Discrete Mathematics 98 (1991) 95–104
North-Holland

Saturated r-uniform hypergraphs

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Received 20 March 1989

Abstract

Erdős, P., Z. Füredi and Z. Tuza, Saturated r-uniform hypergraphs, Discrete Mathematics 98 (1991) 95–104.

The following dual version of Turán's problem is considered: for a given r-uniform hypergraph $\boldsymbol{F}$, determine the minimum number of edges in an r-uniform hypergraph $\boldsymbol{H}$ on n vertices, such that $\boldsymbol{F} \not\subset \boldsymbol{H}$ but a subhypergraph isomorphic to $\boldsymbol{F}$ occurs whenever a new edge (r-tuple) is added to $\boldsymbol{H}$. For some types of $\boldsymbol{F}$ we find the exact value of the minimum or describe its asymptotic behavior as n tends to infinity; namely, for $\boldsymbol{H}_r(r+1, r)$, $\boldsymbol{H}_r(2r-2, 2)$ and $\boldsymbol{H}_r(r+1, 3)$, where $\boldsymbol{H}_r(p, q)$ denotes the family of all r-uniform hypergraphs with p vertices and q edges. Several problems remain open.

1. Introduction

A *hypergraph* $\boldsymbol{H}$ is a pair $(V, \mathscr{H})$, where $\mathscr{H}$ is a family of subsets of V. The elements of V are called *vertices*, the $H \in \mathscr{H}$ are called edges (hyperedges). A hypergraph is called *r-uniform*, or an r-graph, if $|H| = r$ holds for every $H \in \mathscr{H}$. The 2-graphs are called graph. For $X \subset V$ we set $\mathscr{H}[X] = \{H: X \subset H \in \mathscr{H}\}$. The degree $\deg(\boldsymbol{H}, X)$, or briefly $\deg(X)$, of a set X is the cardinality of $\mathscr{H}[X]$. $\deg(\{x\})$ is abbreviated as $\deg(x)$. The family of all r-subsets of a k-set is called the complete r-graph and is denoted by $\boldsymbol{K}_r^k$. For brevity, $\boldsymbol{H}$ usually is identified with $\mathscr{H}$. The hypergraph $\boldsymbol{F} = (U, \mathscr{F})$, $\mathscr{F} = \{F_1, \ldots, F_m\}$ is called a *subhypergraph*

* Research supported in part by the Hungarian National Science Foundation under Grant No. 1812.

of $\boldsymbol{H}$ if there are edges $H_i \in \mathscr{H}$ and a bijection $\varphi : (\bigcup_{1 \leq i \leq s} F_i) \rightarrow (\bigcup H_i)$ such that $\varphi(F_i) = H_i$. The number of such injections is denoted by $\mathrm{NSub}(\boldsymbol{F}, \boldsymbol{H})$. If $\mathrm{NSub}(\boldsymbol{F}, \boldsymbol{H}) > 0$, then we write $\boldsymbol{F} \subset \boldsymbol{H}$.

Let $\boldsymbol{F}$ be a given r-graph. An r-uniform hypergraph $\boldsymbol{H} = (V, \mathscr{H})$ is called *$\boldsymbol{F}$-saturated* if $\boldsymbol{F} \not\subset \boldsymbol{H}$ and whenever a new edge H (an r-tuple of V) is added to $\boldsymbol{H}$, then $\mathscr{H} \cup \{H\}$ contains a subhypergraph isomorphic to $\boldsymbol{F}$. We are interested in the behavior of the function $\mathrm{sat}(n, \boldsymbol{F})$, defined as the *minimum* number of edges in an $\boldsymbol{F}$-saturated hypergraph on n vertices.

Although the first result of this type was published as early as in 1964 [3], very little is known about $\mathrm{sat}(n, \boldsymbol{F})$ for arbitrary $\boldsymbol{F}$. The first attempt to describe general properties of $\mathrm{sat}(n, \boldsymbol{F})$ for graphs was made by Kászonyi and Tuza [6] who gave a linear upper bound $\mathrm{sat}(n, \boldsymbol{F}) \leq cn$ for every graph $\boldsymbol{F}$, where $c = c(\boldsymbol{F})$ is a constant, depending on $\boldsymbol{F}$ but not on n. The corresponding conjecture

$$\mathrm{sat}(n, \boldsymbol{F}) \leq \mathrm{O}(n^{r-1}) \qquad (?)$$

for the r-uniform case has not yet been proved—this seemingly rather simple problem is open even for 3-uniform hypergraphs. For 'weakly' and 'monotonically' saturated hypergraphs, however, the analogous inequalities follow from a more general recent result of the third author [10]. (Those definitions will be given later.)

Those few examples of *graphs* for which the exact value of $\mathrm{sat}(n, \boldsymbol{F})$ is known, are surveyed in [9] where several problems are raised as well. For hypergraphs, however, there is just *one* theorem of this type (with its extensions under weaker assumptions): Bollobás [1] proved that $\mathrm{sat}(n, \boldsymbol{K}_r^t) = \binom{n}{r} - \binom{n-t+r}{r}$. In the present paper we investigate some further particular cases, when $\boldsymbol{F}$ has few edges.

First of all, let us note that the problem can be raised in a more general setting, namely when $\boldsymbol{F}$ stands for a (finite or infinite) collection of r-uniform hypergraphs. Then *$\boldsymbol{F}$-saturated* means that $\boldsymbol{H}$ contains no member of $\boldsymbol{F}$, but this property does not hold anymore when any new edge is added to $\boldsymbol{H}$. (This natural generalization was useful in [6] proving the linear upper bound for graphs.)

Denote by $\boldsymbol{H}_r(p, q)$ the family of all r-uniform hypergraphs with p vertices and q edges. If $p = r + 1$, then $\boldsymbol{H}_r(r + 1, q)$ consists of just one hypergraph. Note that if p is relatively large with respect to q and r, then the problem becomes trivial. As a matter of fact, for $n \geq p$

$$\mathrm{sat}(n, \boldsymbol{H}_r(p, q)) = q - 1 \quad \text{whenever } q - 1 \leq \binom{p - r}{r}.$$

A surprising result of Ruzsa and Szemerédi [7] states that the *maximum* number of edges in a 3-uniform hypergraph on n vertices, not containing any member of $\boldsymbol{H}_3(6, 3)$ is at most $\mathrm{o}(n^2)$ but it grows faster than $n^{2-\varepsilon}$, for all $\varepsilon > 0$. Our first result shows that the *minimum* has a simpler behavior.

Theorem 1. *For $n \geq 5$, $\mathrm{sat}(n, \boldsymbol{H}_3(6, 3)) = \lfloor (n - 1)/2 \rfloor$.*

There are several $\boldsymbol{H}_3(6, 3)$-saturated structures with a minimum number of edges. As it can be seen from the proof, all but at most two connected components of them are isomorphic to $\boldsymbol{H}_3(4, 2)$.

In the case when the forbidden hypergraph $\boldsymbol{F}$ is 'almost complete', we have the following asymptotic result.

Theorem 2. *For* $n > r \geq 2$, $\operatorname{sat}(n, \boldsymbol{H}_r(r+1, r)) = (\frac{1}{2} - o(1))\binom{n}{r-1}$.

We conjecture that the construction given in the proof is best possible; it would yield $\operatorname{sat}(n, \boldsymbol{H}_r(r+1, r)) = \sum_{i \geq 1} \binom{n-2i}{r-2}$. This equality is trivial for graphs ($r = 2$), and we can prove it for $r = 3$ as well.

Theorem 3. *For* $n \geq 4$, $\operatorname{sat}(n, \boldsymbol{H}_3(4, 3)) = \lfloor (n-1)^2/4 \rfloor$. *Moreover, there are* 2 *or* 1 *extremal hypergraphs according as n is odd or even.*

The case of $\boldsymbol{H}_3(4, 2)$ is relatively simple and its solution can be extended for r-uniform hypergraphs as follows. Denote by $t(n, r)$ the minimum number of edges in a graph of order n that does not contain an independent set of r vertices. The complementary form of Turán's theorem [8] states that the unique n-vertex graph of $t(n, r)$ edges without an independent set of size r consists of $r-1$ complete subgraphs of almost equal sizes as connected components, i.e.

$$t(n, r) = \sum_{i=0}^{r-2} \binom{\left\lfloor \frac{n+i}{r-1} \right\rfloor}{2} = \frac{n^2}{2r-2} - O(n).$$

Theorem 4. $t(n, r)/\binom{r}{2} \leq \operatorname{sat}(n, \boldsymbol{H}_r(2r-2, 2)) \leq t(n, r)/\binom{r}{2} + O(n)$, *and the lower bound is sharp for infinitely many values of n.*

Note that the latter three results remain valid for so-called *monotonically saturated* hypergraphs, too, i.e. those which may contain $\boldsymbol{F}$ (or some members of $\boldsymbol{F}$ when $\boldsymbol{F}$ is a hypergraph family) and the requirement is that the addition of any new edge *increases* the number of subhypergraphs isomorphic to $\boldsymbol{F}$ (or $\sum_{G \in \boldsymbol{F}} \operatorname{NSub}(G, \boldsymbol{H}) < \sum_{G \in \boldsymbol{F}} \operatorname{NSub}(\boldsymbol{G}, \boldsymbol{H} \cup \{H\})$, respectively).

Using the terminology of [2], call $\boldsymbol{H}$ *weakly* $\boldsymbol{F}$*-saturated* if the *complement* $\bar{\boldsymbol{H}}$ of $\boldsymbol{H}$ (i.e. the r-tuples not belonging to $\boldsymbol{H}$) can be ordered, say $\bar{\mathcal{H}} = \{H_1, \ldots, H_s\}$, in such a way that for every i $(1 \leq i \leq s)$ the number of subhypergraphs isomorphic to (some members of) $\boldsymbol{F}$ in $\mathcal{H} \cup \{H_j : j \leq i\}$ is strictly larger than that in $\mathcal{H} \cup \{H_j : j < i\}$; i.e.

$$\sum_{G \in \boldsymbol{F}} \operatorname{NSub}(G, \mathcal{H} \cup \{H_j : j \leq i\}) > \sum_{G \in \boldsymbol{F}} \operatorname{NSub}(\boldsymbol{G}, \mathcal{H} \cup \{H_j : j < i\}).$$

Denote by $\operatorname{wsat}(n, \boldsymbol{F})$ ($\operatorname{msat}(n, \boldsymbol{F})$) the minimum number of edges in a weakly (monotonically) $\boldsymbol{F}$-saturated hypergraph on n vertices. It is easy to see that in general

$$\operatorname{wsat}(n, \boldsymbol{F}) \leq \operatorname{msat}(n, \boldsymbol{F}) \leq \operatorname{sat}(n, \boldsymbol{F}).$$

Theorem 5. *For* $n > r \geqslant 2$, $\mathrm{wsat}(n, \boldsymbol{H}_r(r+1, 3)) = n - r + 1$.

In a more general form, a conjecture of the third author [9] states that

$$\mathrm{wsat}(n, \boldsymbol{H}_r(r+1, k)) = \binom{n-r-2+k}{k-2} \qquad (?).$$

We note that the argument presented in the proof for $k = 3$ is purely combinatorial. For the other solved case $k = r + 1$, however, the proofs in [4] and [5] both use algebraic techniques.

It remains open to describe the extremal hypergraphs in Theorem 5.

2. Proofs

Proof of Theorem 1. The simplest example proving the upper bound for $n \geqslant 5$ is the collection $\boldsymbol{H}$ of $\lfloor (n-1)/2 \rfloor$ triplets that contain a fixed vertex $v \in V$ and are pairwise disjoint outside v. This $\boldsymbol{H}$ is easily seen to be $\boldsymbol{H}_3(6, 3)$-saturated.

To prove the upper bound, let $\boldsymbol{H}$ be an $\boldsymbol{H}_3(6, 3)$-saturated hypergraph on n vertices. If $\boldsymbol{H}$ has two edges H, H' with $|H \cap H'| = 2$, then $H \cup H'$ does not meet any further edge of $\boldsymbol{H}$. Moreover, it is easy to see that $\boldsymbol{H} \backslash \{H, H'\}$ is $\boldsymbol{H}_3(6, 3)$-saturated on the remaining $n - 4$ vertices. Thus, in this case we can apply induction from $n - 4$ to n, since the lower bound $\max\{2, \lfloor (n-1)/2 \rfloor\}$ is trivial for $4 \leqslant n \leqslant 6$.

Suppose that any two edges of $\boldsymbol{H}$ share at most one vertex. Denote by $V_1, \ldots, V_t$ the vertex sets of the connected components of $\boldsymbol{H}$. (If v is an isolated vertex, then $\{v\}$ is considered to be a component.) If $t \geqslant 3$, pick a vertex $v_i \in V_i$ for $i = 1, 2, 3$. The hypergraph $\boldsymbol{H} \cup \{v_1, v_2, v_3\}$ does not contain any member of $\boldsymbol{H}_3(6, 3)$, contradicting the assumption that $\boldsymbol{H}$ is saturated. Thus, $t \leqslant 2$. Since $\boldsymbol{H}$ is 3-uniform, a connected component on n' vertices must have at least $\lceil (n'-1)/2 \rceil$ edges. Consequently, $|\mathscr{H}| \geqslant \lceil (|V_1| - 1)/2 \rceil + \lceil (|V_2| - 1)/2 \rceil \geqslant \lceil (n-2)/2 \rceil = \lfloor (n-1)/2 \rfloor$. (For $t = 1$, we artificially put $V_2 = \emptyset$.) □

By a more careful analysis of the above proof, one can describe the structure of all $\boldsymbol{H}_3(6, 3)$-saturated hypergraphs $\boldsymbol{H}$ having $\lfloor (n-1)/2 \rfloor$ edges. For n even, $n \geqslant 6$, $\boldsymbol{H} = t\boldsymbol{H}_3(4, 2) + \boldsymbol{S}_a + \boldsymbol{S}_b$ with $a, b \geqslant 1$, $t \geqslant 0$, $4t + 2a - 1 + 2b - 1 = n$, where $+$ in the definition of $\boldsymbol{H}$ means vertex-disjoint union of those hypergraphs, and $\boldsymbol{S}_a$ is a *star* with $2a - 1$ vertices and $a - 1$ triples having a common element (the *center*).

Let $\boldsymbol{S}_a$ and $\boldsymbol{S}_b$ be stars with centers c_a, c_b over V_a and V_b. If $|V_a \cap V_b| = 1$, then $\mathscr{S}_a \cup \mathscr{S}_b$ is called a *double star*, and is denoted by $\boldsymbol{S}^{(d)}$. For n odd $n \geqslant 5$, every extremal $\boldsymbol{H}$ consists of the (vertex-disjoint) union of some copies of $\boldsymbol{H}_3(4, 2)$ and a double star $\boldsymbol{S}^{(d)}$.

Proof of Theorem 2. The upper bound

$$\mathrm{sat}(n, \boldsymbol{H}_r(r+1, r)) \leqslant \sum_{i \geqslant 1} \binom{n-2i}{r-2}$$

is shown by the following construction. Consider the n-element set $V = \{v_1, \ldots, v_n\}$, and for $i \leqslant n-r+1$ set $\mathcal{F}_i = \{F \subset V: |F| = r, \{v_i, v_{i+1}\} \subset F$, and $v_j \notin F$ for $j < i\}$. Note that $|\mathcal{F}_i| = \binom{n-i-1}{r-2}$ and $\mathcal{F}_i \cap \mathcal{F}_j = \emptyset$ for $i \neq j$. Define $\mathcal{F} = \bigcup_{i \geqslant 1} \mathcal{F}_{2i-1}$. We claim that $\boldsymbol{F} = (V, \mathcal{F})$ is $\boldsymbol{H}_r(r+1, r)$-saturated.

First, we show that $\boldsymbol{F}$ is $\boldsymbol{H}_r(r+1, r)$-free. Let $Y \subset V$ be any $(r+1)$-tuple, and let v_i be the first element of Y. If i is even, then $\mathcal{F} \cap \mathcal{F}_i = \emptyset$, so that $Y \cap \{v_{i+1}, \ldots, v_n\}$ can be the only r-tuple of $\boldsymbol{F}$ contained in Y. A similar situation holds when i is odd and $v_{i+1} \notin Y$. Finally, if i is odd and $v_{i+1} \in Y$, then Y contains $r-1$ members of $\boldsymbol{F}_i$, but it cannot contain any $F \in \mathcal{F}_j$ for $j > i$.

We show that $\boldsymbol{F}$ is saturated. Put $P_i = \{v_{2i-1}, v_{2i}\}$. For each r-tuple $H \notin \mathcal{F}$ there is a minimum i such that $P_i \cap H \neq \emptyset$. It follows that $P_i \not\subset H$ (otherwise $H \in \mathcal{F}_{2i-1} \subset \mathcal{F}$ holds). Then $P_i \cup H$ is an $(r+1)$-tuple containing $r-1$ members of $\boldsymbol{F}_{2i-1}$. Hence, $\boldsymbol{H}_r(r+1, r) \subset \boldsymbol{F} \cup \{H\}$, implying $\mathrm{sat}(n, \boldsymbol{H}_r(r+1, r)) \leqslant |\mathcal{F}|$.

To prove the lower bound, let $\boldsymbol{H}$ be an $\boldsymbol{H}_r(r+1, r)$-saturated (monotonically $\boldsymbol{H}_r(r+1, r)$-saturated) r-uniform hypergraph on the n-element vertex set V with $\mathrm{sat}(n, \boldsymbol{H}_r(r+1, r))$ edges (with $\mathrm{msat}(n, \boldsymbol{H}_r(r+1, r))$ edges, respectively. For every edge $H \in \mathcal{H}$ define $\mathcal{F}(H)$ as the set of r-tuples $F \subset V$, $F \notin \mathcal{H}$, such that H is contained in a subhypergraph $\boldsymbol{H}' \subset \boldsymbol{H} \cup \{F\}$, $\boldsymbol{H}' \cong \boldsymbol{H}_r(r+1, r)$. Such an $\boldsymbol{H}'$ shares precisely $r-1$ edges with $\boldsymbol{H}$ (at least $r-1$ edges in the monotonic case). (From now on we do not mention the monotonic case separately.)

Denote the collection of $(r-1)$-element subsets of a set H by ∂H, and set $\partial\mathcal{H} = \bigcup_{H \in \mathcal{H}} \partial H$. For $E \subset V$, $|E| = r-1$, let $m(E) = |\{H' \in \mathcal{H}: E \subset H'\}| = |\mathcal{H}[E]|$. We have

$$|\partial\mathcal{H}| = \sum_{H \in \mathcal{H}} \left(\sum_{E \in \partial\mathcal{H}} \frac{1}{m(E)} \right).$$

Every r-tuple not belonging to $\mathcal{H}$ can have at most one $(r-1)$-element subset $E \notin \partial\mathcal{H}$. Hence, there are no two E, $E' \notin \partial\mathcal{H}$ with $|E \cap E'| = r-2$, so $\binom{n}{r-1} - |\partial\mathcal{H}| \leqslant \binom{n}{r-2}/(r-1)$. This gives $|\partial\mathcal{H}| = (1 - \mathrm{o}(1))\binom{n}{r-1}$.

Choose a function $k = k(n)$ tending to infinity with n and also satisfying $k(n)/n \to 0$ (say, $k = \sqrt{n}$). Take a subhypergraph $\mathcal{H}_0 \subset \mathcal{H}$, maximal under inclusion, such that $|\mathcal{F}(\mathcal{H}_0)| \leqslant k\,|\mathcal{H}_0|$, where $\mathcal{F}(\mathcal{H}_0) =: \bigcup_{H \in \mathcal{H}_0} \mathcal{F}(H)$. Setting $\mathcal{H}_1 =: \mathcal{H} \setminus \mathcal{H}_0$, we have $|\mathcal{F}(H) \setminus \mathcal{F}(\mathcal{H}_0)| > k$ for all $H \in \mathcal{H}_1$.

Since $|\mathcal{F}(\mathcal{H})| = \binom{n}{r} - |\mathcal{H}| \geqslant \binom{n}{r} - \frac{1}{2}\binom{n}{r-2}$, by the assumptions $k = \mathrm{o}(n)$ and $|\mathcal{H}_0| \leqslant |\mathcal{H}| \leqslant \mathrm{O}(n^{r-1})$ we obtain $|\mathcal{F}(\mathcal{H}_1) \setminus \mathcal{F}(\mathcal{H}_0)| = (1 - \mathrm{o}(1))\binom{n}{r}$.

Next, we show $|\partial\mathcal{F}(\mathcal{H}) \setminus \partial\mathcal{H}_1| = \mathrm{o}(n^{r-1})$. Indeed, suppose to the contrary that the difference Δ of those two collections of sets has cardinality at least $c_1 n^{r-1}$ for some constant $c_1 > 0$. Then the average number of sets $D \in \Delta$ containing an $(r-2)$-tuple is at least $c_2 n$ $(c_2 > 0)$. Observe that $D \cup D' \notin \mathcal{H}_1 \cup (\mathcal{F}(\mathcal{H}_1) \setminus \mathcal{F}(\mathcal{H}_0))$

for any $D, D' \in \Delta$, $|D \cap D'| = r-2$, since in every $F \in \mathscr{F}(\mathscr{H}_1) \setminus \mathscr{F}(\mathscr{H}_0)$ we have $|\partial \mathscr{F} \cap \partial \mathscr{H}| \geq r-1$ by the saturatedness of $\mathscr{H}$. Using the inequality between arithmetic and quadratic means, now we have at least $c_3\binom{n}{r}$ r-tuples not belonging to $\mathscr{H}_1 \cup (\mathscr{F}(\mathscr{H}_1) \setminus \mathscr{F}(\mathscr{H}_0))$. This contradicts the fact $|\mathscr{F}(\mathscr{H}_1) \setminus \mathscr{F}(\mathscr{H}_0)| = (1-\mathrm{o}(1))\binom{n}{r}$ and proves $|\Delta| = \mathrm{o}(n^{r-1})$

For $E \in \partial \mathscr{H}_1$, put $m_1(E) = |\{H' \in \mathscr{H}_1 : E \subset H'\}|$. Choose an $H \in \mathscr{H}_1$ and let $\partial H = \{E_1, \ldots, E_r\}$. We claim that all but at most two of the $m_1(E_i)$'s are larger than $k/2r$. Suppose that $t =: \max_{1 \leq i \leq r} m_1(E_i) = m_1(E_r)$. Since $|\mathscr{F}(H) \setminus \mathscr{F}(\mathscr{H}_0)| > k$, E_r is contained in $>k/r$ sets $F \in \mathscr{F}(H) \setminus \mathscr{F}(\mathscr{H}_0)$. Let those sets F be $E_r \cup \{v_j\}$ $(1 \leq j \leq t, t > k/r)$. Each v_j defines $r-2$ additional edges of $\mathscr{H}_1$ (those forming a $\boldsymbol{H}_r(r+1, r)$ with H and $E_r \cup \{v_j\}$). Hence, $m_1(E_1) + \cdots + m_1(E_{r-1}) \geq (r-2)t$. Since no term is larger than t, all but at most two exceed $t/2$.

We conclude that

$$\sum_{E \in \partial H} \frac{1}{m_1(E)} \leq 2 + (r-2)\frac{2r}{k} = 2 + \mathrm{o}(1)$$

for all $H \in \mathscr{H}_1$. Thus

$$|\partial \mathscr{H}_1| = \sum_{H \in \mathscr{H}_1} \left(\sum_{E \in \partial H} \frac{1}{m_1(E)} \right) \leq (2 + \mathrm{o}(1))\, |\mathscr{H}_1|.$$

Taking into account that

$$(1-\mathrm{o}(1))\binom{n}{r-1} \leq |\partial \mathscr{F}(\mathscr{H})| \leq |\partial \mathscr{H}_1| + \mathrm{o}\left(\binom{n}{r-1}\right),$$

we obtain that

$$(1-\mathrm{o}(1))\binom{n}{r-1} \leqq |\partial \mathscr{H}_1| \leq (2+\mathrm{o}(1))\, |\mathscr{H}_1|,$$

implying the theorem. □

Proof of Theorem 3. For $r=3$ the construction given in the previous proof yields $\mathrm{sat}(\boldsymbol{H}_3(4,3)) \leq \sum_{1 \leq i \leq \lfloor n/2 \rfloor}(n-2i) = \lfloor (n-1)^2/4 \rfloor$.

To prove the lower bound, let $\boldsymbol{H}$ be an $\boldsymbol{H}_3(4,3)$-saturated 3-uniform hypergraph on the n-element vertex set V with $\mathrm{sat}(n, \boldsymbol{H}_3(4,3))$ edges. For each pair $\{v, v'\} \subset V$, denote by $m(v, v')$ the number of triples $H \in \mathscr{H}$ containing both v and v'; $m(v, v')$ is the *multiplicity* of this pair. A *special pair* is a pair that either has zero multiplicity or is contained in an edge $H \in \mathscr{H}$ the other two pairs of which have multiplicity 1. (If *all* pairs of H have multiplicity 1, then we fix one of them and call it a special pair, but the other two pairs will not be considered special ones.)

Claim. *The special pairs are pairwise disjoint.*

Proof. Suppose to the contrary that some vertex v forms a special pair with two distinct vertices $v_1, v_2 \in V$. First we define two pairs E_1, E_2 containing v as follows. If $\{v, v_i\}$ has multiplicity 1, then we put $E_i =: \{v, v_i\}$. Otherwise, choose an $H_i = \{v, v_i, x_i\} \in \mathcal{H}$ such that both $\{x_i, v\}$ and $\{x_i, v_i\}$ have multiplicity 1, and put $E_i = \{v, x_i\}$. In either case, adding the 3-element set $E_1 \cup E_2$ to $\mathcal{H}$ no $\boldsymbol{H}_3(4, 3)$ can occur, contradicting saturatedness. Hence the Claim follows. □

Denote by $\mathscr{E}$ the collection of 2-element subsets $E \subset V$ which are not special pairs. For $E \in \mathscr{E}$ we put $w(E) = 1/m(E)$. Moreover, define the *weight* $w(H)$ of an edge $H \in \mathcal{H}$ as $w(H) = \sum \{w(E): E \subset H \text{ and } E \in \mathscr{E}\}$. Observe that $w(H) \leqslant 2$ for every $H \in \mathcal{H}$. Indeed, if H contains a special pair, then $w(H)$ has just two terms, both equal to 1. If H does not contain any special pair, then at most one of its pairs can have multiplicity 1, so $w(H) \leqslant (1/1) + (1/2) + (1/2) = 2$.

The previous Claim implies that there are at most $\lfloor n/2 \rfloor$ special pairs. Since every $E \in \mathscr{E}$ is contained in $1/w(E)$ edges of $\mathcal{H}$, we obtain

$$\left\lfloor \frac{(n-1)^2}{2} \right\rfloor = \binom{n}{2} - \left\lfloor \frac{n}{2} \right\rfloor \leqslant |\mathscr{E}| = \sum_{H \in \mathcal{H}} w(H) \leqslant 2\,|\mathcal{H}|.$$

This implies the lower bound for $|\mathcal{H}| = \text{sat}(n, \boldsymbol{H}_3(4, 3))$. □

We note that the extremal hypergraphs in Theorem 3 are not always unique. In the case $n = 5$, $V = \{1, 2, 3, 4, 5\}$, the following triple-systems both are $\boldsymbol{H}_3(4, 3)$-saturated:

$$\{123, 124, 125, 345\} \quad \text{and} \quad \{135, 145, 235, 245\}.$$

The first example is isomorphic to the construction given at the beginning of the proof of Theorem 2, and the special pairs can be chosen in three different ways, but $\{12\}$ always is a special pair. In the second example both special pairs $\{12\}$ and $\{34\}$ have multiplicity 0.

Considering an extremal $\boldsymbol{H}_3(4, 3)$-saturated hypergraph on the underlying set $\{v_3, \ldots, v_n\}$ and adding to it $\mathscr{F}_1$ $(= \{\{v_1 v_2 v_i\}: 3 \leqslant i \leqslant n\})$ one obtains again an extremal construction. In this way, the above two examples on 5 vertices yield 2 extremal configurations for all odd n. Below we prove that there are no more $\boldsymbol{H}_3(4, 3)$-saturated hypergraphs with n vertices and $\lfloor (n-1)^2/4 \rfloor$ edges.

Proof of the extremal cases. Denote by $\mathscr{C}$ the set of special pairs. Let $m = \lceil n/2 \rceil$. In the case when n is even, the special pairs $C_1, \ldots, C_m$ define a partition of V. In the case when n is odd, let $C_m =: V \setminus (\bigcup \mathscr{C})$, and $\mathscr{C} = \{C_1, \ldots, C_{m-1}\}$.

Split $\mathcal{H}$ into two parts $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2$, where each $H_1 \in \mathcal{H}_1$ contains a pair $C(H_1) \in \mathscr{C}$ (the other two of its pairs have multiplicity 1), and each $H_2 \in \mathcal{H}_2$ intersects three C_i's. Then every H_2 contains a pair $U(H_2)$ with multiplicity 1, while the other two pairs contained in it have multiplicity 2. Observe that $|H_1 \cap H_2| \leqslant 1$ for all $H_1 \in \mathcal{H}_1$ and $H_2 \in \mathcal{H}_2$. Hence, any four-element subset of V can contain only one type of triples.

(1) If $\{ab\}, \{cd\} \in \mathscr{C}$, $\{abc\} \in \mathscr{H}_1$, then $\{abd\} \in \mathscr{H}_1$. Suppose to the contrary that $\{abd\} \notin \mathscr{H}$. Then $\{abcd\}$ contains just one triple. So $\mathscr{H} \cup \{bcd\}$ induces a $\boldsymbol{H}_3(4, 3)$ on $\{bcde\}$ $(e \notin \{abdc\})$. As $\{bce\} \notin \mathscr{H}$, we have that $\{bde\}$ and $\{cde\} \in \mathscr{H}$. Then $m(de) \geqslant 2$, contradicting the fact $\{cde\} \in \mathscr{H}_1$.

(2) If $\{xa\} \in \mathscr{C}$ and $\{abc\} \in \mathscr{H}_2$ with $m(ab) = 1$, then $\{xbc\} \in \mathscr{H}_2$. Indeed, $\{xab\} \notin \mathscr{H}$. Consider $\boldsymbol{H} \cup \{xab\}$. There is an $\{xaby\}$ containing two members of $\mathscr{H}$. If $y \neq c$, then $m(ab) = 1$ implies $\{xya\}, \{xyb\} \in \mathscr{H}$. Since $\{xa\} \in \mathscr{C}$, we have $\{xya\} \in \mathscr{H}_1$, $m(xy) = 1$, a contradiction. Thus $y = c$, and $\{xbc\} \in \mathscr{H}_2$.

(3) If $H, H' \in \mathscr{H}_2$ and $|H \cap H'| = 2$, then they meet the same three C_i's. Indeed, suppose to the contrary that $H = \{abc\}$, $H' = \{bcd\}$ with $x \in C_x$ $(x \in \{abcd\})$. We may suppose that $m(ab) = 1$ and $C_a = \{aa'\}$. Then (2) implies that $\{a'bc\} \in \mathscr{H}$, which yields the contradiction $m(bc) \geqslant 3$, unless $a' = d$, proving (3).

If there exists an $H \in \mathscr{H}_2$, and $H \cap C_i$, $H \cap C_j$, $H \cap C_k \neq \emptyset$, then $\{x_i x_j x_k\} \in \mathscr{H}_2$ for all $x_i \in C_i$, $x_j \in C_j$, $x_k \in C_k$. Indeed, consider $\boldsymbol{H} \cup \{x_i x_j x_k\}$. There exists an x such that $\{x x_i x_j x_k\}$ contains two edges of $\boldsymbol{H}$. (1) implies $x \notin (C_i \cup C_j \cup C_k)$ and (3) implies $x \notin (V \setminus (C_i \cup C_j \cup C_k))$, a contradiction.

If there exists an $H \in \mathscr{H}_2$, $H \subset (C_i \cup C_j \cup C_k)$, then one of these C's is a singleton, namely the one not meeting the pair $\{a, b\} \subset H$, $m(ab) = 1$. Hence, in this case n is odd, and $\bigcap \mathscr{H}_2 = C_m$.

Define the following partial ordering of the members of $\mathscr{C}$: $C_i < C_j$ if for some $x \in C_j$ one has $(C_i \cup \{x\}) \in \mathscr{H}_1$. We have that $(C_i \cup (C_j \setminus \{x\})) \in \mathscr{H}_1$, too, by (1).

It is easy to see that the relation '$<$' really is a partial order, i.e. $C_i < C_j$, $C_j < C_k$ imply $C_i < C_k$. (By adding a triple T which intersects all these three pairs we first can rule out the case $C_i > C_k$, and then we can exclude the case when $C_i \cup C_k$ is intersected in two elements by a three-tuple of $\mathscr{H}_2$.) From now on we may suppose that the ordering $\{C_1, C_2, \ldots\}$ is a linear extension of the partial order "$<$" (i.e. $C_i > C_j$ is impossible for $i < j$).

If $\mathscr{H}_2 = \emptyset$, then the partial order '$<$' is a complete ordering of the pairs, so that $\mathscr{H}$ is isomorphic to the example given at the beginning of the proof of Theorem 2.

Finally, consider the case $\mathscr{H}_2 \neq \emptyset$. Suppose that there is a triple $T \in \mathscr{H}_2$ such that $T \subset (C_u \cup C_v \cup C_m)$, $u < v < m$. Let $i \notin \{uvm\}$. Then $C_i < C_u$ (and $C_i < C_v$). (This can be seen by adding a triple to $\mathscr{H}$ which intersects C_i, C_u, and C_v.)

This implies that all members of $\mathscr{H}_2$ are contained in $C_u \cup C_v \cup C_m$. Moreover, we obtain $u = m - 2$, $v = m - 1$, i.e. $\mathscr{H}$ is isomorphic to the second example. □

Proof of Theorem 4. The lower bound can easily be seen: If $\boldsymbol{H}$ is $\boldsymbol{H}_r(2r-2, 2)$-saturated, then every r-element subset of the vertex set V meets some edge of $\boldsymbol{H}$ in at least two vertices. Thus, the edges of $\boldsymbol{H}$ should cover at least $t(n, r)$ pairs. Since each edge covers $\binom{r}{2}$ of them, the estimate follows.

To prove the upper bound we apply Wilson's theorem [11]. It states that for any given r and sufficiently large t $(t > t_0(r))$, if $(r-1) \mid (t-1)$ and $r(r-1) \mid t(t-1)$, then the edges of the complete graph $\boldsymbol{K}_2^t$ can be partitioned into edge-disjoint complete subgraphs isomorphic to K_2^r. For those t that satisfy the divisibility conditions we consider $n=(r-1)t$. Dividing the n-element vertex set into $r-1$ t-element parts and taking a decomposition guaranteed by Wilson's theorem in each of those parts, we obtain an $\boldsymbol{H}_r(2r-2,2)$-saturated hypergraph with $t(n,r)/\binom{r}{2}$ edges. This example settles the case $t=ur(r-1)+1$, i.e. $n=ur(r-1)^2+r-1$, $u>u_0(r)$.

For the other n's the upper bound follows from the following

Claim. $\operatorname{sat}(n+1, \boldsymbol{H}_r(2r-2,2)) \leq \operatorname{sat}(n, \boldsymbol{H}_r(2r-2,2)) + n/(r-1)$.

Proof. Let $\boldsymbol{H}=(V,\mathcal{H})$ be an extremal $\boldsymbol{H}_r(2r-2,2)$-saturated hypergraph with $|V|=n$, $v \notin V$. Let $E_1,\ldots,E_s$ be a maximal family of pairwise disjoint $(r-1)$-element subsets of V such that $|H \cap E_i| \leq 1$ holds for all $H \in \mathcal{H}$ and i. Then $\mathcal{H} \cup \{E_i \cup \{v\} : 1 \leq i \leq s\}$ is also a saturated family on $V \cap \{v\}$. □

Proof of Theorem 5. Let $\boldsymbol{H}$ be the collection of the $n-r+1$ r-tuples containing a fixed $(r-1)$-element subset Y of an n-element underlying set V. We claim that $\boldsymbol{H}$ is a weakly $\boldsymbol{H}_r(r+1,3)$-saturated hypergraph. Indeed, it is easily seen, that any ordering of the edges H of the complement of $\boldsymbol{H}$, in which the cardinalities $|Y \cap H|$ form a decreasing sequence, satisfies the requirements.

To prove the lower bound, suppose to the contrary that there is a weakly $\boldsymbol{H}_r(r+1,3)$-saturated r-uniform hypergraph $\boldsymbol{H}$ on the n-element vertex set V with at most $n-r$ edges. Define a partition of the edge set $\mathcal{H}=\mathcal{H}_1 \cup \cdots \cup \mathcal{H}_s$ with the properties:

(1) $|\bigcup \mathcal{H}_i| \leq |\mathcal{H}_i| + r - 1$ for $1 \leq i \leq s$, and

(2) s is minimal with respect to (1).

As there are such partitions (e.g., when each part contains just one edge), there is a minimal one. Moreover, $s>1$, otherwise (1) implies $|V|=|\bigcup \mathcal{H}| \leq |\mathcal{H}|+r-1<n$.

Let $U_i=\bigcup \mathcal{H}_i$. (2) implies that $|U_i \cap U_j| \leq r-2$, otherwise replacing $\mathcal{H}_i$ and $\mathcal{H}_j$ by their union one can get an appropriate partition with a smaller number of parts. Let $\mathcal{H}^*$ be the $\partial_r \mathcal{U}$, i.e.

$$\mathcal{H}^* =: \{E \subset V : |E|=r,\ E \subset U_i \text{ for some } i\}.$$

By definition, $\mathcal{H} \subset \mathcal{H}^*$. If $|E \cap U_i| = r-1$ $(E \subset V, |E|=r)$, then E is not contained in any U_i, $E \notin \mathcal{H}^*$.

Now consider an ordering of the edges of the complement $\bar{\mathcal{H}}$ that establishes weak saturatedness. Let E be the first r-tuple not contained in any U_i. Since $\boldsymbol{H}$ is weakly saturated, there are $H', H'' \in \mathcal{H}^*$ such that $\{E, H', H''\}$ is isomorphic to $\boldsymbol{H}_r(r+1,3)$. Then, $|H' \cap H''| = r-1$ implies that H' and H'' are covered by the same U_i. However, $E \subset (H' \cup H'')$, so that $E \subset U_i$, a contradiction. □

To describe the extremal families, the following reformulation of Theorem 5 might be useful: suppose that $\boldsymbol{H}$ is an r-uniform hypergraph with n vertices and

$n - r + 1$ edges such that $\boldsymbol{H}_r(r+1, 3) \subset \boldsymbol{H}$. Then $\boldsymbol{H}$ is not weakly saturated. The reason is the following simple observation.

Proposition. *Let $\boldsymbol{H}$ be a weakly $\boldsymbol{F}$-saturated hypergraph on n vertices with* wsat$(n, \boldsymbol{F})$ *edges. Then $\boldsymbol{F} \not\subset \boldsymbol{H}$.*

Proof. Assuming $F \in \mathcal{F} \subset \mathcal{H}$, the hypergraph $\mathcal{H} \setminus \{F\}$ is weakly saturated, as well. Hence, $|\mathcal{H}|$ cannot be minimal. □

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