Maximal Triangle-Free Graphs with Restrictions on the Degrees

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ABSTRACT

We investigate the problem that at least how many edges must a maximal triangle-free graph on n vertices have if the maximal valency is $\leq D$. Denote this minimum value by $F(n, D)$. For large enough n, we determine the exact value of $F(n, D)$ if $D \geq (n - 2)/2$ and we prove that $\lim F(n, cn)/n = K(c)$ exists for all $0 < c$ with the possible exception of a sequence $c_k \to 0$. The determination of $K(c)$ is a finite problem on all intervals $[\gamma, \infty)$. For $D = cn^{\varepsilon}$, $1/2 < \varepsilon < 1$, we give upper and lower bounds for $F(n, D)$ differing only in a constant factor. (Clearly, $D < (n - 1)^{1/2}$ is impossible in a maximal triangle-free graph.)

1. INTRODUCTION

A triangle-free graph G on n vertices is called *maximal* if adding any edge to the edge set of G will create a triangle in G. Obviously, a maximal triangle-free graph has at least $n - 1$ edges and the only extremal graph is the star on n vertices. However, the star contains a vertex with valency $n - 1$. In this paper, we are interested in the problem that at least how many edges must a maximal triangle-free graph have if the maximal degree of vertices is $\leq D$ for some $D < n - 1$. We denote the minimal number of edges of a maximal triangle-free graph with maximal valency $\leq D$ by $F(n, D)$.

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The problem is partly motivated by a version of András Hajnal's triangle-free game (cf. [13]). Starting with the empty graph on n points for some $n \geq 3$, two players build a graph on this vertex set by alternatingly picking edges. They are not allowed to choose an edge that would complete a triangle in the graph and the game ends when it is impossible to choose an edge without violating this rule. The first player's aim is to finish the game as soon as possible; on the other hand, the second player tries to ensure that the resulting graph has a lot of edges. It can be shown that the second player can achieve that no vertex has degree $>(n + 1)/2$ in the graph hence $F(n, (n + 1)/2)$ is a lower bound for the length of the game.

Clearly, a maximal triangle-free graph has diameter 2. Analogously to $F(n, D)$, it is possible to define the function $f(n, D)$ as the minimal number of edges in a graph of diameter 2 and maximal valency $\leq D$. Erdős and Rényi [5] started the investigation of the function $f(n, D)$ and Erdős, Rényi, and T. Sós [6] determined the exact value of $f(n, D)$ for $D \geq \lfloor (n + 1)/2 \rfloor$. Pach and Surányi [12] proved that $\lim f(n, cn)/n$ exists for all $0 < c$ with the possible exception of a sequence $c_k \to 0$. We shall prove analogous results for the function $F(n, D)$.

For $D \geq n/2$, the complete bipartite graph $K_{D,n-D}$ provides an example of a maximal triangle-free graph with maximal valency $\leq D$. However, there are maximal triangle-free graphs with much less edges.

Example 1.1. Let $(n - 2)/2 < D \leq n - 3$ and let $V = \{x, y\} \cup V_1 \cup V_2 \cup V_3$ be a partition of a set V of cardinality n into parts of size $|V_1| = |V_2| = n - 2 - D$ and $|V_3| = 2D - (n - 2)$. Let the graph $\mathcal{G}$ have the following set of edges: x is connected to each vertex in $V_1 \cup V_3$; y is connected to each vertex in $V_2 \cup V_3$; finally, each $z \in V_1$ is connected to each $w \in V_2$. Then $\mathcal{G}$ is a maximal triangle-free graph with maximal valency D and $2n - 5 + (n - 3 - D)^2$ edges.

Example 1.2. Let $V = \{x_1, x_2, \ldots, x_{10}\}$ be the vertex set of the Petersen graph P such that x_1, x_2, x_3, x_4 are pairwise nonadjacent. Note that the Petersen graph itself is a maximal triangle-free graph with $3 \times 10 - 15$ edges. Let $n \geq 10$ be given. Let V_1, V_2, V_3, V_4 be pairwise disjoint sets, also disjoint from V, of size $|V_i| = \lfloor (n - 6 + (i - 1))/4 \rfloor$. Then $\sum_{i=1}^{4} |V_i| = n - 6$. For $1 \leq i \leq 4$, replace x_i by the independent set V_i in P, connecting the vertices in V_i to the original neighbors of x_i in P. The resulting graph $\mathcal{G}$ has n vertices, $3n - 15$ edges, and maximal valency

$$D = \begin{cases} (n - 2)/2, & n \equiv 0 \pmod{4}, \\ (n - 3)/2, & n \equiv 1, 3 \pmod{4}, \\ (n - 4)/2, & n \equiv 2 \pmod{4}. \end{cases}$$

Also, it is easy to see that the vertex-duplication procedure described above maintains the maximal triangle-free property so $\mathcal{G}$ is maximal triangle-free.

Theorem 1.3. Let $n > 2^{2^{28}}$. Then

$$F(n,D) = \begin{cases} 2n - 5, & D = n - 2, \\ 2n - 5 + (n - 3 - D)^2, & n - 3 - \sqrt{n - 10} \leq D \leq n - 3, \\ 3n - 15, & (n - 2)/2 \leq D < n - 3 - \sqrt{n - 10}. \end{cases}$$

We shall prove Theorem 1.3 in Section 5. The proof is an easy corollary of the tools developed in the next three sections and the following result of Duffus and Hanson [2]: Let $\mathcal{G}$ be a maximal triangle-free graph on n vertices with minimum degree 3. Then

$$|E(\mathcal{G})| \geq 3n - 15. \tag{1.1}$$

In [2], the following more general problem was investigated: Determine $E(n,k,\delta)$, the minimum number of edges of a maximal K_k-free graph on n vertices with minimum degree δ.

2. RESULTS FOR $D < n/2$

The following simple lemma provides a lower bound for $F(n,D)$.

Lemma 2.1. Suppose that $\mathcal{G}$ is a maximal triangle-free graph on n vertices with maximal degree $\leq D$. Then $|E(\mathcal{G})| > (n^2/2D) - n$.

Proof. Let x be an arbitrary vertex of $\mathcal{G}$ and denote its valency by $d(x)$. The number of vertices reachable from x via a path of length 1 or 2 is $\leq d(x) + d(x)(D - 1) = d(x)D$. Since the diameter of $\mathcal{G}$ is 2, $d(x)D \geq n - 1$ implying $d(x) \geq (n - 1)/D$. Hence $|E(\mathcal{G})| = \frac{1}{2}\sum_{x \in V(\mathcal{G})} d(x) \geq (n^2 - n)/2D > n^2/2D - n$. ∎

Let us note that this argument also shows that $D < \sqrt{n - 1}$ is impossible in a maximal triangle-free graph. Three graphs are known with $D = \sqrt{n - 1}$: the pentagon, the Petersen graph, and the Hoffman-Singleton graph [11].

Our next goal is to give an upper bound for $F(n,D)$.

Example 2.2. Let $q \geq 3$ be a prime power and let $\mathcal{P}$ be a projective plane on $W_1 = \{x_1,\ldots,x_{q^2+q+1}\}$ with line set $\{L_1,\ldots,L_{q^2+q+1}\}$. We can suppose that the lines containing x_{q^2+q+1} are L_i for $q^2 + 1 \leq i \leq q^2 + q + 1$. Let $W_2 = \{y_1,\ldots,y_{q^2+q}\}$ be a set disjoint from W_1. First, we define a set system $\mathcal{H}$ and a graph G on the $2(q^2 + q)$ vertices $V = \{x_i, y_i\colon 1 \leq i \leq q^2 + q\}$. $\mathcal{H}$ consists of q^2 sets of size $2(q + 1)$; namely, let $H_i = L_i \cup \{y_j : x_j \in L_i\}$ $(1 \leq i \leq q^2)$. G is a $(q - 1)$-regular bipartite graph defined as follows. The sets $W_1 \backslash \{x_{q^2+q+1}\}$ and W_2 are independent in G. The x_i and y_j are connected if and only if $i \neq j$ and $\{x_i, x_j\} \subset L_k$ for some $q^2 + 1 \leq k \leq q^2 + q + 1$.

Based on $\mathcal{H}$ and G, we can build a maximal triangle-free graph $\mathcal{G}$. Let $n \geq 3q^2 + 2q$. For $1 \leq i \leq q^2$, we choose sets V_i disjoint from each other and from V such that $|V_i| = \lfloor (n - 2(q^2 + q) + (i - 1))/q^2 \rfloor$ for all i. Then the sets V_i are nonempty and $|V| + \sum_{i=1}^{q^2} |V_i| = n$. We define $\mathcal{G}$ on the vertex set $V \cup V_1 \cup \cdots \cup V_{q^2}$. The $x, y \in V$ are adjacent in $\mathcal{G}$ if and only if they are adjacent in G. The set $V_1 \cup \cdots \cup V_{q^2}$ is independent in $\mathcal{G}$. Finally, $x \in V$ and $y \in V_i$ are connected if and only if $x \in H_i$.

It is easy to check that $\mathcal{G}$ is really a maximal triangle-free graph. If $x \in V_i$ for some $1 \leq i \leq q^2$, then $d(x) = 2(q + 1)$. If $x \in V$ then $d(x) \leq q(\lfloor (n - 2(q^2 + q))/q^2 \rfloor + 1) + q - 1 \leq (n/q)$. Finally, $|E(\mathcal{G})| = 2(q + 1)(n - 2q^2 - 2q) + (q - 1)(2q^2 + 2q)/2 = 2(q + 1)n - q(q + 1) \times (3q + 5) < 2(q + 1)n$.

Remark 2.3. In Definition 2.6, we shall formulate the abstract properties of $\mathcal{H}$ and G, which ensured that the graph $\mathcal{G}$ built on them is maximal triangle-free. The construction in Example 1.2 can also be described in this setting: in this case, $\mathcal{H}$ and G are defined on six vertices $\{y_1, y_2, \ldots, y_6\}$. $\mathcal{H}$ consists of four sets, $H_1 = \{y_1, y_2, y_3\}$, $H_2 = \{y_1, y_4, y_5\}$, $H_3 = \{y_2, y_4, y_6\}$, and $H_4 = \{y_3, y_5, y_6\}$. The graph G has three edges, $\{y_1, y_6\}$, $\{y_2, y_5\}$, and $\{y_3, y_4\}$.

Lemma 2.4. Let $D \geq 5\sqrt{n}$. Then $F(n, D) < (4n^2/D) + 2n$.

Proof. Let q be a prime satisfying $n/D \leq q \leq 2n/D$. Then $n > 3q^2 + 2q$ and $2(q + 1) < n/q \leq D$ so the maximal triangle-free graph $\mathcal{G}$ constructed in Example 2.2 has maximal valency $\leq D$. Moreover, $|E(\mathcal{G})| < ((4n/D) + 2)n$, proving the assertion of the lemma. ∎

Theorem 2.5. Let $c > 0$ and $1/2 < \varepsilon < 1$ be fixed. Then $(1 + o(1)) \times (1/2c)n^{2-\varepsilon} < F(n, cn^\varepsilon) < (1 + o(1))(2/c)n^{2-\varepsilon}$.

Proof. Lemma 2.1 proves the lower bound for $F(n, cn^\varepsilon)$. We can improve the upper bound provided by Lemma 2.4 by noticing that, for large enough n, there is a prime q satisfying $(1/c)n^{1-\varepsilon} < q < (1/c)n^{1-\varepsilon} + ((1/c)n^{1-\varepsilon})^{7/12}$. (Such prime exists if n is sufficiently large since, by a theorem of Huxley [10], for large enough x there exists a prime between x and $x + x^{7/12}$.) Using this prime q, the graph $\mathcal{G}$ constructed in Example 2.2 has maximal valency $< cn^\varepsilon$ and $(1 + o(1))(2/c)n^{2-\varepsilon}$ edges. ∎

In the case $D = cn$, Lemmas 2.1 and 2.4 give linear upper and lower bounds for $F(n, D)$. We can strengthen this result considerably; however, the proof requires the exploration of the structure of maximal triangle-free graphs.

Certain hypergraph–graph pairs are intimately related to maximal triangle-free graphs. A *hypergraph* is a pair $\mathcal{H} = (V, E(\mathcal{H}))$ with $E(\mathcal{H}) \subset 2^V$. $\mathcal{H}$ is *intersecting* if $H_i \cap H_j \neq \emptyset$ for all $H_i, H_j \in E(\mathcal{H})$. $\mathcal{H}$ is a *sunflower* if $H_i \cap H_j = \bigcap_{H \in E(\mathcal{H})} H$ for all $H_i, H_j \in E(\mathcal{H})$. The edges of $\mathcal{H}$ are called the *petals* of the sunflower. For an arbitrary hypergraph $\mathcal{H}$, a weight function w: $E(\mathcal{H}) \to \mathbb{R}^{+,0}$ is a *fractional edge packing* of $\mathcal{H}$ if

$$\sum_{x \in H} w(H) \leq 1$$

holds for all $x \in V$. The *value* of a fractional edge packing is $\sum_{H \in E(\mathcal{H})} w(H)$. The maximal value of fractional edge packings of $\mathcal{H}$ is denoted by $\nu^*(\mathcal{H})$. The $\nu^*(\mathcal{H})$, as a solution of a problem in linear programming with rational coefficients, is a rational number for all $\mathcal{H}$.

Definition 2.6. Let $\mathcal{H} = (V, E(\mathcal{H}))$ be a hypergraph and $G = (V, E)$ be a graph on some set V. The pair $\mathcal{H}, G$ is a *core* if it satisfies the following properties:

(1) $\mathcal{H}$ is intersecting;
(2) G is triangle-free;
(3) for all $e \in E$ and $H \in E(\mathcal{H})$, $e \not\subset H$;
(4) for all $x \in V$ and $H \in E(\mathcal{H})$, $x \notin H$, there exists $y \in H$ such that $\{x, y\} \in E$;
(5) for all $x, y \in V$, if $\{x, y\} \not\subset H$ for any $H \in E(\mathcal{H})$ then either $\{x, y\} \in E$ or there exists $z \in V$ with $\{x, z\} \in E$ and $\{z, y\} \in E$.

Definition 2.7. Let $\mathcal{H} = (V, E(\mathcal{H}))$ be a hypergraph with edge set $E(\mathcal{H}) = \{H_1, H_2, \dots, H_m\}$ and $c \geq 1/\nu^*(\mathcal{H})$. We define $A(\mathcal{H}, c)$ as the minimal value of the objective function in the linear programming problem, $A(\mathcal{H}, c) = \min \sum_{i=1}^m |H_i| y_i$, under the restrictions

$$\sum_{x \in H_i} y_i \leq c, \quad \text{for all } x \in V, \tag{2.1}$$

$$y_i \geq 0, \quad \text{for all } 1 \leq i \leq m, \tag{2.2}$$

$$\sum_{i=1}^m y_i = 1. \tag{2.3}$$

Note that $c \geq 1/\nu^*(\mathcal{H})$ is the necessary and sufficient condition to ensure that there are feasible solutions.

Now we are ready to state our last results.

Theorem 2.8. For $c > 0$, let the function $K(c)$ be defined as $K(c) = \inf\{A(\mathcal{H}, c)\}$ where the infimum is taken over all hypergraphs $\mathcal{H}$, which occur in a core with an appropriate graph G and $c \geq 1/\nu^*(\mathcal{H})$. Then $K(c)$ is monotone decreasing, piecewise linear, and right-continuous. The points of discontinuities of $K(c)$ are all rational and included in a sequence $c_1 > c_2 > \cdots \to 0$. Moreover, for each $\gamma > 0$, the determination of $K(c)$ on the interval $[\gamma, \infty)$ is a finite problem (by solving finitely many linear programming problems).

Definition 2.9. For $c > 0$, we define a function $B(c)$ the following way. If $c > 1$ then $B(c) := 1$. For $0 < c \leq 1$, let

$$B_0(c) := 2^{2((1/c)+(1/c^2))+1}, \qquad B_{k+1}(c) := 2^{2((1/c)+(1/c^2))+1+\sum_{i=0}^{k} B_i(c)},$$
$$\text{and} \quad B(c) := B_{\lfloor 2((1/c)+(1/c^2)) \rfloor}(c).$$

Theorem 2.10. (a) For all $c > 0$, $F(n, cn + B(c)) = K(c)n + o(n)$.
(b) If $K(c)$ is continuous at c then $\lim_{n\to\infty} F(n, cn)/n = K(c)$.

We shall prove Theorems 2.8 and 2.10 in the next two sections. A standard argument shows that for determining $K(c) = \inf\{A(\mathcal{H}, c)\}$, it is enough to consider hypergraphs $\mathcal{H}$ with a bounded number of edges. The crucial point in the proof of Theorem 2.8 is Lemma 3.4, where we show that the number of vertices can be bounded as well. Hence $K(c)$ can be determined as the minimum of finitely many values $A(\mathcal{H}, c)$.

The connection between $F(n, cn)$ and $K(c)$ is explained in Section 4. We show that if a maximal triangle-free graph $\mathcal{G}$ has a linear number of edges, then its vertex set V can be partitioned into two parts $V = V_1 \cup V_2$ such that $|V_1| = o(|V|)$ and V_2 is independent. Then the neighborhoods of vertices in V_2 define a hypergraph $\mathcal{H}$ on V_1 that, together with the restriction of $\mathcal{G}$ to V_1, is a core. Moreover, the percentage of vertices of V_2 with a given edge of $\mathcal{H}$ as neighborhood defines a weight function on $\mathcal{H}$, which is a feasible solution of the linear programming problem (2.1)–(2.3).

Finally, in Section 6, we apply one of our constructions to obtain an improved bound for a problem of Erdős and Fajtlowicz on the maximal valency of a maximal triangle-free graph.

3. THE FUNCTION $K(c)$

In this section, we prove Theorem 2.8. From the definition of $A(\mathcal{H}, c)$ (cf. Definition 2.7), it is clear that $c \geq 1$ implies that $A(\mathcal{H}, c)$ is equal to the size of the smallest edge of $\mathcal{H}$ for any hypergraph $\mathcal{H}$. Thus $K(c) = 1$ for all $c \geq 1$. Throughout this section, we suppose that $0 < c < 1$. For these values of c, the core described in Example 2.2 provides an upper bound for $K(c)$. The next construction gives a slightly better bound.

Example 3.1. Let p be a prime satisfying $(p + 1)/(p^2 + p + 1) < c$. We define a core on a set $V = V_1 \cup V_2$ of cardinality $2(p^2 + p + 1)$. Let $E_1, \ldots, E_{p^2+p+1}$ be the line set of the projective plane on a set V_1, $|V_1| = p^2 + p + 1$. Let $V_2 = \{x_1, \ldots, x_{p^2+p+1}\}$. We define $E(\mathcal{H}) = \{E_i \cup \{x_i\}: 1 \leq i \leq p^2 + p + 1\}$. The graph G is bipartite with classes V_1 and V_2; we connect x_i to all points in $V_1 \backslash E_i$. It is clear that the pair $(\mathcal{H}, G)$ is a core. Also, the weight function $w(H_i) := 1/(p^2 + p + 1)$ gives a feasible solution of the linear program (2.1)–(2.3) and $\sum w(H_i)|H_i| = p + 2$.

Corollary 3.2. $K(c) \leq 2(1 + (1/c))$.

Proof. Let p be a prime satisfying $1/c \leq p \leq 2/c$. Then $p + 2 \leq 2 + 2/c$, and the previous example proves the assertion of the corollary. ∎

Lemma 3.3. Suppose that $A(\mathcal{H}, c) \leq 2(1 + (1/c))$ for some hypergraph $\mathcal{H}$. Then there exists a subhypergraph $\mathcal{H}_1$ such that $A(\mathcal{H}_1, c) = A(\mathcal{H}, c)$ and $|E(\mathcal{H}_1)| \leq 2((1/c) + (1/c^2)) + 1$. Moreover, if $(\mathcal{H}, G)$ was a core with an appropriate graph G, then there exists a graph G_1 such that $(\mathcal{H}_1, G_1)$ is a core.

Proof. Let $|E(\mathcal{H})| = m$. The system of inequalities (2.1)–(2.3) defines a convex polytope P in the m-dimensional Eucledian space. P is bounded and nonempty; hence the function $\sum |H_i| y_i$ takes its minimum at a vertex p of P. This vertex is the intersection of m hyperplanes of the type $\sum_{x \in H_i} y_i = c$ for some $x \in V$; or $y_i = 0$ for some $1 \leq i \leq m$; or $\sum_{i=1}^{m} y_i = 1$. Since

$$\sum_{x \in V} \sum_{x \in H_i} y_i = \sum_{i=1}^{m} |H_i| y_i = A(\mathcal{H}, c) \leq 2(1 + (1/c)),$$

at most $2((1/c) + (1/c^2))$ equations of hyperplanes of the first type can occur. Thus, for at least $m - 2((1/c) + (1/c^2)) - 1$ values of i, the equation $y_i = 0$ occurs. Let $E(\mathcal{H}_1)$ be the set of those edges H_j of $\mathcal{H}$ for which $y_j = 0$ does *not* occur among the equations defining p. Clearly, $A(\mathcal{H}_1, c) = A(\mathcal{H}, c)$ and $|E(\mathcal{H}_1)| \leq 2((1/c) + (1/c^2)) + 1$.

Suppose that $(\mathcal{H}, G)$ is a core for a graph G. The pair $(\mathcal{H}_1, G)$ satisfies the first four points of Definition 2.6. Notice that if the vertices $x, y \in V$ violate (5) in Definition 2.6, then the edge $\{x, y\}$ can be added to G without violating (1)–(4). Thus G can be extended to a graph G_1 such that $(\mathcal{H}_1, G_1)$ is a core. ∎

Lemma 3.4. Suppose that a hypergraph $\mathcal{H}$ has $m \leq 2((1/c) + (1/c^2)) + 1$ edges and $(\mathcal{H}, G)$ is a core for some graph G. Then there exists a hypergraph $\mathcal{H}_1$ and a graph G_1 on $\leq B(c)$ points such that $A(\mathcal{H}_1, c) \leq A(\mathcal{H}, c)$, $|E(\mathcal{H}_1)| = m$, and $(\mathcal{H}_1, G_1)$ is a core.

Proof. The function $B(c)$ is defined in Definition 2.9. We can suppose that each vertex of the underlying set V of $\mathcal{H}$ and G occurs in at least one edge of $\mathcal{H}$; otherwise we delete these vertices from V and the edges of G incident to them from G. The obtained hypergraph $\mathcal{H}'$ and graph G' satisfies $A(\mathcal{H}', c) = A(\mathcal{H}, c)$ and the first four points of Definition of 2.6, so G' can be extended to obtain a core.

Let $H_1, \ldots, H_m$ be the edges of $\mathcal{H}$ ordered the following way. H_1 is an edge of minimal size. If $H_1, \ldots, H_k$ are already defined, we choose H_{k+1} such that $H_{k+1} \backslash (H_1 \cup \cdots \cup H_k)$ is as small as possible. Let V_k be the union of the first k edges for $0 \leq k \leq m$; $V_0 = \emptyset$ and $V_m = V$. If $|H_{k+1} \backslash V_k| \leq 2^{|V_k|+m}$ for all $0 \leq k \leq m-1$, then $|V| \leq B(c)$ and there is nothing to prove. If $|H_{k+1} \backslash V_k| > 2^{|V_k|+m}$ for some k, then we describe a method to construct a core $(\mathcal{H}', G')$ such that $A(\mathcal{H}', c) \leq A(\mathcal{H}, c)$, $|\mathcal{H}'| = m$, and $\sum_{H \in \mathcal{H}'} |H| < \sum_{H \in \mathcal{H}} |H|$. By repeated application of this procedure, we can obtain the desired pair $(\mathcal{H}_1, G_1)$.

Suppose that $|H_{k+1} \backslash V_k| > 2^{|V_k|+m}$. We partition the points of $V \backslash V_k$ into equivalence classes. The x, y are equivalent if and only if (a) they belong to the same edges of $\mathcal{H}$ and (b) they are adjacent (in G) with the same vertices in V_k. We denote the equivalence class of $x \in V \backslash V_k$ by $\langle x \rangle$. We define $\mathcal{H}'$ and G' on the underlying set $V' = V_k \cup \{a_{\langle x \rangle}, b_{\langle x \rangle} : x \in V \backslash V_k\}$. The edges of $\mathcal{H}'$ will be $H_1', \ldots, H_m'$. Let $H_i' = H_i$ for $i \leq k$. For $i > k$, let $H_i' = \{x \in V_k : x \in H_i\} \cup \{a_{\langle x \rangle}, b_{\langle x \rangle} : x \in H_i\}$. Since each edge of $\mathcal{H}$ is partitioned into $\leq 2^{|V_k|+m-k-1}$ classes, $|H_i'| < |H_i|$ for $i > k$. Let $x, y \in V_k$ be adjacent in G' if and only if they were adjacent in G. The $x \in V_k$ is adjacent to $a_{\langle y \rangle}$ and $b_{\langle y \rangle}$ for some $y \in V \backslash V_k$ in G' if and only if x and y were adjacent in G. The sets $\{a_{\langle x \rangle} : x \in V \backslash V_k\}$ and $\{b_{\langle x \rangle} : x \in V \backslash V_k\}$ are independent in G'. Finally, $a_{\langle x \rangle}$ and $b_{\langle y \rangle}$ are connected if and only if there exist $x' \in \langle x \rangle$ and $y' \in \langle y \rangle$ such that $\{x', y'\}$ is an edge of G.

Using the fact that $(\mathcal{H}, G)$ is a core, it is easy to check that the pair $(\mathcal{H}', G')$ satisfies the first four conditions in Definition 2.6. Hence it is possible to add some edges to G' to obtain a core. The weight function w giving the value $A(\mathcal{H}, c)$ of the objective function in the linear programming problem (2.1)–(2.3) for $\mathcal{H}$ is also a feasible solution for $\mathcal{H}'$; hence $A(\mathcal{H}', c) \leq A(\mathcal{H}, c)$. (In fact, $A(\mathcal{H}', c) < A(\mathcal{H}, c)$ if $w(H_i) > 0$ for any $i > k$.) ∎

Corollary 3.5. *$K(c)$ is the minimum of the finitely many values $A(\mathcal{H}, c)$ where $\mathcal{H}$ is a hypergraph on $\leq B(c)$ points with $\leq ((1/c) + (1/c^2)) + 1$ edges and $(\mathcal{H}, G)$ is a core for some graph G.* ∎

Lemma 3.6. *Let $\mathcal{H}$ be an arbitrary hypergraph. Then $A(\mathcal{H}, c)$ is a continuous, convex, piecewise linear, monotone decreasing function on $[(1/(\nu^*(\mathcal{H}))), \infty)$.*

Proof. Proposition 3.1 in [12] states that there exist real numbers $0 = \tau_0 < \tau_1 < \cdots < \tau_k = \nu^*(\mathcal{H})$, $0 < a_0 < \cdots < a_{k-1}$, and $0 = b_0 > \cdots > b_{k-1}$ such that $a(\mathcal{H}, T)$, the minimal value of $\sum_{H \in E(\mathcal{H})} w(H)\,|H|$ for fractional edge packings w with value T, is equal to $a_i T + b_i$ if $T \in [\tau_i, \tau_{i+1}]$. Substituting $T = 1/c$, we obtain immediately that $A(\mathcal{H}, c) = a_i + b_i c$ for $c \in [1/\tau_{i+1}, 1/\tau_i]$, proving the assertions of the lemma. ∎

Proof of Theorem 2.8. Let $\gamma > 0$ be fixed. By Corollary 3.5 and Lemma 3.6, $K(c)$ can be obtained on $[\gamma, \infty)$ as the minimum of finitely many piecewise linear, monotone decreasing functions; hence $K(c)$ itself has these properties. The only possible discontinuities are at points of the form $1/\nu^*(\mathcal{H})$ for some hypergraph $\mathcal{H}$ from this finite collection; in particular, there are only finitely many discontinuities in $[\gamma, \infty)$. ∎

4. PROOF OF THEOREM 2.10

The next two lemmas utilize ideas from [12].

Lemma 4.1. Let $n > B(c)$. Then there exists a maximal triangle-free graph $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$ with maximal degree $\leq cn + B(c)$, $|V(\mathcal{G})| = n$, and $|E(\mathcal{G})| \leq K(c)n + B(c)^2$.

Proof. By Corollary 3.5, there exists a core $(\mathcal{H}, G)$ on some set V, $|V| \leq B(c)$, and a weight function w on the edges of $\mathcal{H}$ such that $|E(\mathcal{H})| = m \leq 2((1/c) + (1/c^2)) + 1$, w is a solution of the linear programming problem (2.1)–(2.3), and $A(\mathcal{H}, c) = \sum_{i=1}^{m} w(H_i)\,|H_i| = K(c)$. Let $n > B(c)$ be given. For $1 \leq i \leq m$, we choose disjoint sets V_i with $\lfloor (n - |V|)w(H_i) \rfloor \leq |V_i| \leq \lfloor (n - |V|)w(H_i) \rfloor + 1$ such that $\sum_{i=1}^{m} |V_i| = n - |V|$. The graph $\mathcal{G}$ is defined on $V(\mathcal{G}) = V \cup V_1 \cup \cdots \cup V_m$. $x, y \in V$ are adjacent in $\mathcal{G}$ if and only if they are adjacent in G. The set $V_1 \cup \cdots \cup V_m$ is independent in $\mathcal{G}$. Finally, $x \in V$ and $y \in V_i$ are connected if and only if $x \in H_i$.

It is easy to check that $\mathcal{G}$ is a maximal triangle-free graph. If $x \in V_i$ for some i then $d(x)$, the valency of x, is $\leq |H_i| \leq B(c)$. If $x \in V$ then $d(x) \leq c(n - |V|) + m + |V| - 1 \leq cn + B(c)$. Finally, the number of edges in $\mathcal{G}$ is $\leq K(c)(n - |V|) + m|V| + |V|^2/2 \leq K(c)n + B(c)^2$. ∎

Lemma 4.2. Let $n > 2^{64}$ and $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$ a maximal triangle-free graph with maximal degree $\leq cn + B(c)$, $|V(\mathcal{G})| = n$, and $|E(\mathcal{G})| \leq K(c)n + B(c)^2$. Then $V(\mathcal{G})$ can be partitioned in the form $V(\mathcal{G}) = V_1 \cup V_2$ satisfying the following properties.

(a) $|V_1| \leq ((2K(c) + 1)n + 2B(c)^2)/\log\log n$. (The logarithms are of base 2.)
(b) V_2 is an independent set in $\mathcal{G}$.

(c) For $x \in V_2$, let $H(x) = \{y \in V_1: \{x,y\} \in E(\mathcal{G})\}$. Let $\mathcal{H}$ be a hypergraph on V_1 with edge set $\{H(x): x \in V_2\}$ and let G be the spanned subgraph of $\mathcal{G}$ on V_1. Then $(\mathcal{H}, G)$ is a core.

Proof. Let $X = \{x \in V(\mathcal{G}): d(x) \geq \log\log n\}$. Then $|X| \log\log n \leq \sum_{x \in V(\mathcal{G})} d(x) \leq 2K(c)n + 2B(c)^2$, hence

$$|X| \leq \frac{2K(c)n + 2B(c)^2}{\log\log n}. \tag{4.1}$$

For $x \in V(\mathcal{G})\backslash X$, let $H(x) = \{y \in X: \{x,y\} \in E(\mathcal{G})\}$. Clearly, $|H(x)| < \log\log n$ for all $x \in V(\mathcal{G})\backslash X$. Let $Y = \{y \in V(\mathcal{G})\backslash X: \exists\, z \in V(\mathcal{G})\backslash X$ such that $H(y) \cap H(z) = \emptyset\}$. We claim that the partition $V_1 = X \cup Y$, $V_2 = V(\mathcal{G})\backslash(X \cup Y)$ satisfies (a)–(c). Since $\mathcal{G}$ is triangle-free, the definition of Y ensures that (b) holds. Also, in a maximal triangle-free graph, (b) implies (c). So, in view of (4.1), it is enough to prove that $|Y| \leq (n/\log\log n)$.

The key observation is that the set system $\{H(y): y \in Y\}$ cannot contain a sunflower with more than $\log\log^2 n + \log\log n$ petals. To see this, let us suppose that $\{H(y_i): 0 \leq i \leq \lfloor \log\log^2 n + \log\log n \rfloor\}$ is a sunflower for some $y_i \in Y$. By the definition of Y, there exists $z \in Y$ such that $H(y_0) \cap H(z) = \emptyset$. Since the vertices in V_2 have $\mathcal{G}$-degree $<\log\log n$, less than $\log\log^2 n$ of the y_i can be reached from z via a $\leq$2-long path in $V\backslash X$. Hence $H(z)$ must intersect $>\log\log n$ of the $H(y_i)$. The points of intersection must be different since $H(z) \cap \bigcap H(y_i) = \emptyset$. However, this is a contradiction with $|H(z)| < \log\log n$.

By a well-known theorem of Erdős and Rado [4], if a hypergraph has at least $r!\,m^r$ edges of size $\leq r$, then some subhypergraph is a sunflower with m petals. This implies that the set system $\{H(y): y \in Y\}$ has $<(\log\log n)!\,(\log\log^2 n + \log\log n)^{\log\log n} < (n/\log\log^3 n)$ members. (This is the point where we use that $n > 2^{64}$.) Finally, for each $H \subset X$, $|\{y \in Y: H(y) = H\}| \leq \log\log^2 n$ because these y must be reached via a $\leq$2-long path in $V\backslash X$ from the vertex $z \in Y$ satisfying $H(z) \cap H = \emptyset$. ∎

Proof of Theorem 2.10. (a) Lemma 4.1 proves that $F(n, cn + B(c)) \leq K(c)n + o(n)$. Fix $c > 0$ and $\varepsilon > 0$. We have to prove that for large enough n, if $\mathcal{G}_n$ is a maximal triangle-free graph on n points with maximal degree $\leq cn + B(c)$ then $|E(\mathcal{G}_n)| > (K(c) - \varepsilon)n$. Suppose, on the contrary, that there is an infinite set M of integers such that, for $n \in M$, there exists such $\mathcal{G}_n$ with $|E(\mathcal{G}_n)| \leq (K(c) - \varepsilon)n$.

By Theorem 2.8, $K(c)$ is continuous from the right for all c. Hence we can choose $\delta > 0$ such that $K(c + \delta) > K(c) - \varepsilon$. After that, we choose

$n \in M$, satisfying the following inequalities:

$$n > 2^{64}; \tag{4.2}$$

$$K(c+\delta)\left(n - \frac{(2K(c)+1)n + 2B(c)^2}{\log\log n}\right) > (K(c)-\varepsilon)n; \tag{4.3}$$

$$\frac{cn + B(c)}{n - ((2K(c)+1)n + 2B(c)^2)/\log\log n} < c + \delta. \tag{4.4}$$

By Lemma 4.2, $\mathcal{G}_n(V)$ can be partitioned into two parts V_1, V_2 satisfying (a)–(c). We use the notation introduced in Lemma 4.2(c). Let the weight function w on $E(\mathcal{H})$ defined as

$$w(H) = \frac{|\{x \in V_2 \colon H(x) = H\}|}{|V_2|}.$$

Clearly, $\sum_{H \in E(\mathcal{H})} w(H) = 1$. For $z \in V_1$, using that $d_{\mathcal{G}_n}(z) \le cn + B(c)$ and (4.4), we obtain

$$\sum_{z \in H} w(H) \le \frac{cn + B(c)}{|V_2|} \le c + \delta.$$

Hence w is a feasible solution of the linear programming problem (2.1)–(2.3) for the parameter value $c + \delta$ implying that $\sum w(H)|H| \ge K(c+\delta)$. However, by (4.3), this means that $\mathcal{G}_n$ has $>(K(c)-\varepsilon)n$ edges, a contradiction.

(b) Suppose that $K(c)$ is continuous at c and let $\varepsilon > 0$ arbitrary small. Since $F(n, D)$ is a monotone deceasing function of D,

$$F(n, cn + B(c)) \le F(n, cn) \le F(n, (c-\varepsilon)n + B(c-\varepsilon))$$

if n is large enough. Dividing by n and taking the limit, we obtain $K(c) \le \lim F(n, cn)/n \le K(c-\varepsilon)$. Since $K(c)$ is continuous at c and ε is arbitrary, this implies $\lim F(n, cn)/n = K(c)$. ∎

5. THE CASE $D \ge n/2$

The aim of this section is to prove Theorem 1.3.

Lemma 5.1. Let $n > 2^{2^{28}}$ and suppose that $d(x) \ge 4$ for each vertex of a maximal triangle-free graph $\mathcal{G}$, $|V(\mathcal{G})| = n$. Then $|E(\mathcal{G})| > 3n - 15$.

Proof. We repeat the argument of the proof of Lemma 4.2. Suppose that $|E(\mathcal{G})| \leq 3n - 15$. Let $X = \{x \in V(\mathcal{G})\colon d(x) \geq \log\log n\}$. Then, by our assumption, $|X| \log\log n \leq \sum_{x \in V(\mathcal{G})} d(x) \leq 2(3n - 15)$, hence $|X| < (6n/\log\log n)$. For $x \in V(\mathcal{G})\backslash X$, let $H(x) = \{y \in X\colon \{x,y\} \in E(\mathcal{G})\}$ and let $Y = \{y \in V(\mathcal{G})\backslash X\colon \exists\, z \in V(\mathcal{G})\backslash X$ such that $H(y) \cap H(z) = \emptyset\}$. As in the proof of Lemma 4.2, we can see that $|Y| < (n/\log\log n)$ and the set $V(\mathcal{G})\backslash(X \cup Y)$ is independent. Hence $|E(\mathcal{G})| \geq \sum_{x \in V(\mathcal{G})\backslash(X \cup Y)} d(x) \geq 4(n - (7n/\log\log n))$. Since $n > 2^{2^{28}}$, $4(n - (7n/\log\log n)) > 3n$, contradicting our assumption that $|E(\mathcal{G})| \leq 3n - 15$. ∎

Lemma 5.2. Let $\mathcal{G}$ be a maximal triangle-free graph with maximal degree M and let $|V(\mathcal{G})| = n$.

(a) If $\min\{d(x)\colon x \in V(\mathcal{G})\} = 1$, then $M = n - 1$ and $\mathcal{G}$ is a star.
(b) If $\min\{d(x)\colon x \in V(\mathcal{G})\} = 2$, then either $M = n - 2$ and $|E(\mathcal{G})| = 2n - 4$ or $M \leq n - 3$ and $|E(\mathcal{G})| \geq 2n - 5 + (n - 3 - M)^2$.

Proof. (a) Obvious.

(b) Let x be a vertex of degree 2 with neighbors y, z. Let $A = \{w \in V(\mathcal{G})\colon \{w,y\} \in E(\mathcal{G}) \wedge \{w,z\} \in E(\mathcal{G})\}$; $B = \{w \in V(\mathcal{G})\colon \{w,y\} \in E(\mathcal{G}) \wedge \{w,z\} \notin E(\mathcal{G})\}$; and $C = \{w \in V(\mathcal{G})\colon \{w,y\} \notin E(\mathcal{G}) \wedge \{w,z\} \in E(\mathcal{G})\}$. Since each vertex can be reached from x via a path of length ≤ 2, $A \cup B \cup C$ is a partition of $V(\mathcal{G})\backslash\{y,z\}$. Since $\mathcal{G}$ is a maximal triangle-free graph, the sets A, B, C are independent, there are no edges between A and $B \cup C$, and each pair $u \in B$, $v \in C$ is connected. Moreover, $B = \emptyset \Leftrightarrow C = \emptyset$. If $B = C = \emptyset$, then $M = n - 2$ and $|E(\mathcal{G})| = 2n - 4$. If $B \neq \emptyset$, then, because of symmetry, we can suppose $|B| \leq |C|$. In this case, $M = n - |B| - 2$ and $|E(\mathcal{G})| = |B|\,|C| + |B| + |C| + 2(n - 2 - |B| - |C|) = 2n - 5 + (|B| - 1)(|C| - 1) \geq 2n - 5 + (n - 3 - M)^2$. ∎

Proof of Theorem 1.3. By Examples 1.1, 1.2 and Lemmas 5.1, 5.2, and finally (1.1). ∎

Conjecture 5.3. Example 3.1 is optimal in the range $(p + 1)/(p^2 + p + 1) < c < 1/p$ for $p \geq 3$.

The above conjecture would imply that $F(n,D) = 4n - 28$ for $(3/7)n < D < n/2 - O(1)\,(n > n_0(D))$. D. Hanson (private communication) have found an example showing that $F(n,D) \leq 4n - 25$ for a slightly larger range, for $D > .4n$ $(n \geq 10)$.

6. MAXIMAL TRIANGLE-FREE GRAPHS WITH SMALL DEGREES

In this section, we consider the following problem. Determine the minimum number $D_2(n)$ such that there exists a triangle-free graph of diameter 2 on n vertices and maximum degree $D_2(n)$. Obviously, $D_2(n) \geq \sqrt{n-1}$. Erdős and Fajtlowicz (cf. [1]) pointed out that the random method gives only the upper bound $D_2(n) \leq O(\sqrt{n} \log n)$. The true order of the magnitude of $D_2(n)$ was determined by Hanson and Seyffarth [7], who constructed some circular graphs showing $D_2(n) \leq (2 + o(1))\sqrt{n}$. Another circular graphs were found by Hanson and Strayer [8], but their method cannot give a better upper bound. Example 2.2 indicates that in fact

Theorem 6.1. $D_2(n) \leq 2/\sqrt{3}\,(\sqrt{n} + n^{7/24})$ (for all $n > n_0$).

To get this bound, we have to choose the prime q as large as possible satisfying $3q^2 + 2q \leq n$. Then $r := n - 3q^2 - 2q < 2q^{19/12}$ for $n > n_0$. The only new idea we need is the well-known fact from finite geometry, that one can choose the sizes of V_i such a way that the size of every $|V_i|$ is either $1 + \lfloor r/q^2 \rfloor$ or $1 + \lceil r/q^2 \rceil$, and each vertex from V has degree at most $2q - 1 + 2\lceil r/q \rceil$ (see [9]).

Further generalizations were investigated by Erdős and Pach [3], who considered graphs *with property* I_k, that is graphs in which every independent set of size k has a common neighbor.

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References

[1] A Bondy, P. Erdős, and S. Fajtlowicz, Domination in graphs of diameter 2. Technical Report, University of Waterloo, 1979.

[2] D. A. Duffus and D. Hanson, Minimal k-saturated and color critical graphs of prescribed minimum degree. *J. Graph Theory* **10** (1986) 55–67.

[3] P. Erdős and J. Pach, Remarks on stars and independent sets. *Aspects of Topology*, Vol. 93, London Mathematics Society Lecture Notes Series (1985) 307–313.

[4] P. Erdős and R. Rado, Intersection theorems for systems of sets. *J. London Math. Soc.* **35** (1960) 85–90.

[5] P. Erdős and A Rényi, On a problem in the theory of graphs [in Hungarian]. *Publ. Math. Inst. Hung. Acad. Sci.* **7** (1962) 623–641.

[6] P. Erdős, A Rényi, and V. T. Sós, On a problem of graph theory. *Studia Sci. Math. Hung.* **1** (1966) 215–235.

[7] D. Hanson and K. Seyffarth, k-saturated graphs of prescribed maximum degree. *Congres. Numer.* **42** (1984) 169–182.
[8] D. Hanson and H. J. Strayer, 3-saturated circulant graphs. *Congres. Numer.* **71** (1990) 147–152.
[9] James W. P. Hirschfeld, *Projective Geometries over Finite Fields.* Oxford Mathematics Monographs, The Clarendon Press, Oxford University Press, New York (1979).
[10] M. Huxley, The difference between consecutive primes. Proceedings of the Symposium in Pure Mathematics, Vol. 24, American Mathematics Society (1973) 141–146.
[11] A. J. Hoffman and R. R. Singleton, On Moore graphs with diameters 2 and 3. *IBM J. Res. Devel.* **4** (1960) 497–504.
[12] J. Pach and L. Surányi, Graphs of diameter 2 and linear programming. *Algebraic Methods in Graph Theory, Colloquia mathematica Societatis János Bolyai,* Vol. 25. North Holland (1978) 599–629.
[13] Á. Seress, On Hajnal's triangle-free game. *Graphs Combinat.* **8** (1992) 75–79.