mathscr{C}$, and apply the appropriate petal modifications. Then the following lemma states, that the obtained sets X^*, X^*_α, X^*_β are compatible covers with the following possible deficiencies. (A similar statement holds for Y^*, Y^*_α, Y^*_β.)

Lemma 8. (i) $|X_\alpha| \le |X|$, $|X^*_\alpha| \le |X^*|$ *etc., and* $|X^*| + |Y^*| \le |C|$.
(ii) *If* $v \in X$ *is not a petal vertex, then* $v \in X^*_\alpha$ *and* $v \in X^*_\beta$.
(iii) X^*, X^*_α, X^*_β *meet all triangles of type* (8.1)-(8.2).
(iv) *If* X^*_α (X^*, *or* X^*_β) *misses a triangle* $\{v_i, v_{i+1}, x\}$ *of type* (8.3), *then* $\{v_i, v_{i+1}\} \subset Y$ *and* x *is not the designated special vertex of* X_{in}.
(v) *If* X^*_α *misses a triangle* $\{v_i, w, v_{i+2}\}$ *of type* (8.4), *then* $\{v_i, v_{i+2}\} \subset Y$, *and either* $v_{i+1} \in Y$, *too, or* $v_{i+1} \in X$ *with* $\alpha = \chi([v_i, v_{i+1}]) = \chi([v_{i+1}, v_{i+2}])$.

(vi) *If X^*_α misses a triangle $\{v_i, v_{i+1}, v_{i+2}\}$ of type* (8.5) *(petal), then all these three vertices belong to Y.*
(vii) *If X^*_α misses an edge $\{v_i, v_{i+1}\}$ with $\alpha \in \chi([v_i, v_{i+1}])$, then $\{v_i, v_{i+1}\} \subset Y$.* ∎

During the induction proving Theorem 4 beside the simple reductions discussed above we use the following method, what we call *basic reduction.*

Basic Reduction. *It consists of the following steps: Select an appropriate 2-connected simple subcomplex $\mathcal{C}$. Put $C := V(\partial\mathcal{C})$ in X and Y (in most cases alternately), then extend them to X^* and Y^* in the interior of $\mathcal{C}$. Change X^* to X^*_α and X^*_β, and do so with Y^*. We will construct the sets A_1 and B_1 using the above four sets. Consider the planar triangulated figure $\mathcal{Q}$, (it is usually $\mathcal{R}\setminus C$ with eventually modifying the labeling on its boundary). Apply induction for $\mathcal{Q}$ to get A_2 and B_2. Finally, $A = A_1 \cup A_2$, and $B = B_1 \cup B_2$ will satisfy the conditions of Theorem 4.*

The precise meaning of these reductions is that any pair (A_2, B_2) that obeys the conditions of Theorem 4 on the reduced region $\mathcal{Q}$ can be extended to one that obeys the same conditions on the entire region $\mathcal{R}$. The crucial point in the basic reduction is, that one could not use simply the induction hypothesis for the two parts of $\mathcal{R}$, because in general $|V(\mathcal{C})| + |V(\mathcal{Q})| > n$, so we have to guarantee somehow that the covers of $\mathcal{C}$ and $\mathcal{Q}$ fit together.

By virtue of Lemma 7, we need to distinguish only four cases (7.1)-(7.4) to complete the proof. We have already provided reductions for the case of the pendant edge (7.1) in Section 7. The case of the cut vertex (7.2) is the subject of the next Section 9. It is divided into subcases according to the parities of $|C|$ and of $|D|$. Our reduction in case (7.2) will apply to case (7.4) as well, simply set $v_1 = q$ and follow the same argument. In the case of the cut segment (7.4) we again use four separate reductions depending on the parities of $|C|$ and of $|D|$. Each of these cases might be subdivided into several subcases. The case (7.4) when both $|C|$ and $|D|$ are even is especially complicated; it is considered in the last Section 12.

9. The case of a cut vertex

We now turn to the problem of reducing a simple figure $\mathcal{C}$, connected to $\mathcal{R}\setminus\mathcal{C}$ only at vertex v_1, with q not in $\mathcal{C}\setminus v_1$ (or $\mathcal{C} = \mathcal{R}$ itself is simple). Let C denote its vertex set, and let D be the set of vertices of the boundary, $D = \{v_1, v_2, \dots, v_d\}$, $d \geq 3$. We may also suppose that $\mathcal{C}$ is a minimal subcomplex supplied by Lemma 7, so if the special vertex $q \in D$, then $q = v_1$.

Consider first the case when $E = [v_d, v_2] \in \mathcal{E}(\mathcal{C})$ and it is a **chord**. The minimality of $|C|$ implies that E does not cut off more than one vertex from the rest of $\mathcal{C}$ (and so from $\mathcal{R}$), because we then use reduction (7.3) for $\mathcal{C}\setminus v_1$ (or for a subfigure of that). Hence, $C = D = \{v_1, v_2, v_3, v_4\}$ with the only chord E. Let $A_1 = B_1 = \{v_1, v_3\}$ and apply induction to $\mathcal{Q} = \mathcal{R}\setminus C$.

Consider the case when a **chord** of $\mathcal{C}$ starts from v_1, $F = [v_1, v_i] \in \mathcal{E}(\mathcal{C})$, $1 < i < d$. The minimality of $|C|$ implies that F does not cut off more than one vertex from $\mathcal{C}$ (on either side), because we then use reduction (7.3). Hence, $C = D =$

$\{v_1, v_2, v_3, v_4\}$ with the only chord $F = [v_1, v_3]$. Let $A_1 = B_1 = \{v_1, v_3\}$ and apply induction to $\mathcal{Q} = \mathcal{R} \setminus C$.

From now on, we may suppose that v_1, v_2 and v_d are not petal vertices of $\mathcal{C}$. Moreover, there is no triangle in $\mathcal{T}(\mathcal{C})$ containing more than one chord. Indeed, suppose, on the contrary, that T has at least two chords. Then, similarly as we did above, one can use one of these chords to cut off a smaller simple figure (containing T) from $\mathcal{C}$. This contradicts the minimality of $|C|$.

From now on we may apply Lemma 8. Suppose that $|D|$ **is even**. Assign every second vertex on D to X and the rest to Y, with v_1 in X, i.e. $X = \{v_1, v_3, \dots, v_{d-1}\}$, $Y = \{v_2, v_4, \dots, v_d\}$. These assignments can be extended to all of $\mathcal{C}$ as described in Section 8 and both $X^* = X_{\text{in}} \cup X$ and Y^* meet all triangles having at most one vertex from D (types of (8.1)-(8.2)). Our alternating assignment assures that there is a member of X and of Y in every edge in D, triangles of $\mathcal{C}$ that contain an edge of D intersect both X and Y automatically. So X and Y cover all triangles of types (8.3) and (8.5). If a triangle $T \in \mathcal{T}(\mathcal{C})$ is not covered by X, then it must have exactly one chord, say $[v_i, v_{i+2}]$, with both endpoints in X, $v_i, v_{i+2} \in X$ and an inner vertex $w \notin D$, $T = \operatorname{conv}(v_i, v_{i+2}, w)$, i.e. of type (8.4). So after the appropriate modifications for petals X^*_α and Y^*_α are α-compatible covers of $\mathcal{C}$, while X^*_β and Y^*_β are β-compatible covers.

If $|X| \le |C|/2$, then we may use X^*_α for A_1 and X^*_β for B_1 which will assign v_1 to both. This will reduce the problem to $\mathcal{Q} = \mathcal{R} \setminus \mathcal{C}$. **Otherwise**, we must have $|Y^*| \le (|C|-1)/2$, and we may use Y^*_α for A_1 and Y^*_β for B_1, and reduce the problem to $\mathcal{R} \setminus (C \setminus v_1)$.

In the rest of this section we suppose that $|D|$ **is odd**. Assign vertices of D alternately to X and Y again with v_1 in X, but now two adjacent vertices v_1, v_d of D must be assigned to the same set, which we choose to be X, $X := \{v_1, v_3, \dots, v_d\}$, $Y := D \setminus X$. This assignment can be extended to $\mathcal{C} \setminus D$ as before, but now the vertex of $\mathcal{C} \setminus D$ whose incident boundary triangle contains $[v_1, v_d]$ must be a member of Y^* in this extension. Our induction hypothesis proves the existence of extensions obeying this condition. Again, similarly as above, we get the compatible covers X^*_α and X^*_β after appropriate modifications for petals. However, the analogously obtained Y^*_α and Y^*_β are not necessarily compatible covers of $\mathcal{C}$, they might miss the edge $[v_1, v_d]$.

If $|X^*| \le |C|/2$, then the resulting sets X^*_α can be A_1, while X^*_β can be B_1 and we can reduce to $\mathcal{Q} = \mathcal{R} \setminus C$. **If** $|Y^*| \le (|C|/2) - 1$, then we can reduce to $\mathcal{Q} = \mathcal{R} \setminus C$ with $A_1 := Y^*_\alpha + v_1$, $B := Y^*_\beta + v_1$.

The only missing case is $|Y^*| = (|C|-1)/2$. From now on we may suppose that $|C|$ **is odd**. Redefine X as $X = \{v_1, v_3, \dots, v_{d-2}\}$, and let $Y = D \setminus X$. Then the only homogeneous boundary edge is $[v_{d-1}, v_d]$. We explain the subcase when its **label** is β. (The other subcase follows similarly by replacing β and α, and X^*_β by X^*_α.) Using the induction hypothesis extend X and Y to get X^* and Y^*, and modify them at the petals. Then Y^*_α, Y^*_β and X^*_α are compatible covers of $\mathcal{C}$, while X^*_β might miss $[v_{d-1} v_d]$.

In this case, **if** $|Y^*| \leq (|C|-1)/2$, then we can use Y^*_α for A_1 and Y^*_β for B_1 and reduce to $\mathcal{Q}=\mathcal{R}\setminus(C\setminus v_1)$. **If not**, we must have $|X^*| \leq (|C|-1)/2$ and we can use X^*_α as A_1 and $X^*_\beta + v_d$ as B_1, reducing to $\mathcal{Q}=\mathcal{R}\setminus C$.

10. The case of a cut segment, beginning

We now suppose that our simple 2-connected figure $\mathcal{C}$ with boundary $D = \{v_1,\dots,v_d\}$ $(v_{d+1}=v_1)$ is connected to the rest of $\mathcal{R}$ through the vertices v_1 and v_2 of the edge $[v_1,v_2]$, the other vertices from D are ordinary (in $\mathcal{R}$, and so in $\mathcal{C}$). The labels α and β alternate around $\mathcal{C}$ (except $[v_1,v_2]$ has no label). Again, C denotes the vertex set of $\mathcal{C}$. The special vertex $q \notin C\setminus\{v_1,v_2\}$. We may suppose again, that $|C|$ is minimal among such sets (and $|C|\geq 4$).

Consider first the case when one of the connecting vertices v_1,v_2, say v_2, is a **petal** vertex in $\mathcal{C}$, $\{v_1,v_2,v_3\}\in\mathcal{T}(\mathcal{C})$. The minimality of $|C|$ implies that the chord $[v_1,v_3]$ does not cut off more than one vertex from the rest of $\mathcal{C}$ (and so from $\mathcal{R}$). Hence, $C=D=\{v_1,v_2,v_3,v_4\}$. Suppose that the (identical) **labels** on the the boundary edges $[v_2,v_3]$ and $[v_4,v_1]$ are α, (the case β is similar). Apply induction to the complex $\mathcal{Q}=\mathcal{R}\setminus\{v_3,v_4\}$ with label α on the new boundary edge $[v_1,v_2]$. We obtain the compatible covers A_2 and B_2, with $|A_2|+|B_2|\leq n-2$. Suppose, that the vertex v_i belongs to the cover A_2 $(i\in\{1,2\})$. Then the sets $A:=A_2\cup\{v_{3-i}\}$, $B:=B_2\cup\{v_3\}$ yield compatible covers for the whole $\mathcal{R}$.

Consider the case when a triangle $T=\{a,b,c\}\in\mathcal{T}(\mathcal{C})$ contains more than one **chords** of $\mathcal{C}$, say $[a,b]$ and $[a,c]$ are chords. The three vertices of T cut $\partial\mathcal{C}$ into tree arcs. The cut segment $[v_1,v_2]$ can not lie on the arc ac (neither on ab), because then the segment $[a,c]$ cut off a smaller simple figure (containing T) from $\mathcal{C}$. Thus it lies on the arc bc. The segment $[b,c]$ cannot be a non-trivial chord, so it is either cuts off a petal, or it is an external edge of $\mathcal{C}$ (i.e., $[b,c]=[v_1,v_2]$). The first case is impossible, since we may suppose that neither of v_1,v_2 is a petal vertex. In the later case $\mathcal{C}$ is a pentagon, $D=\{v_1,\dots,v_5\}$ with the chords $[v_1,v_4]$ and $[v_2,v_4]$ (i.e. $a=v_4$). As this figure is symmetric, we may suppose that $\chi([v_2,v_3])=\alpha$, hence $\chi([v_5,v_1])=\beta$.

To find compatible covers of $\mathcal{R}$ we apply induction to the complex $\mathcal{Q}=(\mathcal{R}\setminus\{v_3,v_4,v_5\})\cup\mathcal{C}'$, where $\mathcal{C}'$ is the petal (v_1,x,v_2) with a new vertex x and $[x,v_1]$ is labeled β and $[x,v_2]$ is labeled α. We obtain the compatible covers A_2 and B_2 of $\mathcal{Q}$ with $|A_2|+|B_2|\leq n-2$. We may suppose that none of them contains the artificial vertex x, (otherwise we may replace it by one of its neighbors), so $v_1\in B_2$ and $v_2\in A_2$. Then the sets $A:=A_2\cup\{v_4\}$, $B:=B_2\cup\{v_4\}$ yield compatible covers for the whole $\mathcal{R}$.

From now on, we may suppose that any triangle from $\mathcal{T}(\mathcal{C})$ containing two or more vertices of D is either a petal, or a neighbor of a petal (and then it has exactly one vertex in $C\setminus D$), or contains an external edge $[v_i,v_{i+1}]$ $(1\leq i\leq d)$. So we may apply Lemma 8 if needed. Furthermore, we may suppose that v_1 and v_2 are not petal vertices of $\mathcal{C}$. Let z be the third vertex in the triangle containing $[v_1,v_2]$, we have that $z\in C\setminus D$. The above properties make an alternating assignment of the boundary vertices into X and Y, (and then extending and modifying at the petals) a very efficient way of finding compatible covers.

11. The case of a cut segment, odd boundary

Let us summarize again the method applied in Section 9 in the following Corollary what we will use several times. (We formulate it only for X and α.) Let $\mathscr{C}$ be a simple figure such that there is no triangle $T \in \mathscr{T}(\mathscr{C})$ with two or more chords (i.e. of types (8.6)–(8.7)). Suppose that the there are no two neighboring external edges with the same color. (This implies, that in the case d odd at least one edge is colorless.) Call a partition $X \cup Y = D$ *strictly alternating* if each but at most one boundary edge meets both X and Y. In the case d odd the only homogeneous edge is denoted by H, $H \in \partial\mathscr{C}$, and suppose its endpoints are in X, and its color is $\chi(H) = \alpha$. Extend X and Y in the interior of $\mathscr{C}$, and apply the appropriate petal modifications.

Corollary 11. *X^*_α is an α-compatible cover of $\mathscr{C}$, X^*_β and Y^*_β are β-compatible covers; Y^*_α is an α compatible cover, except that it might miss the edge H. Finally, they fulfill Lemma 8* (i) *and* (ii), *too.* ∎

We now continue the investigation of the case when $\mathscr{C}$ is a simple 2-connected figure connected to the rest of $\mathscr{R}$ through the vertices v_1 and v_2 of the edge $[v_1, v_2]$. In view of the last paragraph of the previous Section we could apply Corollary 11.

If $|D|$ is **odd and** $|C|$ is **even**, then we place the vertices in D alternately in X and Y so that both v_1 and v_2 are in X, $(X = \{v_1\} \cup \{v_{2i} : 2 \le 2i < d\})$. This assignment can be extended into $\mathscr{C}$ by induction (with z being a special vertex of $\mathscr{C} \setminus D$, $z \in Y_{\text{in}}$), and both X^* and Y^* (after some modifications at petals) can be used for either A_1 or B_1 (according to Corollary 11). **If** $|X^*| \le |C|/2$, then let $A_1 := X^*_\alpha$, $B_1 = X^*_\beta$ and reduce to $\mathscr{Q} = \mathscr{R} \setminus C$. (Here we used that the vertices v_1, v_2 were not moved at the petal modification, so $\{v_1, v_2\} \subset X^*_\alpha \cap X^*_\beta$.) **Otherwise**, $2|Y^*| \le |C| - 2$ and we may use Y^*_α for A_1 and Y^*_β for B_1 and reduce to $\mathscr{Q} = \mathscr{R} \setminus (C \setminus \{v_1, v_2\})$.

If $|D|$ and $|C|$ are **both odd**, we make use of two different assignments of vertices of D to X^γ and Y^γ $(\gamma \in \{\alpha, \beta\})$, in each of which one of v_1 and v_2 is in each set, and some adjacent pair $\{v^\gamma, w^\gamma\}$ lies in the same set, say X^γ. In one, $[v^\alpha, w^\alpha]$ is chosen to have the label α; in the other β. E.g., suppose that the label on $[v_2, v_3]$ is α, then the label on $[v_d, v_1]$ is β. Define X^α as $\{v_2\} \cup \{v_{2i+1} : 3 \le 2i+1 \le d\}$, $Y^\alpha := D \setminus X^\alpha$, and let $X^\beta := \{v_{2i+1} : 1 \le 2i+1 \le d\}$, $Y^\beta := D \setminus X^\beta$. Assign the third vertex of the triangle containing the homogeneous edge $[v^\gamma, w^\gamma]$ to a special vertex of $\mathscr{C} \setminus D$ belonging to Y^γ_{in} (if that vertex belongs to $C \setminus D$). Extend X^γ, Y^γ into $\mathscr{C}$ by using the induction hypothesis. Corollary 11 implies, that after petal modifications the sets $X^{\gamma*}_\alpha$, $X^{\gamma*}_\beta$, $Y^{\beta*}_\alpha$, $Y^{\alpha*}_\beta$ are compatible covers of $\mathscr{C}$, while $Y^{\alpha*}_\alpha$ might miss the edge $[v_2, v_3]$, and $Y^{\beta*}_\beta$ might miss the edge $[v_1, v_d]$ (only).

In either case, **if** $|X^{\gamma*}| < |Y^*|$ we may use $X^{\gamma*}_\alpha$ for A_1 and $X^{\gamma*}_\beta$ for B_1. In case of $\gamma = \alpha$ reduce to $\mathscr{Q} = \mathscr{R} \setminus (C \setminus \{v_1\})$ (where v_1 is in Y^α and v_2 in X^α in this assignment); In case of $\gamma = \beta$ reduce to $\mathscr{Q} = \mathscr{R} \setminus (C \setminus \{v_2\})$.

Finally, it **must be** that $|Y^{\gamma*}| \le (|C| - 1)/2$ in each case, and in consequence we may use one of these Y's for $A_1 := Y^{\beta*}_\alpha$ and the other for $B_1 := Y^{\alpha*}_\beta$. Thus v_1 will lie in the compatible cover B_1 and v_2 in set A_1, and the cardinalities of A_1 and

B_1 sum to $|C|-1$. This allows a reduction from $\mathcal{R}$ to $\mathcal{Q}=(\mathcal{R}\setminus(C\setminus\{v_1,v_2\}))\cup\mathcal{C}'$ where $\mathcal{C}'$ is the petal (v_1,x,v_2) with a new vertex x and $[x,v_1]$ labeled β and $[x,v_2]$ labeled α. We proceed as in the last but one paragraph of Section 10. We may suppose that the compatible covers A_2 and B_2 of $\mathcal{Q}$ do not contain the artificial vertex x. Thus $|A|=|A_1\cup A_2|\le|A_1|+|A_2|-1$ and $|A|=|B_1\cup B_2|\le|B_1|+|B_2|-1$. We have $|A_2|+|B_2|\le|V(\mathcal{Q})|=n-|C|+3$, which implies the desired upper bound $|A|+|B|\le n$.

12. The case of a cut segment, even boundary

The final case with $|D|$ **is even** has additional complications. Assign every second vertex of D to X, and the rest to Y, with v_1, say, in X and v_2 in Y. Extend them to compatible covers of $\mathcal{C}$, and modify at the petals to get X^*_α, X^*_β, Y^*_α and Y^*_β. **If** X^* has at most $(|C|-1)/2$ elements, we may use it both for $A_1:=X^*_\alpha$ and $B_1:=X^*_\beta$ and reduce to $\mathcal{Q}=\mathcal{R}\setminus(C\setminus\{v_2\})$. **If** $Y^*\le(|C|-1)/2$, then we let $A_1:=Y^*_\alpha$ and $B_1:=Y^*_\beta$ and reduce to $\mathcal{Q}=\mathcal{R}\setminus(C\setminus\{v_1\})$. (Here we used again that the vertices v_1,v_2 were not moved at the petal modification.) If none of the above, then $|X^*|=|Y^*|=|C|/2$, (so from now on $|C|$ is even). In that case use the notations X_{old}, Y_{old} for these covers. Without loss of generality, we will suppose that $\chi([v_2,v_3])=\alpha$. The reduction(s) in the case $\chi([v_2,v_3])=\beta$ is identical.

If $\mathcal{C}$ **has a petal**, with v_m its petal vertex, (necessarily $2\le m\le d-1$), then let $X_2\cup Y_2$ be a strictly alternating labeling of $\mathcal{C}\setminus\{v_m\}$ with v_1,v_2 in X_2. The extensions assured by Corollary 11 for $\mathcal{C}\setminus\{v_m\}$ are compatible covers of $\mathcal{C}\setminus\{v_m\}$ with $|X_2^*|+|Y_2^*|\le|C|-1$. However, $X^*_{2,\alpha}$, $X^*_{2,\beta}$, $Y^*_{2,\alpha}$, and $Y^*_{2,\beta}$ all meet the petal triangle $\{v_{m-1},v_m,v_{m+1}\}$, too, because v_{m-1},v_{m+1} are not petal vertices in $\mathcal{C}\setminus\{v_m\}$, so they were not shifted out from X_2 and Y_2. Note that (in $\mathcal{C}$) the two external edges adjacent to v_m have distinct colors, and the edge colored α meets X_2, the other edge meets Y_2. (There are two cases to check, according to the parity of m.) So $X^*_{2,\alpha}$, $Y^*_{2,\beta}$ are compatible covers of the whole $\mathcal{C}$, and $X^*_{2,\beta}$, $Y^*_{2,\alpha}$ miss only one edge.

We distinguish two cases. **If** $|X_2^*|\le|C|/2-1$, then let $A_1=X^*_{2,\alpha}$ and $B_1=X^*_{2,\beta}\cup\{v_m\}$ and reduce to $\mathcal{Q}=\mathcal{R}\setminus C$. **Otherwise** $|Y_2^*|\le|C|/2-1$, we can use Y^*_β for B_1, and one of the previous $X^*_{\text{old},\alpha}$ or $Y^*_{\text{old},\alpha}$ for A_1, reducing to $\mathcal{R}\setminus(C\setminus\{v_1,v_2\})$ with the label α on $[v_1,v_2]$ (like in the case $|C|=|D|=4$, see the beginning of Section 10). From now on, we may suppose that $\mathcal{C}$ has no any chord.

We will consider new assignments of the vertices in D to X_3 and Y_3 which have both v_1 and v_2 in X_3 and alternate elsewhere except on one more edge E of $\partial\mathcal{C}$ with both endpoints in X_3. (So $|X_3|=d/2+1$, and E's color is necessarily α, because we have supposed that $\chi([v_2,v_3])=\alpha$.)

Suppose such an assignment on D can be extended into $\mathcal{C}\setminus D$ to sets X_3^* and Y_3^* which obey the conclusion of Lemma 8 in all of $\mathcal{C}$. We can then, if $|X^*|$ is at most $|C|/2$ use $X^*_{3,\alpha}$ for A_1 and $X^*_{3,\beta}$ for B_1 and reduce to $\mathcal{R}\setminus\mathcal{C}$. Otherwise, we can use $Y^*_{3,\beta}$ for B_1, and the previous $X^*_{\text{old},\alpha}$ or $Y^*_{\text{old},\alpha}$ for A_1, reducing to $\mathcal{R}\setminus(C\setminus\{v_1,v_2\})$, like we did it above.

We have now provided reductions for every case of Theorem 4. Our proof is therefore complete if we can prove the supposition of the last paragraph above. This is the content of Lemma 12.

Lemma 12. *Suppose, that the simple, 2-connected figure $\mathscr{C}$ has no chord, and has even number of vertices on its boundary D, $D := \{v_1, v_2, \ldots, v_{2k}, v_{2k+1} = v_1\}$ in order. Then, there exists a j, $1 \le j \le k$ and sets $X_3^*, Y_3^* \subset C = V(\mathscr{C})$ with the following properties*

$|X_3^*| + |Y_3^*| \le |C|$,

each triangle in $\mathscr{T}(\mathscr{C})$ meets X_3^* and meets Y_3^*,

each boundary edge meets X_3^*, $\{v_1, v_2\} \subset X_3^*$, and each boundary edge except $[v_1, v_2]$ and $[v_{2j,2j+1}]$ meets Y_3^*.

Note that in this Lemma the original coloring of the external edges of $\mathscr{C}$ is irrelevant.

Proof. We have defined z to be adjacent to v_1 and v_2 in $\mathscr{C}$, z is an inner point of $\mathscr{C}$. Let $2j+1$ be the smallest odd index of vertices of D adjacent to z, $3 \le 2j+1 \le 2k+1$; and let $v_{2i_1}, v_{2i_2}, \ldots, v_{2i_\ell}$ be the vertices from $v_2, v_3, \ldots, v_{2j}$ also adjacent to z, with $1 = i_1 < i_2 < \ldots < i_\ell < j$. Set

$$X := \{v_1, v_2\} \cup \{v_{2j}, v_{2j+1}\} \cup \{v_{2f+1} : 1 < 2f+1 < 2j+1\} \cup \\ \cup \{v_{2f} : 2j+1 < 2f < 2k+1\},$$

and let $Y = \{z\} \cup D \setminus X$. Cut $\mathscr{C}$ into $\ell+2$ induced subcomplexes along $[z, v_1]$ and the above mentioned $\ell+1$ segments adjacent to z, i.e. let us denote by $\mathscr{C}^t$ $(2 \le t \le \ell)$ the (induced) subcomplex of $\mathscr{C}$ with boundary $z, v_{2i_{t-1}}, v_{2i_{t-1}+1}, \ldots, v_{2i_t}$; let $\mathscr{C}^{\ell+1}$ be the induced subcomplex with boundary $z, v_{2i_\ell}, v_{2i_\ell+1}, \ldots, v_{2j}, v_{2j+1}$, let $\mathscr{C}^{\ell+2}$ be the induced subcomplex with boundary $z, v_{2j+1}, \ldots, v_{2k}, v_1$, and finally let $\mathscr{C}^1$ be the subcomplex generated by the triangle $\{z, v_1, v_2\}$. If z is connected only to v_1 and v_2 from D, then $j = k$ and $v_{2j+1} = v_1$, $\mathscr{C}^{\ell+2}$ is empty, $\{v_{2k}, v_1, v_2\} \subset X$; if $j = 1$ then $\ell = 1$ and $\{v_1, v_2, v_3\} \subset X$.

Let V^i denote the set of inner vertices of $\mathscr{C}^i$, (especially, $V^1 = \emptyset$), $V^1 \cup V^2 \cup \ldots \cup V^{\ell+2}$ is a partition of $(C \setminus D) \setminus \{z\}$. Add a new external vertex w and edges $[w, z]$, $[w, v_1]$, $[w, v_{2j+1}]$ and triangles $\{v_1, z, w\}$, $\{v_{2j+1}, z, w\}$ to $\mathscr{C}^{\ell+2}$ to obtain the 2-connected, chordless complex $\mathscr{C}^{\ell+2,+}$. Join the vertex w to X.

We are going to extend the sets $X \cap V(\mathscr{C}^t)$, $Y \cap V(\mathscr{C}^t)$ into the interior using Corollary 11 as follows. The region $\mathscr{C}^1$ has no interior point, $X_{\text{in}}^1 = Y_{\text{in}}^1 = \emptyset$. A region $\mathscr{C}^t$ for $2 \le t \le \ell$ is 2-connected, chordless, the elements of X and Y are strictly alternating on its boundary. So Corollary 11 implies, that there are sets X_{in}^t, Y_{in}^t such that $X^{t,*} = (X \cap V(\mathscr{C}^t)) \cup X_{\text{in}}^t$ and $Y^{t,*}$ both meet all the triangles and boundary edges of $\mathscr{C}^t$, and $|X^{t,*}| + |Y^{t,*}| \le |V^t|$.

$\mathscr{C}^{\ell+1}$ is chordless, 2-connected, X and Y are strictly alternating along its boundary, so after applying Corollary 11 the extended set $Y^{\ell+1,*}$ will meet all triangles and boundary edges except $[v_{2j}, v_{2j+1}]$.

In case of $\mathscr{C}^{\ell+2,+}$ apply Lemma 8 to the partition on its boundary, i.e. assign labels on the external edges of $\mathscr{C}^{\ell+2,+} \setminus V(\partial \mathscr{C}^{\ell+2,+})$, let $z \in Y$ be the special vertex of that figure, and apply the induction hypothesis to get the sets $X_{\rm in}^{\ell+2}$ and $X_{\rm in}^{\ell+2}$ with total size at most $|V^{\ell+2}|+1$. The complex $\mathscr{C}^{\ell+2,+}$ has no chord at all, so (ii), (v), (vi) do not apply. There are only two triangles of type (8.3) with a homogeneous external edge, namely the two new triangles containing w and z. These are also covered since $z \in Y$.

Finally, let $X_{3,{\rm in}} = \cup_{1\le t\le \ell+2} X_{\rm in}^t$, $X_3^* = (X \setminus \{w\}) \cup X_{3,{\rm in}}$ and $Y_{3,{\rm in}} = \cup_{1\le t\le \ell+2} Y_{\rm in}^t$, $Y_3^* = X \cup Y_{3,{\rm in}}$ The sets X_3^*, Y_3^* have at most $|C|$ elements, both meet all triangles and almost all sides of $\mathscr{C}$, fulfilling the claims of Lemma 12. ∎

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