Discrete Mathematics 101 (1992) 59–64
North-Holland

Indecomposable regular graphs and hypergraphs

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Received 3 August 1990
Revised 18 December 1991

Abstract

Füredi, Zoltán, Indecomposable regular graphs and hypergraphs, Discrete Mathematics 101 (1992) 59–64.

Let G be a d-regular simple graph with n vertices. Here it is proved that for $d > \sqrt{n} - 1$, G contains a proper regular spanning subhypergraph. The same statement is proved for multigraphs with $d > (n-1)/3$. These bounds are best possible if d is odd. The main tool of the proof is a consequence of Tutte's f-factor theorem on the existence of 2-factors, due to Taškinov. Finally, disproving a conjecture of Alon and Berman, an indecomposable d-regular 3-uniform hypergraph is constructed with $d \geqslant 2^{(n-6)/2}$.

1. Introduction, results

A hypergraph H is a pair $\mathrm{H} = (V(\mathrm{H}), \mathscr{E}(\mathrm{H}))$, where V is a finite set, the set of vertices or points, and $\mathscr{E}$, the edge set, is a collection of subsets of V. Note that $\mathscr{E}$ may contain the same set more than once, the multiplicity of $E \in \mathscr{E}$ is denoted by $m_{\mathrm{H}}(E)$. If we want to emphasize that H contains (or might contain) multiple edges, then we call it *multihypergraph.* If H does not contain multiple edges (i.e. $m(E) = 1$ for all $E \in \mathscr{E}$), then it is called a *simple* hypergraph. A hypergraph is a k-graph, or k-*uniform* hypergraph if all edges have k elements. The 2-graphs are called *graphs.* A *pseudograph* is a (multi)graph with *loops*. The edge $B \in \mathscr{E}(\mathrm{G})$ is a *bridge* of the (pseudo)graph G if there is a partition of the vertex set $V(\mathrm{G}) = V_1 \cup V_2$ such that B is the only edge between V_1 and V_2. We denote by $\mathrm{G} \mid A$ the *induced subhypergraph* on A. The *degree* of a vertex v is $\deg_{\mathrm{H}}(v) = \sum \{m(E) \colon v \in E \in \mathscr{E}\}$. H is d-*regular* if $\deg_{\mathrm{H}}(v) = d$ for all vertices $v \in V$. F is a *subhypergraph* of H if $V(\mathrm{F}) \subset V(\mathrm{H})$ and $\mathscr{E}(\mathrm{F}) \subset \mathscr{E}(\mathrm{H})$. F is a *spanning* subhypergraph of H if $\bigcup \mathscr{E}(\mathrm{F}) = \bigcup \mathscr{E}(\mathrm{H})$. H is *indecomposable* if it contains no proper non-empty regular spanning subhypergraph.

* Research supported in part by the Hungarian National Science Foundation under Grant No. 1812.

In this paper we consider the maximum possible degree of regularity of regular indecomposable hypergraphs. More precisely, define for $n \geq k \geq 1$, $D(n, k)$ as the maximum possible d such that there exists a d-regular indecomposable k-uniform hypergraph on n vertices. Moreover, $D_{\text{simple}}(n, k) = \max\{d$: there exists a d-regular indecomposable simple k-graph$\}$, $D(n) = \max\{d$: there exists a d-regular indecomposable hypergraph on n vertices$\}$, where it is understood that $D(n) = \infty$ is a possibility. Huckemann and Jurkat (cf. [5]) are the first to prove that $D(n)$ (and hence $D(n, k)$) is finite for all n. The problem of estimating $D(n)$ is considered by a number of people, since it has applications in Game Theory. Huckemann, Jurkat and Shapley proved that $D(n) \leq (n+1)^{(n+1)/2}$ for all $n \geq 1$ (cf. [5]). Alon and Berman [1] gave new proofs and the following bound

$$D(n, k) \leq \binom{n}{k} k^{n/2}.$$

They conjectured, that in fact $D(n, k) \leq n^{c(k)}$, where $c(k)$ depends only on k. In Section 4 we will give counterexamples, proving the following.

Proposition 1.1. *$D(n, 3) \geq 2^{(n-6)/2}$ for all $n \geq 3$.*

Concerning the case $k = 2$, (simple graphs, multigraphs, and pseudographs) one of our tools is Petersen's classical result (cf. [8]). It says that for even d, every d-regular pseudograph contains a 2-regular spanning subgraph, i.e. a *2-factor*. In the case d odd, we are going to use the following theorem of Taškinov [11], which easily follows from Tutte's f-factor theorem (cf. [8, Thm. 10.2.20, p. 403]): If a d-regular pseudograph does not contain a 2-factor, then it has at least d bridges.

Using the above tools it follows easily (see [1]), that

$$D_{\text{pseudo}}(n, 2) = \begin{cases} n-1 & \text{for even } n; \\ 2 & \text{for odd } n. \end{cases}$$

A cycle of length n is an optimal example for odd n. For even n, the graph obtained from a star with $n-1$ edges by adding $(n-2)/2$ loops at each endvertex is $(n-1)$-regular and indecomposable. Here, verifying two conjectures of Gronau [6–7] the following is proved.

Proposition 1.2. *Let d denote the largest odd integer not exceeding $(n-1)/3$,*

$$D(n, 2) = \begin{cases} d & \text{for even } n,\ n \neq 6, 8; \\ 2 & \text{for odd } n \text{ and for } n = 6, 8. \end{cases}$$

Proposition 1.3. *Let d denote the largest odd integer not exceeding $\sqrt{n} - 1$, $n \geq 3$. Then*

$$D_{\text{simple}}(n, 2) = \begin{cases} d & \text{for even } n,\ n = 4 \text{ or } n \geq 16; \\ 2 & \text{for odd } n \text{ and for all } 5 \leq n \leq 15. \end{cases}$$

The finiteness of $D(n, 2)$ is one of the first problems of Graph Theory. It was investigated in the classical paper of Petersen [10] in 1891, and he proved an almost exact upper bound $n/3+1$. Indeed, the construction yielding the lower bound $(n/3)-2$ is due to Petersen and Sylvester (see the recent review of Mulder [9]). One of our aims here is to show how easily the above mentioned results follow from the tools developed since then.

2. An indecomposable regular multigraph

We will prove Propositions 1.2 and 1.3 in a slightly more general form. In this section we give a construction yielding the lower bound. As the case d even is trivial from Petersen's theorem in both Propositions, we suppose that $d \geq 3$ is odd.

For $1 \leq m \leq d$, let $\mathcal{G}_m(d)$ denote the class of multigraphs with maximum edge multiplicity at most m, and degree sequence $d, \ldots, d, d-1$. That is, each graph in $\mathcal{G}_m(d)$ is almost d-regular, all degrees are d except one vertex has degree $d-1$. Set $R_m(d) =: \min\{|V(\mathrm{G})|: \mathrm{G} \in \mathcal{G}_m(d)\}$, the minimum cardinality of the vertex set of such an almost regular multigraph, and set $\mathcal{R}_m(d) =: \{\mathrm{G} \in \mathcal{G}_m(d): |V(\mathrm{G})| = R_m(d)\}$, the set of graphs with minimum number of vertices. E.g. for $m > d/2$, $\mathcal{R}_m(d)$ has only one member, a triangle with edge multiplicities $(d-1)/2$, $(d-1)/2$ and $(d+1)/2$. $\mathcal{R}_1(d)$ consists of only one graph, too, a simple graph on $d+2$ vertices obtained from the complete graph by deleting from its edge set a path of length 2 and $(d-1)/2$ disjoint edges. We have

$$R_m(d) = 1 + 2\left\lceil \frac{d}{2m} \right\rceil. \tag{2.1}$$

To prove (2.1) one can use the following theorem of Chungphaisan (cf. [2, p. 121]). There exists a multigraph with degree sequence $d_1 \geq d_2 \geq \cdots \geq d_n \geq 0$ with no loops and maximum edge multiplicity at most m if and only if $\sum d_i$ is even and the following inequality holds for each $1 \leq t \leq n$.

$$\sum_{i=1}^{t} d_i \leq mt(t-1) + \sum_{i=t+1}^{n} \min\{mt, d_i\}. \tag{2.2}$$

Consider a graph $\mathrm{G} \in \mathcal{G}_m(d)$ with n vertices. As d is odd and $d > 1$ we have that n is odd and $n > 1$. Apply (2.2) with $t = n$. We get

$$(n-1)d \leq m(n-1)(n-2) + \min\{m(n-1), d-1\}.$$

Here the right-hand side is at most $m(n-1)^2$, so we get $d \leq m(n-1)$, i.e. $(n-1)/2 \geq d/(2m)$. Here $(n-1)/2$ is an integer, so we get $(n-1)/2 \geq \lceil d/(2m) \rceil$, the correct lower bound for n. Finally, it is similarly easy to obtain from (2.2), that for all odd $n \geq 2\lceil d/(2m) \rceil + 1$ there exists a graph $\mathrm{G} \in \mathcal{G}_m(d)$ with n vertices.

We will need the following simple property of the minimal graphs.

$$\mathrm{G} \in \mathcal{R}_m(d) \text{ does not contain any bridge.} \tag{2.3}$$

Suppose, on the contrary, that the edge $\{x, y\}$ is a bridge with parts $x \in V_x$, $y \in V_y$. Then x is connected to at least $(\deg(x) - 1)/m$ additional vertices of V_x, hence

$$n \geqslant 1 + \left\lceil \frac{d-2}{m} \right\rceil + 1 + \left\lceil \frac{d-1}{m} \right\rceil .$$

Here the right-hand side is larger than $1 + 2\lceil d/(2m) \rceil$, for $(d, m) \neq (3, 1)$. This contradicts (2.1). Finally, the case $d = 3$, $m = 1$ is trivial.

Let V be an n-element set, n even, with $n \geqslant 1 + dR_m(d)$. Consider a partition of V into $d + 1$ parts,

$$V = V_0 \cup V_1 \cup \cdots \cup V_d,$$

such that $|V_0| = 1$, and $|V_i| \geqslant R_m(d)$ is an odd integer for all $1 \leqslant i \leqslant d$. Choose $v_i \in V_i$, i.e. $V_0 = \{v_0\}$. Let $\mathrm{G}_i \in \mathscr{G}_m(d)$ be a graph with vertex set V_i such that all degrees are d but $\deg_{\mathrm{G}_i}(v_i) = d - 1$. Finally, join v_0 to each v_i.

Proposition 2.1. *The obtained d-regular graph,* G, *is indecomposable.*

Proof. Suppose, on the contrary, that $\mathscr{E}(\mathrm{G})$ is decomposable into an a-regular and a b-regular part. As $a + b = d$ is odd, either a or b is even. Hence, by Petersen's theorem, the corresponding component contains a 2-factor F. Say, $\{v_0, v_1\} \in \mathscr{E}(\mathrm{F})$. Then, the degrees of $\mathrm{F} \mid V_1$ are all 2, i.e. even, except $\deg_{\mathrm{F}|V_1}(v_1) = 1$, which is impossible. □

3. The minimal indecomposable graph

Theorem 3.1. *Let $d \geqslant 3$ be an odd integer and let* G *be a d-regular multigraph without a 2-factor. Suppose that the maximum edge multiplicity is at most m. Then,* $|V(\mathrm{G})| \geqslant 1 + d + 2d\lceil d/(2m) \rceil$.

Proof. We may suppose that $|V(\mathrm{G})|$ is minimal, so it is connected. Taškinov's theorem supplies at least d bridges. Delete d of them, and consider the obtained $d + 1$ connected components with vertex sets $V_0, V_1, \ldots, V_d$. The set of d deleted bridges in denoted by $\mathscr{B}$. We may suppose that some of these partition classes, say V_0, does not contain any 2-factor.

If V_0 is adjacent to all bridges of $\mathscr{B}$, then all the other V_i are adjacent to exactly one of them. Hence $\mathrm{G} \mid V_i \in \mathscr{G}_m(d)$, implying $|V_i| \geqslant R_m(d)$. Altogether, $|V| = |V_0| + \sum |V_i| \geqslant 1 + dR_m(d)$, as stated.

Suppose now that for some t, $1 \leqslant t < d$, $B_1, B_2, \ldots, B_t$ are the bridges of $\mathscr{B}$ adjacent to V_0. These bridges define a partition of the vertex set $V = V_0 \cup U_1 \cup \cdots U_t$ such that the only edge between U_i and $V \setminus U_i$ is B_i. Erase all vertices of G outside V_0 and $\bigcup_{1 \leqslant i \leqslant t} B_i$, and replace each $\mathrm{G} \mid U_i$ by a graph from $\mathscr{R}_m(d)$, such that B_i is adjacent to the vertex of degree $d - 1$.

The obtained new d-regular multigraph G′ has no 2-factor either. But the cardinality of its vertex set $V(\mathrm{G}')$ is smaller than $|V|$, by (2.3), a contradiction. □

It is easy to see, that the extremal graph in Theorem 3.1 is isomorphic to the example given in Proposition 2.1.

4. The case of 3-uniform hypergraphs

Here we prove Proposition 1.1. We define a 3-uniform 3×2^k-regular indecomposable (multi)hypergraph H over $2k+5$ vertices ($k\geqslant 0$, $k\not\equiv 2 \pmod 3$). Let

$$V(\mathrm{H})=\{x_1, x_2, \ldots, x_{k+1}, y_1, y_2, \ldots, y_{k+1}, z_1, z_2, z_3\}.$$

Set $A_i=\{x_i, y_i, x_{i+1}\}$, $B_i=\{x_i, y_i, y_{i+1}\}$, for $1\leqslant i\leqslant k$, $C_j=\{x_{k+1}, y_{k+1}, z_j\}$ ($j=1, 2, 3$), $D=\{z_1, z_2, z_3\}$. Define $\mathscr{E}(\mathrm{H})$ as follows.

$$\mathscr{E}(\mathrm{H})=\{A_i, B_i, C_1, C_2, C_3, D\colon 1\leqslant i\leqslant k\},$$

with multiplicities $m(A_i)=m(B_i)=2^k-(-1)^i2^{k-i}$, $m(C_j)=\frac{1}{3}(2^{k+1}+(-1)^k)$, $m(D)=\frac{1}{3}(7\times 2^k-(-1)^k)$.

It is easy to see that H is indeed indecomposable. In any d-regular subhypergraph F the multiplicities of the three edges C_j should be the same, denote this by $f(C_j)$. The multiplicity of each A_i equals that of B_i, $f(A_i)=f(B_i)$. Then the regularity (at the points $x_1, \ldots, x_{k+1}$ and z_1) gives $k+2$ independent linear equations. We get $f(A_1)=d/2$, $f(A_2)=d/4$, $f(A_3)=(3/8)d\cdots$ and in general

$$f(E)=\frac{d}{3\times 2^k}m(E).$$

As $m(C_j)$ is odd, and for $k\not\equiv 2 \pmod 3$, it is not divisible by 3 we have (from the above equation with $E=C_j$) that d is divisible by 3×2^k, a contradiction.

For $n=2k+5$, $k\equiv 2 \pmod 3$, we can use the above construction again, but with multiplicities $m(E)/3$.

For even n, $n=2k+6$, $k\geqslant 1$, let

$$V(\mathrm{F})=\{x_1, x_2, \ldots, x_{k+1}, y_1, y_2, \ldots, y_{k+1}, w, w_1, w_2, w_3\}.$$

Set $E=\{x_{k+1}, y_{k+1}, w\}$, $F_j=\{w, w_1, w_2, w_3\}\setminus\{w_j\}$ ($j=1, 2, 3$), $G=\{w_1, w_2, w_3\}$. Define $\mathscr{E}(\mathrm{F})$ as follows.

$$\mathscr{E}(\mathrm{F})=\{A_i, B_i, E, F_1, F_2, F_3, G\colon 1\leqslant i\leqslant k\},$$

with multiplicities $m(A_i)=m(B_i)=2^k-(-1)^i2^{k-i}$, $m(E)=2^{k+1}+(-1)^k\cdot m(F_j)=\frac{1}{3}(2^k-(-1)^k)$, $m(G)=\frac{1}{3}(7\times 2^k+2\times(-1)^k)$. Then, F is a 3×2^k-regular, indecomposable 3-graph for $k\not\equiv 0 \pmod 3$. In the case $k=0 \pmod 3$ we can use the same construction with multiplicities $m(E)/3$, again. □

Similar constructions can be given to prove that $\liminf_{n\to\infty} (D(n, k))^{1/n} \geq \sqrt{2}$. It is very likely that this limit exists, and probably it is close to $\sqrt{k}$.

Remark 4.1. The above constructions are related to an example from [3], where a 3-uniform hypergraph is given with a fractional matching number with arbitrarily large denominator.

Remark 4.2. Another related result was proved by Engel [4]. First a definition. An $S_\lambda(c, k, t)$ design is *decomposable* if there is a partition of the blocks into a $S_\alpha(v, k, t)$ and a $S_\beta(v, k, t)$ designs with $\alpha + \beta = \lambda$, $\alpha, \beta \geq 1$. In [4] it was proved that for any given v, k and t there exists a $D(v, k, t)$ such that every design with $\lambda > D(v, k, t)$ is decomposable.

Of course, $D(v, k, 1) = D(v, k)$. On the other hand, $D(v, k, t) \leq D(\binom{v}{t}, \binom{k}{t})$.

Remark 4.3. For $k \geq 3$, the true order of $D_{\text{simple}}(n, k)$ is certainly much less than $\binom{n-1}{k-1}$. Since there is no f-factor theorem for hypergraphs, this problem remains open.

Acknowledgements

The author is indebted to H.-D.O.F. Gronau for communicating the problem, and for N. Alon and L. Lovász for fruitful discussions.

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