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# Decomposition of a convex region by lines

By

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**1. Definitions, results.** For two points $x, y \in \mathbb{R}^2$ denote by $l(x, y)$ (or $l(xy)$) the line through them, $|xy|$ their Euclidean distance, $xy$ the closed segment with these endpoints. For a pointset $P \subset \mathbb{R}^2$ let us denote its diameter by diam$(P)$, i.e. diam$(P) = \sup\{|xy| : x, y \in P\}$. The *width* of $P$, $w(P)$, is the infimum of the widths of the parallel strips containing $P$. For the distance we also use the notation $d(A, B)$, i.e. $d(A, B) =: \inf\{|ab| : a \in A, b \in B\}$. bd$(R)$ denotes the *boundary* of $R$, and $\angle(AOB)$ stands for the angle between the segments $OA$ and $OB$.

Let $R$ be a convex, closed region with non-empty interior. Suppose that the (distinct) lines $\mathscr{L} = \{l_1, \ldots, l_n\}$ cut $R$, (i.e. they intersect its interior Int $R$). $\mathscr{L}$ defines a *cell-decomposition*, $\mathscr{C}$, in the following natural way. $C \in \mathscr{C}$ if Int $C \neq \emptyset$, $C \cap l_i = \emptyset$ for all $l_i$, and it is the intersection of Int $R$ and some of the open halfplanes defined by the $l_i$'s. (Actually, $\mathscr{C}$ is not really a *partition* of Int $R$, but I hope it does not cause any confusion.)

**Theorem 1.1.** *If the number of cutting lines $n > 20d$, then there exists a cell $C \in \mathscr{C}$ of width less than* 1.

The main tool will be an upper bound on the area of the $x$-ring of $R$, (it is defined before Theorem 1.6). The proof of 1.1 is postponed to Section 5.

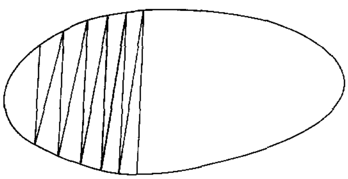

Figure 1.1

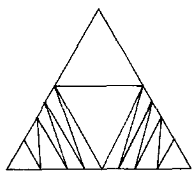

Figure 1.2

Let $n(R)$ denote the maximum number of cutting lines such that all the members of the cell-decomposition have width at least 1. Using cutting lines orthogonal to a diameter $xy$,

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it is obvious that $n(R) > d - (2d/w)$, where $w$ stands for the width. Even more, if we first use the parallel segments $x_i y_i$, and then the segments $x_i y_{i+1}$, then one can obtain that

$$n(R) > 2d - \frac{10d}{w^{1/3}} \tag{1.2}$$

(see Fig. 1.1). The same type of construction gives for the regular triangle of sides $d$ that (see Fig. 1.2) $n(T) \geqq (3\sqrt{3}/2)d - 20d^{2/3} \sim 2.598\ldots d$.

**Conjecture 1.3.** *Theorem* 1.1 *holds for* $n > (3\sqrt{3}/2)d$, *as well.*

It is not true, however, that the maximum number of cutting lines can be constructed by using noncrossing segments in $R$. E.g., the triangular cell-decomposition of the regular triangle shows that

$$n(T) \geqq \frac{3\sqrt{3}}{2} d - O(1), \tag{1.4}$$

(see Fig. 1.3). Using lines with slopes 0, $\pi/4$, $\pi/2$ and $3\pi/4$ a lattice like construction gives (see Fig. 1.4) that for the square $S$ one has

$$n(S) \geqq \frac{3}{\sqrt{2}} \operatorname{diam}(S) - O(1). \tag{1.5}$$

It seems to me that these are the best constructions (at least if $d$ is large enough), both in (1.4) and (1.5) equality holds.

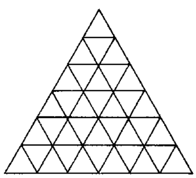
Figure 1.3

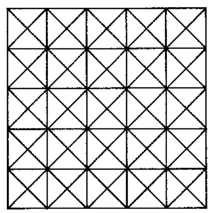
Figure 1.4

The *x-ring*, $R(x)$, of the region $R$ is defined as the set of the points having a cutting segment of length at most $x$, i.e. $R(x) =: \{p \in R : \exists u, v \in \operatorname{bd} R \text{ such that } p \in uv, \text{ and } |uv| \leqq x\}$. So $R(0)$ is $\operatorname{bd} R$, and $R(x) = R$ for $x \geqq w$.

The area of $R(x)$ is abbreviated as $|R(x)|$. For example, in the case $0 \leqq x \leqq d$ the $x$-ring of a circular disc $D_d$ of diameter $d$ is a circular ring of ring-width $\frac{1}{2}(d - \sqrt{d^2 - x^2})$. Then $|D_d(x)| = x^2 \pi/4$, independent of $d$. We will see that this is not an accident, one can obtain an upper bound on $|R(x)|$ depending only on $x^2$ and on the *excentricity* of $R$ (the ratio of the diameter and the width).

**Theorem 1.6.** $|R(x)| < \left(\frac{d}{w} + 10\right) x^2$.

**Theorem 1.7.** $n(R) < 3 \int\limits_0^\infty \frac{|R(x)|}{x^2} dx.$

Theorem 1.1 will be an easy corollary of 1.6 and 1.7.

**2. Basic properties and a lower bound on $|R(x)|$.** For $p \in R$ let

$$w_0(p, R) =: \inf \{|uv| : p \in uv,\ u, v \in \mathrm{bd}(R)\},$$

the length of the shortest cutting segment through $p$. With a little effort one can prove that $w_0(p, R)$ is a concave, continuous function on Int $R$. This implies that $R \backslash R(x)$ is a bounded, convex, open region, so $R(x)$ has area, the notation $|R(x)|$ was justified. It also follows that $0 \leqq x < y$ and $R(x) \neq R$ imply $|R(x)| < |R(y)|$. Define the *cut-width*, $w_0(R)$, as $\inf\{y : R(y) = R\}$. So the function $|R(x)|$ is continuous and strictly monotone increasing in the interval $[0, w_0)$.

It is well-known [1] that every convex region of width $w$ contains a circle $k$ of radius $\varrho$ such that

$$(2.1) \qquad \varrho \geqq \frac{w}{3}.$$

For the center $O$ of $k$ we have $w_0(O, R) \geqq \frac{2}{3} w$, hence $w_0 \geqq \frac{2}{3} w$. From now on $\varrho = \varrho(R)$ stands for the maximum radius of an (open) circular disc inscribed in $R$.

**Theorem 2.2.** $\varrho \geqq \frac{\sqrt{3}}{4} w_0.$

**Theorem 2.3.** $w_0 \geqq \frac{4}{3\sqrt{3}} w.$

Here $\sqrt{3}/4 = 0.433\ldots$ and $4/(3\sqrt{3}) = 0.7698\ldots$. Both bounds are best possible, and their proofs postponed to Sections 7 and 8, resp. Theorems 2.2 and 2.3 together imply (2.1) yielding a new (and more complicated) proof.

**Theorem 2.4.** *For* $0 \leqq x \leqq w_0$ *one has* $\frac{\pi}{4} x^2 \leqq |R(x)|.$

This does not hold in general for $x > w_0$, because area$(R) \geqq (\pi/4) w^2$ is not true. For the regular triangle $T$ one has

$$\text{area } T = \frac{w^2}{\sqrt{3}} \sim w^2\, 0.577\ldots < w^2 \frac{\pi}{4} \sim w^2\, 0.785\ldots.$$

The main tool of the proof of 2.4 is the following lemma. Suppose that $O \in C \subset R$, where $O$ is the center of the coordinate system, and $C$ is a convex, open region, (see Fig. 2.1). For $0 \leqq \alpha < 2\pi$ let $h(\alpha)$ be the halfline starting from $O$ and with direction $\alpha$. Let $s(\alpha)$ be the cutting segment of $R$ such that it is perpendicular to $h(\alpha)$, the line of $s(\alpha)$ meets

$h(\alpha)$ and touches $C$. Finally, let $T(\alpha) \in \operatorname{bd} C \cap s(\alpha)$ (in at most countably many $\alpha$'s $T(\alpha)$ is not uniquely determined, but this does not cause any problem), and let $s^+(\alpha)$ be the subsegment of $s(\alpha)$ starting at $T(\alpha)$ with direction $\alpha + \pi/2$.

**Lemma 2.5.** $\operatorname{area}(R \backslash C) = \frac{1}{2} \int_0^{2\pi} |s^+(\alpha)|^2 \, d\alpha.$

The proof is straightforward. Some hint can be seen on Fig. 2.1.

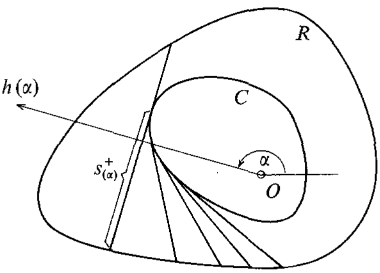


Figure 2.1

P r o o f of 2.4. The case of $x = w_0$ follows from the monotonicity of $|R(x)|$, so we may suppose that $x < w_0$. Then $\emptyset \neq R \backslash R(x) =: C$, so one can apply Lemma 2.5 with both $s^+(\alpha)$ and with $s^-(\alpha) =: s(\alpha) \backslash s^+(\alpha)$. Using the inequality $u^2 + v^2 \geqq (u+v)^2/2$ for $u, v \geqq 0$, and the fact that $|s(\alpha)| \geqq x$ we have that

$$\operatorname{area} R(x) = \frac{1}{4} \int_0^{2\pi} (|s^+(\alpha)|^2 + |s^-(\alpha)|^2) \, d\alpha$$

$$\geqq \frac{1}{4} \int_0^{2\pi} \frac{1}{2} |s(\alpha)|^2 \, d\alpha \geqq \frac{1}{8} x^2 \, 2\pi. \quad \square$$

It is clear that equality can hold only if $s^+(\alpha) = x/2$ for almost all $\alpha$. Then $C$ is strictly convex, bd $(C)$ is smooth, $s^+(\alpha) \equiv x/2$, and with a short analysis one can show that $C$ and $R$ are both circular discs.

**3. Upper estimate on the area of the $x$-ring.** In this section we prove Theorem 1.6 in the following slightly stronger form.

**Theorem 3.1.** $|R(x)| < \left( \frac{d}{w + \sqrt{w^2 - x^2}} + 9.44 \right) x^2.$

First, two lemmas on the area of 2 cuts.

From now on we always suppose that $x < w$. For a segment $AB$, $|AB| = x$, $A, B \in \operatorname{bd} R$ the region $C(AB)$ is defined as the intersection of the (closed) halfplane $H$ and $R$, where $\operatorname{bd} H = l(AB)$, and $H \cap (R \backslash R(x)) = \emptyset$. The region $C(AB)$ *has been cut* by $AB$.

**Lemma 3.2.** *Suppose that $A_1, B_1, A_2, B_2 \in \mathrm{bd}(R)$ are such that $|A_1 B_1| = |A_2 B_2| = x$ and these two segments are parallel. (To avoid trivialities we also suppose that $l(A_1 B_1) \neq l(A_2 B_2)$.) Then*

$$\operatorname{area} C(A_1 B_1) + \operatorname{area} C(A_2 B_2) \leqq \frac{d}{w + \sqrt{w^2 - x^2}} x^2 .$$

P r o o f. Suppose that $l_i$ is the touching line of $C(A_i B_i)$ parallel to $A_i B_i$ and avoiding $\mathrm{Int}\, R$, $i = 1, 2$. Let $AB$ be the longest segment parallel to $A_1 B_1$ such that $A, B \in \mathrm{bd}\, R$. (We may also suppose that not only their slopes but the directions of the segments $A_1 B_1$, $AB$ and $A_2 B_2$ are identical. Moreover, that these three lines are distinct.)

Then $C(A_i B_i)$ is contained in the trapezoid with sides $A_i B_i$, $l(AA_i)$, $l(BB_i)$, and $l_i$. Let $M_i = d(l_i, AB)$, and $u_i$ the length of the shorter base of the trapezoid (lying on $l_i$). Then the area of the trapezoid containing $C(A_i B_i)$ is $\frac{1}{2} M_i (x^2 - u_i^2)/(|AB| - u_i)$. Considering this fraction as a function of $u_i$ it takes its maximum over $0 \leqq u_i < x$ at the value $u_i = |AB| - \sqrt{|AB|^2 - x^2}$. Using the facts that $|AB| \geqq w$, and $M_1 + M_2 \leqq d$ the statement follows. □

**Lemma 3.3.** *Suppose that the disjoint regions $C_1$ and $C_2$ have been cut by the segments $A_1 B_1$, $A_2 B_2$ of lengths at most $x$. Then*

$$\operatorname{area} C_1 + \operatorname{area} C_2 \leqq \frac{d}{w + \sqrt{w^2 - x^2}} x^2 + \frac{1}{2} x^2 .$$

P r o o f. Consider the convex quadrilateral $A_1 B_1 B_2 A_2$, and let $P = l(A_1 A_2) \cap l(B_1 B_2)$. (One can proceed a similar way if these lines are parallel.) Suppose that $A_2$ lies between $A_1$ and $P$, and $B_2$ lies between $B_1$ and $P$. Consider the lines $l(A_2)$ and $l(B_2)$ through $A_2$ and $B_2$ parallel to $A_1 B_1$. We may suppose that $l(A_2)$ lies closer to $P$ then $l(B_2)$. Let $A_2 B_2'$ be a cutting segment of $R$ parallel to $A_1 B_1$, and let $B_2''$ be the intersection of $l(A_2 B_2')$ with $l(B_1 P)$. If $B_2'$ lies between $A_2$ and $B_2''$, then $|A_2 B_2'| \leqq |A_2 B_2''| \leqq |A_1 B_1| \leqq x$, and

$$C(A_2 B_2) \subset C(A_2 B_2') \cup \operatorname{Conv}(A_2 B_2 B_2'') .$$

The area of this triangle is at most $x^2/2$, and for the regions $C_1$ and $C(A_2 B_2')$ one can apply Lemma 3.2. Finally, if $A_2$ lies between $B_2'$ and $B_2''$, then $C(A_2 B_2) \subset \operatorname{Conv}(A_2 B_2 B_2'')$, so we get the same upper bound. □

P r o o f o f T h e o r e m 1.6. As in the previous section, let $O \in R \backslash R(x)$ be the origin of the coordinate system, and let $h(\alpha)$ be a halfline starting from $O$ with slope $\alpha$, where $0 \leqq \alpha < 2\pi$. Let $A(\alpha) B(\alpha)$ be the cutting segment of $R$ such that $A(\alpha)$, $B(\alpha) \in \mathrm{bd}\, R$, $|A(\alpha) B(\alpha)| = x$, the slope of the (directed) line $l(A(\alpha) B(\alpha))$ is $\alpha + \pi/2$, and it meets $h(\alpha)$. (For finitely many $\alpha$'s the definition of $A(\alpha) B(\alpha)$ might be not unique.) The lines $a(\alpha)$ and $b(\alpha)$ touching $R$ at $A(\alpha)$ and $B(\alpha)$, resp., meet at the point $M(\alpha)$. The angle $\angle (A(\alpha) M(\alpha) B(\alpha))$ is denoted by $\kappa(\alpha)$, and called the *angle* of the segment $A(\alpha) B(\alpha)$. If, e.g. at the point $A(\alpha)$ there are more touching lines, then $a(\alpha)$ is defined in such a way that $\kappa(\alpha)$ is maximal. The segment $A(\alpha) B(\alpha)$ is abbreviated as $s(\alpha)$, and $C(A(\alpha) B(\alpha))$ as $C(\alpha)$.

The cutting segment $s(\alpha)$ is called *maximal* if $C(\alpha) \subset C(\beta)$ implies $\alpha = \beta$. (Every cutting segment has length $x$ except if otherwise stated.) Every $C(\alpha)$ is either maximal, or a part of a maximal one. It is obvious that $C(\alpha) \subset C(\beta)$ (and $\alpha \neq \beta$) implies that

$$\kappa(\beta) < \pi/2. \tag{3.4}$$

Let $\operatorname{Int} C(\alpha_i)$ $(i \in I)$ be a family of maximal cuts with pairwise disjoint interiors, having $\kappa(\alpha_i) < \pi/2$. By definition, $R \cup \{\operatorname{Conv}(A(\alpha_i)\, B(\alpha_i)\, M(\alpha_i)) : i \in I\}$ is a convex region as well. As no convex region has more than three acute angles we obtain that

$$|I| \leqq 3.$$

Let $\varepsilon > 0$, and $0 < \gamma_1 < \ldots < \gamma_m < 2\pi$ and arbitrary $\varepsilon$-fine pointset on $[0, 2\pi)$, (i.e. $\gamma_1 < \varepsilon$, $\gamma_{j+1} - \gamma_j < \varepsilon$, and $2\pi - \gamma_m < \varepsilon$). The endpoints of $s_j =: s(\gamma_j)$ are abbreviated as $A_j$ and $B_j$. We are going to give an upper bound for the area of

$$\Pi =: \cup \{C(\alpha_i) : i \in I\} \cup \{\operatorname{Conv}(A_j B_j A_{j+1} B_{j+1}) : 1 \leqq j < m\}.$$

First of all, if $s_j \cap s_{j+1} \neq \emptyset$, then area $\operatorname{Conv}(A_j B_j A_{j+1} B_{j+1}) < (\gamma_{j+1} - \gamma_j)\, x^2/2$, so

$$\text{area}\, \Pi_0 =: \text{area}\,(\cup \{\operatorname{Conv}(A_j B_j A_{j+1} B_{j+1}) : A_j B_j \cap A_{j+1} B_{j+1} \neq \emptyset\}) \leqq x^2 \pi. \tag{3.5}$$

Suppose that $\kappa(\alpha_1) \leqq \kappa(\alpha_2) \leqq \kappa(\alpha_3)$ (in the case of $|I| = 3$). Then $\kappa(\alpha_3) \geqq \pi/3$, so

$$\text{area}\, C(\alpha_3) \leqq \frac{\sqrt{3}}{4} x^2. \tag{3.6}$$

Let $D_i =: \{p \in R \backslash C(\alpha_i) : d(p, s(\alpha_i)) \leqq x(1 + \delta)\}$. Here $\delta$ will tend to 0 when $\varepsilon \to 0$. Then

$$\text{area}\, D_i \leqq x^2 (1 + \delta) + \frac{1}{2} x^2 (1 + \delta)^2 \kappa(\alpha_i) < x^2 \left(1 + \frac{\pi}{4} + 3\delta\right). \tag{3.7}$$

We claim that the union of the $C(\alpha_i)$'s, $D_i$'s and $\Pi_0$ covers $\Pi$. Indeed, if $p \in \operatorname{Conv}(A_j B_j A_{j+1} B_{j+1})$, and $s_j \cap s_{j+1} = \emptyset$, then one of these two cut regions contains the other, say, $C(\gamma_{j+1}) \subset C(\gamma_j)$. Hence $\kappa(\gamma_j) < \pi/2$ by (3.4). Then either $C(\gamma_j) \subset C(\alpha_i)$ for some $i \in I$, and we are done, or $s(\gamma_j)$ intersects $s(\alpha_i)$, because of the maximality of the system $\{\alpha_i : i \in I\}$. Then $\operatorname{Conv}(A_j B_j A_{j+1} B_{j+1}) \subset C(\alpha_i) \cup D_i$. (Eventually, there might be finitely many exceptions, but that does not cause any problem.) Here we used that the function $A(\alpha)$ (and $B(\alpha)$) is piecewise continuous for given $x$ and $R$, so $d(A_{j+1}, A_j) < \delta$, and $d(B_{j+1}, B_j) < \delta$ whenever $\varepsilon$ is sufficiently small (with at most $4d/x$ exceptions).

Using Lemma 3.3 for the area $C(\alpha_1)$ + area $C(\alpha_2)$ and adding the upper bounds for the parts of $\Pi$ obtained in (3.5)–(3.7), one has

$$\text{area}\, \Pi \leqq \frac{d}{w + \sqrt{w^2 - x^2}} x^2 + \left(\frac{1}{2} + \pi + \frac{\sqrt{3}}{4} + 3\left(1 + \frac{\pi}{4}\right) + 9\delta\right) x^2.$$

Here the coefficient of $x^2$ in the second term is $9.430\ldots + 9\delta$.

Finally, it is clear that lim area $\Pi$ = area $R(x)$, as $\varepsilon \to 0$. □

**4. Remarks and problems on $R(x)$.** If for given $w$ and $x$ one looks for the smallest possible coefficients yielding a general upper bound of the form $|R(x)| \leqq \left(a\frac{d}{w}+b\right)x^2$, then $a$ could not be smaller than $w/(w+\sqrt{w^2-x^2})$, as we have seen in the proof of Lemma 3.2. In this sense Theorem 3.1 is sharp. However, there is a room to improve $b$. Considering a trapezoid of bases about $w$, and $w-\sqrt{w^2-x^2}$, and sides of length $\sim d$, one can see that $b$ could not be smaller than 1.03….

It is easy to prove with the above methods that if $\kappa(\alpha) \geqq \pi/2$ for all $\alpha$, then

$$|R(x)| \leqq \frac{\pi}{2}x^2.$$

For example, if $R$ is a rectangle of sides $a \leqq b$, and $x \leqq a/2$, then

$$|R(x)| = \frac{3\pi}{8}x^2$$

the area of the astroid.

**Proposition 4.1.** *Suppose that the boundary of $R$ is sufficiently smooth. Then* $\lim_{x\to 0} |R(x)|/x^2 = \pi/4$.

This proposition, as I. Bárány pointed out, can be proved by standard methods for regions having continuous curvature.

Rounding with very small quarter-circles the corner of a rectangle, one can see that the function $|R(x)|/x^2$ is not necessarily monotone increasing or decreasing.

Is it true that $|R(x)|/x^2$ is monotone increasing? Is it a convex function of $x$?

Can we obtain a similar upper bound for non-convex regions?

Is there an analog of Theorem 3.1 in higher dimensions? The volume of the points of a ball with radius $r$ having a secant of length at most $x$ (for $0 \leqq x \leqq 2r$) is

$$\frac{\pi}{3}x^2\left(r+\frac{r^2-(x^2/4)}{r+\sqrt{r^2-(x^2/4)}}\right) \sim \frac{\pi}{2}rx^2$$

whenever $x = o(r)$.

**5. Upper bounds on the number of cutting lines.** Here we prove Theorem 1.7 and 1.1.

Suppose that $l_1, \dots, l_n$ cut $R$ with each obtained cell having width at least 1. Then by (2.1) each cell $C$ contains a circle of radius at least 1/3, so we have

$$\text{area } C \geqq \tfrac{1}{6} \text{ per } C, \tag{5.1}$$

where per $C$ stands for the length of bd $C$. We may suppose that $h_i = |l_i \cap R|$ is ordered monotone increasingly, $0 < h_1 \leqq \dots \leqq h_n$.

Claim 5.2. $\sum_{h_i \leqq x} h_i \leqq 3\,|R(x)|$.

Proof. Consider the cell-decomposition obtained by $\{l_i : h_i \leqq x\}$ without the cell containing $R \backslash R(x)$. Then for the area of these cells we have

$$3\,|R(x)| \geqq \sum_{C \subset R(x)} 3 \text{ area } C \geqq \sum \tfrac{1}{2} \text{ per } C$$
$$= \tfrac{1}{2} \text{ per } R + \sum_{h_i \leqq x} h_i - \tfrac{1}{2} \text{ per (the cell containing } R \backslash R(x)) \geqq \sum_{h_i \leqq x} h_i .$$

**Lemma 5.3.** *Suppose that* $0 < y_1 \leqq y_2 \leqq \dots \leqq y_n$, *and* $\sum_{i \leqq k} y_i \leqq f_k$ *hold for all k. Then* $n \leqq \sum_{k \leqq n} (f_k - f_{k-1})/y_k$, $(f_0 =: 0)$.

Proof. We have that $\sum (f_k - f_{k-1})/y_k = \sum_{k<n} f_k (y_k^{-1} - y_{k+1}^{-1}) + f_n/y_n$. Here the right hand side is larger then $\sum_{i \leqq n} y_i/y_n + \sum_{k<n} \left( \sum_{i \leqq k} y_i \right) (y_k^{-1} - y_{k+1}^{-1}) = n$. □

Proof of Theorem 1.7. By Claim 5.2 we have that

$$\sum_{i \leqq k} h_i \leqq 3\,|R(h_k)|.$$

Then Lemma 5.3 gives that

$$n \leqq 3 \sum_{k \leqq n} \frac{|R(h_k)| - |R(h_{k-1})|}{h_k}. \tag{5.4}$$

Rearranging the right hand side and using the equation $\int_a^b x^{-2}\,dx = (1/a) - (1/b)$ for $0 < a < b$, we obtain that

$$n \leqq 3 \left( \frac{|R(h_n)|}{h_n} + \sum_{k=1}^{n-1} |R(h_k)| \left( \frac{1}{h_k} - \frac{1}{h_{k+1}} \right) \right)$$
$$= 3 \int_{h_n}^{\infty} \frac{|R(h_n)|}{x^2}\,dx + 3 \sum_{k=1}^{n-1} \int_{h_k}^{h_{k+1}} \frac{|R(h_k)|}{x^2}\,dx \leqq 3 \int_0^{\infty} \frac{|R(x)|}{x^2}\,dx.$$

In the last step we used the monotonicity of $|R(x)|$. □

Proof of Theorem 1.1. It is well-known that

$$\text{area } R \leqq dw. \tag{5.5}$$

(Even more, area $R \leqq$ area $(D \cap S)$, where $D$ is a circular disc of diameter $d$, and $S$ is a strip of width $w$ with a common symmetry.) Then first Theorem 1.7, then the following corollary of 1.6

$$|R(x)| \leqq 11 \frac{d}{w} x^2,$$

and finally the inequality $|R(x)| \leqq |R|$ give that

$$n \leqq 3 \int_0^\infty \frac{|R(x)|}{x^2}\, dx \leqq 3 \int_0^{cw} 11 \frac{d}{w}\, dx + 3 \int_{cw}^\infty \frac{dw}{x^2}\, dx$$
$$= 33\, cd + 3 \frac{d}{c} = 3\, d \left(11\, c + \frac{1}{c}\right).$$

Choosing $c = 1/\sqrt{11}$ we obtain that the right hand side is at most $6\sqrt{11}\, d < 20\, d$. □

**6. Another measure for small cuts.** Suppose that $\{l_1, \ldots, l_\alpha\}$ are cutting lines having cells with inscribed circle radius at least 1/2. Let $a(R)$ denote the largest possible $a$. Similarly to (1.2), 1.7 and 1.1 we have that

$$a(R) > 2\, d - \frac{10\, d}{w^{1/3}},$$

$$a(R) < 2 \int_0^\infty \frac{|R(x)|}{x^2}\, dx,$$

$$a(R) < 13\, d.$$

For example, for the circle $D$ of diameter $d$, for the square $S$ of sidelength $s$, and for the rectangle $R$ with sides $a > b$ we have

$$2\, d - o(d) \leqq a(D) < \pi\, d,$$

$$2\sqrt{2}\, s < a(S) < 4.35\, s,$$

$$2\, a - \frac{2\, a}{2\, b - 1} < a(R) < 2\, a + \frac{3\pi}{4}\, b.$$

**7. Cutting segments and the largest inscribed circle.** In this section we prove Theorem 2.2.

Suppose that $k$ is the largest inscribed circle of the closed convex region $R$. Denote its center by $O$, its radius by $\varrho$. If there are two opposite points of $\mathrm{bd}\,(k)$ touching $\mathrm{bd}\,(R)$, then $w_0(R) = 2\,\varrho(R)$, and we are done. So we may suppose that there are three points $A_1, B_1, C_1 \in \mathrm{bd}\,(k) \cap \mathrm{bd}\,(R)$ such that $O \in \mathrm{Int\,Conv}\,(A_1 B_1 C_1)$. The three lines touching $k$ at the points $A_1, B_1$ and $C_1$ form a triangle $ABC$ such that $A_1 \in BC$, etc. Then the $ABC$ triangle contains $R$, so $w_0(\mathrm{Conv}\, ABC) \geqq w_0(R)$. So it is sufficient to prove that $w_0(ABC) \leqq (4/\sqrt{3})\,\varrho$.

Denote the angles of the $ABC$ triangle by $2\alpha$, $2\beta$, $2\gamma$, e.g., $\angle\,(ACB) = 2\gamma$. Considering the lines orthogonal to $OX$ at the point $X$, where $X \in \{A, B, C\}$, one can obtain an (acute) triangle $A_2 B_2 C_2$ such that $C \in A_2 B_2 \ldots, O$ is the meeting point of the three heights $AA_2$, $BB_2$ and $CC_2$. (See Fig. 7.1.) Then $\angle(OB_2C) = \alpha$, and $\angle(OC_2A) = \beta$, so we have

$$|CC_2| = |CB_2| \cot\beta = |OC| \cot\alpha \cot\beta = \varrho(\sin\gamma)^{-1} \cot\alpha \cot\beta. \tag{7.1}$$

Of course, similar equations hold for the other two heights of $A_2B_2C_2$.

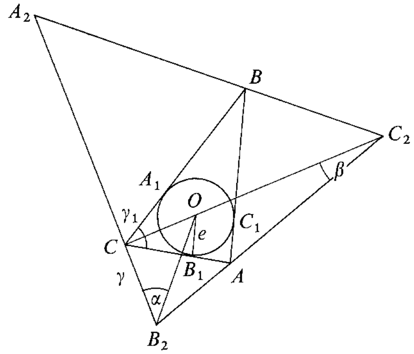

Figure 7.1

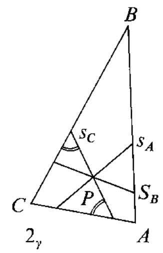

Figure 7.2

Suppose that $P \in \mathrm{Conv}(ABC)$ is a point (actually, the only point) having no secants through it shorter than $w_0 =: w_0(ABC)$. Then there are three segments $s_A$, $s_B$ and $s_C$ through $P$ of length $w_0$, such that the endpoints of $s_C$ lie on $CA$ and $CB$ and it is parallel to $A_2B_2$, and so on for $s_A$ and $s_C$, (see Fig. 7.2). The distance of $P$ from $A_2B_2$ is $\frac{1}{2} w_0 \cot\gamma$. It is easy to prove that for an arbitrary triangle $XYZ$ with $P \in \mathrm{Int}(XYZ)$, one has

$$\frac{d(P, l(XY))}{h_Z} + \frac{d(P, l(YZ))}{h_X} + \frac{d(P, l(ZX))}{h_Y} = 1,$$

where $h_X$ is the height of the triangle from $X$, etc. Applying this to $A_2B_2C_2$ and $P$ (7.1) implies that

$$\frac{\cot\gamma \sin\gamma}{\cot\alpha \cot\beta} + \frac{\cot\alpha \sin\alpha}{\cot\beta \cot\gamma} + \frac{\cot\beta \sin\beta}{\cot\gamma \cot\alpha} = \frac{2\varrho}{w_0}. \tag{7.2}$$

So we are done if we prove that the left hand side is at least $\sqrt{3}/2$. This is equivalent to the following

$$\frac{(\cos\gamma)^2}{\sin\gamma} + \frac{(\cos\alpha)^2}{\sin\alpha} + \frac{(\cos\beta)^2}{\sin\beta} \geqq \frac{\sqrt{3}}{2} \cot\alpha \cot\beta \cot\gamma. \tag{7.3}$$

As $\alpha + \beta + \gamma = \pi/2$ we have that

$$\cot\alpha \cot\beta \cot\gamma = \cot\alpha + \cot\beta + \cot\gamma.$$

Hence, (7.3) is equivalent to

$$f(\alpha) + f(\beta) + f(\gamma) \geqq 0,$$

where

$$f(\alpha)=\frac{(\cos\alpha)^2}{\sin\alpha}-\frac{\sqrt{3}}{2}\cot\alpha .$$

The second derivative of $f(\alpha)$ is $f''(\alpha)=(\sin\alpha)^{-3}(u^4-u^2-\sqrt{3}u+2)$, where $u=\cos\alpha$. This is positive for $u\leqq 1$, so $f(\alpha)$ is strictly convex for $0<\alpha<\pi$. Jensen's inequality yields

$$f(\alpha)+f(\beta)+f(\gamma)\geqq 3f\left(\frac{\alpha+\beta+\gamma}{3}\right)=3f(\pi/6)=0. \quad \square$$

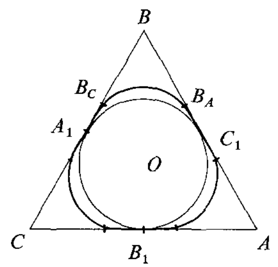


Figure 7.3

Equality holds only if $\alpha=\beta=\gamma=\pi/6$. It is easy to see that $w_0=(4/\sqrt{3})\varrho$ implies that $R_1\subset R\subset \operatorname{Conv}(ABC)$, where the triangle $ABC$ is regular, and the boundary of the region $R_1$ consists of the middle third of the sides of the $ABC$ (see Fig. 7.3), and of three *shell-lines*, where e.g. $B_C B_A =: \{P : \exists X\in AC$ such that $|PX|=(4/\sqrt{3})\varrho,\ O\in PX\}$.

**8. Width and cutting width.** In this section we prove Theorem 2.3 in the following slightly stronger form. Let $O$ be the center of the largest inscribed circle $k$ into $R$, the radius of $k$ is $\varrho$.

**Theorem 8.1.** $w_0(O,R)\geqq \dfrac{4}{3\sqrt{3}}\,w.$

Proof. If $\varrho>(2/3\sqrt{3})\,w$, then we are done, because $w_0(O,R)\geqq 2\varrho$. So from now on we may suppose that

(8.1) $$w\geqq (3\sqrt{3}/2)\,\varrho>2.598\ldots\varrho .$$

There are points $A_1$, $B_1$ and $C_1\in \operatorname{bd}k\cap\operatorname{bd}R$ such that $O\in\operatorname{Int}\operatorname{Conv}(A_1B_1C_1)$. The three lines touching $k$ at the points $A_1, B_1$ and $C_1$ form a triangle $A'B'C'$ such that $A_1\in B'C'$, etc. (See Fig. 8.1.) The triangle $A'B'C'$ contains $R$.

Let $A$ be a point of $R$ such that $d(A, l(B'C'))$ is maximum. Then $A$ lies in the triangle $B_1A'C_1$. Consider the secants of $k$ from $A$, denote their touching points by $A_B$ and $A_C$ and their angle by $\alpha$. We have

(8.2) $$w\leqq \varrho+|OA|=\varrho+\varrho\left(\sin\frac{\alpha}{2}\right)^{-1}.$$

By (8.1) we have that $|OA| > 1.5\,\varrho$, so $\alpha < \pi/2$. One can define $B$ and $C$ in a similar way. The region $R_1$ bounded by the six secants and the arcs $B_A A_B$, $A_C C_A$ and $C_B B_C$ is contained in $R$.

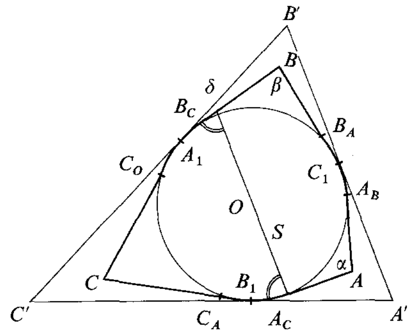

Figure 8.1

Obviously, $w_0(O, R) \geqq w_0(O, R_1)$. As $\alpha$, $\beta$ and $\gamma$ are acute angles, $w_0(O, R_1) > 2\varrho$. Even more, it is easy to show that a shortest cutting segment $s$ of $R_1$ intersects two secants, say $BB_C$ and $AA_C$ in the same angle $\delta$. (This follows from the fact that if $s$ is the shortest secant segment through a point $O$ lying on the bisector, then $s$ is perpendicular to this bisector.) Suppose that $\alpha \geqq \beta$. Then, as $2\delta \leqq \alpha + \beta \leqq 2\alpha$ we have that

(8.3) $$w_0(O, R_1) \geqq |s| = 2\varrho(\sin\delta)^{-1} \geqq 2\varrho(\sin\alpha)^{-1}.$$

The ratio of (8.3) and (8.2) gives

(8.4) $$\frac{w_0}{w} \geqq \frac{2(\sin\alpha)^{-1}}{1 + \left(\sin\frac{\alpha}{2}\right)^{-1}} = \frac{2}{\sin\alpha + 2\cos\frac{\alpha}{2}}.$$

Here the denominator takes its maximum over $0 < \alpha < \pi$ at the value $\alpha = \pi/3$, so the right hand side of (8.4) is at least $4/(3\sqrt{3})$. □

It is quite clear that equality is possible only in the case $\alpha = \beta = \gamma = \pi/3$, then $R$ is a regular triangle.

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