On Zero-Trees

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ABSTRACT

Consider an integer-valued function on the edge-set of the complete graph $\mathbf{K}_{m+1}$. The weight of an edge-subset is defined to be the sum of the associated weights. It is proved that there exists a spanning tree with weight 0 modulo m.

1. 0-TREES AND 0-SUM SUBSETS

Let $\mathbf{K}_{m+1}$ be a complete graph with vertex-set V and edge-set $\mathscr{E}$. Let Z be an m-element group. The binary group operation is denoted by the plus sign and is called addition. Let $\varphi:\mathscr{E} \to \mathrm{Z}$ be a function on the edges, which we call a *weight function*. The weight $\varphi(\{x, y\})$ of an edge $\{x, y\}$ in $\mathscr{E}$ is abbreviated as $\varphi(xy)$. For a subset $\mathscr{F} \subset \mathscr{E}$ denote

$$\varphi(\mathscr{F}) = \sum_{f \in \mathscr{F}} \varphi(f),$$

so that $\varphi(\mathscr{F})$ is the sum of the weights of the edges in $\mathscr{F}$. (If Z is not Abelian, then $\varphi(\mathscr{F})$ means all possible sums, and $\varphi(\mathscr{F}) = 0$ means that there is an ordering with sum 0.)

Theorem 1.1. There is a spanning tree $\mathbf{T}$ of $\mathbf{K}_{m+1}$ such that the sum of the weights of its edges is 0 modulo Z, i.e., $\varphi(\mathbf{T}) = 0$.

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This statement was conjectured for the cyclic group $Z = Z_m$ by Bialostocki, and was proved by Bialostocki and Dierker [2] when m is a prime. Fan Chung [3] proved the case when φ takes only at most three diffent values.

There are many results on subset sums; for a survey see Alon [1]. Theorem 1.1 implies that $\binom{m+1}{2}$ elements of Z contain an m-element 0-sum. Indeed, for Abelian groups much more is true. Erdös, Ginzburg, and Zvi [4] showed that every sequence of length $2m - 1$ contains a substring of length m with 0 sum. The sequence consisting of $m - 1$ 0's and $m - 1$ 1's shows that their result is sharp. More on this is described in [1].

It would be interesting to extend the scope of Theorem 1.1 to other graphs. Call a graph $\mathbf{G}$ *0-weighted* (with respect to the group Z), if every $\varphi:\mathscr{E}(\mathbf{G}) \to Z$ induces a 0-weighted spanning tree. It is possible that φ induces only one 0-tree of $\mathbf{K}_{m+1}$, for example, if $\varphi \equiv 0$ except for the edges of a complete claw $\mathbf{C}(x, V\backslash\{x\})$, where $\varphi \equiv 1$. (The edge-set of the claw $\mathbf{C}(x, A)$ is defined by $\mathscr{E}(\mathbf{C}(x, A)) = \{\{x, y\} : y \in A\}$.) Hence deleting any edge from $\mathbf{K}_{m+1}$ we obtain a graph that is not 0-weighted with respect to Z_m.

Suppose that the graph $\mathbf{G}$ is 0-weighted mod m ($m \geq 2$). Let $C \subset V(\mathbf{G})$ be an arbitrary k-element set ($0 < k < m$) and $V_1 \cup V_2$ a partition of $V(\mathbf{G})\backslash C$ such that there is no edge joining V_1 and V_2. Then defining $\varphi \equiv 0$ for the edges in $C \cup V_2$ and $\varphi \equiv 1$ otherwise, it is easy to see that for one of the following k integers,

$$|V_1|, |V_1| + 1, \ldots, |V_1| + k - 1 \equiv 0 (\mathrm{mod}\ m). \tag{1.2}$$

In the case $m = 2$ this constraint simple means that every 2-connected block of $\mathbf{G}$ has odd number of vertices.

Conjecture 1.3 (Seymour [5]). $\mathbf{G}$ is 0-weighted mod m if and only if (1.2) holds for all C.

He proved the cases $m = 2, 3$.

2. STABILIZERS AND QUASI-STABILIZERS

Here we start the proof of Theorem 1.1 by introducing notations. For a set $A \subset V$ we denote the weights of the spanning trees in A by $\Phi(A)$. Let $\mathscr{A} = (A \subset V : |\Phi(A)| \geq |A| - 1\}$. Suppose that $A \in \mathscr{A}$ has maximum cardinality. As all the 2-element subsets belong to $\mathscr{A}$, we may suppose that $|A| \geq 2$.

If $A = V$, then $\Phi(A) = Z$, so there is a tree of weight 0, and we are done. From now on we suppose that $V\backslash A \neq \emptyset$. The edges joining A with $V\backslash A$ are called *crossing* edges. We use the notation $N(A)$ for the set of weights of crossing edges, $N(A) =: \{\varphi(ax) : a \in A, x \in V \backslash A\}$. Denote by $C(A)$ the subgroup of Z generated by the elements of $N(A)$. Let $\Gamma =: N \backslash N = \{\varphi(ax) - \varphi(by) : a, b \in A \text{ and } x, y \in V \backslash A\}$. Usually the set $\Phi(A)$ is abbreviated as Φ. Define $\mathrm{Stab}_l(\Delta)$, the *left stabilizer* of $\Delta \subset Z$ as follows: $\mathrm{Stab}_l(\Delta) = \{\alpha \in Z : \alpha + \Delta = \Delta\}$. The definition of *right stabilizers* is similar, and finally, let $\mathrm{Stab}(\Delta) = \mathrm{Stab}_l \cap \mathrm{Stab}_r = \{\alpha \in Z : \Delta + \alpha = \alpha + \Delta = \Delta\}$.

The maximality of A implies that

$$|(\Phi(A) + \varphi(ax)) \cup (\varphi(ax) + \Phi(A))| \leq |\Phi(A \cup x)| = |\Phi(A)|;$$

hence

Fact 2.1. $\varphi(ax) + \Phi = \Phi + \varphi(ax)$ holds for all $a \in A, x \in V \backslash A$. ∎

Fact 2.2. If $a, b \in A$ and $x \in V \backslash A$, then $\varphi(ax) - \varphi(bx)$ belongs to $\mathrm{Stab}(\Phi)$.

Proof. For every spanning tree of A, we can add either the edge ax or bx to get a spanning tree of $A \cup \{x\}$. Hence $\Phi(A \cup \{x\})$ contains both $\Phi + \varphi(ax)$ and $\Phi + \varphi(bx)$. As the set A was chosen to be maximal in $\mathcal{A}$, we have

$$|A| - 1 = |\Phi| \leq |(\Phi + \varphi(ax)) \cup (\Phi + \varphi(bx))| \leq |\Phi(A \cup \{x\})| < |A|.$$

Hence the two translates of Φ coincide, and $\varphi(ax) - \varphi(bx)$ is a stabilizer. ∎

If Z is an Abelian group, and $\gamma \in Z$ is an arbitrary element, then φ induces a 0-tree if and only if the function $\varphi - \gamma$ induces a 0-tree. Hence we may suppose that there is a crossing edge ax with $\varphi(ax) = 0$. Then most of the technicalities in the rest of this paper are slightly simpler.

Define $\mathrm{QStab}(\Delta)$ as the set of *quasi-stabilizers*, $\alpha \in Z$ belongs to $\mathrm{QStab}(\Delta)$ if $\Delta + \alpha = \alpha + \Delta$, and $(\Delta + \alpha) \backslash \Delta$ contains at most one element. Let $\mathrm{QStab}_l(\Delta) = \{\alpha \in Z : |(\alpha + \Delta) \backslash \Delta| \leq 1\}$; the definition of QStab_r is analogous. Note that QStab is not necessarily the intersection of the set of left and right quasistabilizers, but $\mathrm{QStab} = \mathrm{QStab}_l \cap \mathrm{QStab}_r \cap C(\Delta)$, where $C(\Delta) = \{\alpha \in Z : \Delta + \alpha = \alpha + \Delta\}$. We have $\mathrm{Stab}(\Delta) \subset \mathrm{QStab}(\Delta)$ but $\mathrm{QStab}(\Delta)$ is not necessarily a subgroup of Z, unlike $\mathrm{Stab}(\Delta)$.

Fact 2.3. If $a, b \in A$ and $x, y \in V \backslash A$, then $\varphi(ax) - \varphi(by)$ belongs to $\mathrm{QStab}(\Phi)$.

Proof. First, Fact 2.1 implies that $\varphi(ax) - \varphi(by)$ commutes Φ. Next, we may proceed exactly as in the proof of 2.2. We have that $\Phi(A \cup \{x, y\})$ contains both $\Phi + \varphi(ax) + \varphi(xy)$ and $\Phi + \varphi(by) + \varphi(xy)$, and the size of their union is at most $|\Phi| + 1$. Hence $(\Phi + \varphi(ax) + \varphi(xy)) \backslash (\Phi + \varphi(by) +$

$\varphi(xy))$ contains at most one element, so

$$(\Phi + \varphi(ax) + \varphi(xy))\backslash(\Phi + \varphi(by) + \varphi(xy))$$
$$= ((\Phi + \gamma)\backslash\Phi) + \varphi(by) + \varphi(xy),$$

where $\gamma = \varphi(ax) - \varphi(by)$. This gives $\gamma \in \mathrm{QStab}_r(\Phi)$. ∎

If $A \subset D \subset V$, then Φ(D) contains a spanning tree **T** (over D) with a weight that is a sum of $|D| - 1$ members of $N(A)$,

$$\Phi(D) \cap (|D| - 1)N(A) \neq \varnothing. \tag{2.4}$$

Indeed, this is trivial for $D\backslash A \neq \varnothing$, as there is a spanning tree consisting of crossing edges only. For the case $D = A$, let $a_0 \in A, x \in V\backslash A$ and consider the claw $\mathbf{C} = \mathbf{C}(x, A)$ with center x and end points A. Then $\Sigma_{a\in A}\varphi(ax) \in \Phi(A \cup x) = \Phi(A) + \varphi(a_0x)$, implying that

$$\sum_{a\in A\backslash a_0} \varphi(ax) \in \Phi(A). \tag{2.5}$$

3. A GENERAL CONSTRUCTION FOR 0-TREES

The proof of Theorem 1.1 uses two ideas. First, we consider the maximal set A defined in the previous section. Second, we will see that Φ (or something similar) contains a coset of Stab(Φ) containing a value from $(n - |A|)N(A)$. These imply the existence of a 0-tree. We have to overcome some technical difficulties, namely, to handle the cosets of a subgroup properly we need a *normal* subgroup. Stab(Φ) is not necessarily a normal subgroup of Z, but as we will see, it is normal in $C(A)$. The next lemma formulates these ideas.

Lemma 3.1. Suppose that $A \subset D \subset V$, and for $\Delta \subset Z$, the following hold:

(1) $\Delta \subset \Phi(D)$,
(2) $\gamma \in \mathrm{Stab}_l(\Delta)$ for all $\gamma \in \Gamma$,
(3) $\Delta \cap (|D| - 1)N(A) \neq \varnothing$, and
(4) $v \in C(A)$ and $\alpha \in \mathrm{Stab}_l(\Delta) \cap C(A)$ imply $v + \alpha - v \in \mathrm{Stab}_l(\Delta)$.

Then there is a 0-tree with vertex set V.

Proof. Let $I =: \{\alpha \in C(A): \alpha + \Delta = \Delta\} = C(A) \cap \mathrm{Stab}_l(\Delta)$. Then I is a subgroup of $C(A)$, and (4) implies that it is a normal subgroup, $I \lhd C(A)$. Consider the coset-decomposition of $C(A)$:

$$C(A) = (I + \eta_1) \cup (I + \eta_2) \cup \dots .$$

The second constraint (more exactly, the fact $\Gamma \subset I$) implies that all the members $\varphi(ax) \in N(A)$ are contained in the same coset. Then the normality of I implies that for some η_0 we have

$$(|D| - 1)N \subset (I + \eta_0). \tag{3.2}$$

As $(n - 1)z = 0$ for all $z \in Z$, it follows that

$$(-n + |D|)N \subset (I + \eta_0). \tag{3.3}$$

As $I < \mathrm{Stab}_l(\Delta)$, Δ is also a union of cosets of I, $\Delta = (I + \theta_2) \cup (I + \theta_2) \cup \ldots$. Then (3) and (3.2) imply that the coset $I + \eta_0$ is contained in Δ and (1) implies that it is contained in $\Phi(D)$,

$$(I + \eta_0) \subset \Phi(D). \tag{3.4}$$

Let $\mathbf{C} = \mathbf{C}(a_0, V\backslash D)$ for some $a_0 \in A$. We have that $-\varphi(\mathbf{C}) \in (-n + |D|)N(A)$, and then $-\varphi(\mathbf{C}) \in \Phi(D)$ by (3.3) and (3.4). So there is a tree $\mathbf{T}$ over D with weight $\varphi(\mathbf{T}) = -\varphi(\mathbf{C})$. Then $\mathbf{T} \cup \mathbf{C}$ is a 0-weighted spanning tree. ∎

The usefulness of the above construction is shown by the following:

Case 3.5. Suppose that $|\mathrm{Stab}_r(\Phi)| > 1$. Then there is a 0-tree over V.

Proof. We are going to apply Lemma 3.1 with $D = A$ and $\Delta = \Phi(A)$. As Φ is the union of some cosets of $\mathrm{Stab}_r(\Phi)$,

$$\Phi = (\eta_1 + \mathrm{Stab}_r) \cup \cdots \cup (\eta_s + \mathrm{Stab}_r),$$

we have that $|(\alpha + \Phi)\backslash\Phi|$ is always divisible by $|\mathrm{Stab}_r(\Phi)|$. In our case this implies that $\mathrm{QStab}_l(\Phi) = \mathrm{Stab}_l(\Phi)$. Then Fact 2.3 yields the second constraint in Lemma 3.1. (The first one is obvious.) The third one is immediate from (2.5). Finally, to check (4) let $v \in C(A)$, $\alpha \in \mathrm{Stab}_l(\Phi)$. We obtain from 2.1 that

$$(v + \alpha - v) + \Phi = v + \alpha + \Phi - v = v + \Phi - v = \Phi. \quad ∎$$

4. TWO 0-TREES OF CROSSING EDGES

From now on we may suppose that $\mathrm{Stab}(\Phi) = \{0\}$. (Even more, that $\mathrm{Stab}_l(\Phi) = \mathrm{Stab}_r(\Phi) = \{0\}$.) Then Fact 2.2 implies that each claw $\mathbf{C}(x, A)$ is homogeneous ($x \in V\backslash A$), i.e., there is a function $\varphi(x): V\backslash A \to Z$ such that of all crossing edge $\{a, x\}$ one has

$$\varphi(ax) = \varphi(x).$$

If all the crossing edges have the weight $\varphi(a_1x_1)$, then one can easily find a spanning tree with all of its edges having the same weight. So from now on, we also suppose that Γ has nonzero elements as well, e.g., $\varphi_1 = \varphi(ax_1) \neq \varphi(ax_2) = \varphi_2$.

Claim 4.1. If $|\Phi(A) \cap (|A| - 1)N| \leq 1$ and $|A| > 2$, then $\varphi_i + \varphi_j = \varphi_j + \varphi_i$, $2(\varphi_j - \varphi_i) = 0$ and $2\varphi_i = 2\varphi_j$ for all $\varphi_i, \varphi_j \in N(A)$.

Proof. This proposition is a sharpening of (2.4). Let $K \subset A$ be an k-element subset, and let $\mathbf{T}_k$ be the tree consisting of the union of $\mathbf{C}(x_j, K)$ and $\mathbf{C}(A\backslash K) \cup \{\omega_k\})$ where $\omega_k \in K$, for $k = 1, 2, 3$. Considering the weights of these trees, we obtain that $\varphi_i + \varphi_i + \theta$, $\varphi_i + \varphi_j + \theta$, $\varphi_j + \varphi_i + \theta$ and $\varphi_j + \varphi_j + \theta$ all belong to $((|A| - 1)N + \varphi_i + \varphi_j) \cap \Phi(A \cup \{x_i, x_j\})$, where $\theta = (|A| - 2)\varphi_i + \varphi_j$. Here $|\Phi(A \cup \{x_i, x_j\})\backslash(\Phi + \varphi_i + \varphi_j)| = 1$, so our condition says that these four expressions take only at most 2 values. However, for $\varphi_i \neq \varphi_j$ we have $2\varphi_i \neq \varphi_i + \varphi_j$ and $2\varphi_i \neq \varphi_j + \varphi_i$. So the right-hand sides must coincide, $\varphi_i + \varphi_j = \varphi_j + \varphi_i$. Similarly, we have $2\varphi_i = 2\varphi_j$. Finally, these two equations imply the third one. ∎

Case 4.2. Suppose that $\varphi_i + \varphi_j = \varphi_j + \varphi_i$, $2\varphi_i = 2\varphi_j$ hold for all $\varphi_i, \varphi_j \in N(A)$, and $|N(A)| \leq |A|$. Then there exists a 0-tree:

Proof. We handle this case by the following construction: As $2(\varphi_j - \varphi_i) = 0$, the order of Z $(= n - 1)$ is even. Consider the claw $\mathbf{C} = \mathbf{C}(a_1, V\backslash A)$ for some $a_1 \in A$, and let $\varphi^1, \ldots, \varphi^t \in N(A)$ be the elements appearing an odd number of times as a weight of an edge of $\mathbf{C}$. (The case $\{\varphi^1, \ldots, \varphi^t\} = \emptyset$ is even simpler.) Let $x^i \in V\backslash A$ be an element with $\varphi(a_1x_i) = \varphi_i$. As $|A| \geq t$, there are distinct elements $a_1, \ldots, a_t \in A$. Then the following union is a spanning 0-tree:

$$\mathbf{T} = \mathbf{C}(a_1, V\backslash A) \cup \left(\bigcup_{i=2}^{t} \{a_i, x_i\} \right) \cup \mathbf{C}(x_1, A\backslash\{a_1, \ldots, a_t\}).$$

Indeed, in the sum $\Sigma_{e\in\mathbf{T}}\varphi(e)$ every weight $\varphi(e) \neq \varphi^1$ appears even times, so the total sum equals to $(n - 1)\varphi^1 = 0$. ∎

Case 4.3. Suppose that $N(A) = \{\varphi_1, \varphi_2\}$. Then there is a 0-tree.

Proof. Let $K \subset A$ be an k-element subset $(0 \leq k \leq |A|)$, and let $\mathbf{T}'_k$ be the tree consisting of the union of $\mathbf{C}(x_2, K)$ and $\mathbf{C}(x_1, A\backslash K)$ joining with the edge $\{x_1, x_2\}$. Consider $\varphi(\mathbf{T}'_k) = \varphi(x_1x_2) + (|A| - k)\varphi_1 + k\varphi_2$. As A is maximal, the above $|A| + 1$ weights are not all distinct, implying $c\varphi_1 = c\varphi_2$ for some $(2 \leq)$ $c \leq |A|$.

Suppose that the number of crossing edges $\{\{a_1, x\} : x \in V\backslash A$ with weight $\varphi_2\}$ is $lc - k$, where $0 \leq k < c$. Let $K \subset A$, $a_1 \notin A$ with $|K| = k$.

Then the following union is a spanning 0-tree.

$$\mathbf{T}_k^* = \mathbf{C}(a_1, V\backslash A) \cup \mathbf{C}(x_2, K) \cup \mathbf{C}(x_1, A\backslash K).$$

The weight of $\varphi(\mathbf{T}_k^*) = lc\varphi_2 + (n - 1 - lc)\varphi_1 = (n - 1)\varphi_1 = 0.$ ∎

5. THE STRUCTURE OF QUASI-STABILIZERS

We are going to use Lemma 3.1 four more times.

Suppose that $\beta \in \Gamma$ has *maximum* order, e.g., $b\beta = 0, b'\beta \neq 0$ for $0 < b' < b$, and for all $\gamma \in \Gamma$ there exists some $0 < c(\gamma) \leq b$ such that $c\gamma = 0$. Let $Z(\beta)$ denote the subgroup of Z generated by β; we have $|Z(\beta)| = b \geq 2$. We may suppose that $\beta = \varphi(ax_2) - \varphi(ax_1)$.

Consider the orbits $C_0, C_1, \ldots, C_s$ of the elements of Φ in the repeated applications of the operation $\xi \to \xi + \beta$. As $\beta \in \mathrm{QStab}(\Phi)\backslash\mathrm{Stab}(\Phi)$, we have that all but one of these orbits are contained in Φ. Say, $\Phi = C_0' \cup C_1 \cup \cdots \cup C_s$, where each C_i is a coset of $Z(\beta)$ (of the form $C_i = \xi_i + Z(\beta)$) and $C_0' \subset C_0$, $0 < |C_0'| < b$. Say,

$$C_0' = \{\xi_0, \xi_0 + \beta, \ldots, \xi_0 + (u - 1)\beta\}.$$

Here the case $s = 0$ is of course possible.

We classify three cases according to $u = b - 1, u = 1$ or $1 < u < b - 1$. In this and in the next two sections the case $b \geq 3$ is settled. First, we consider the case $u = b - 1$ in a more general setting.

Case 5.1. Suppose that there is a subgroup $M < Z$, $|M| = m \geq 3$, and a coset decomposition

$$\eta_0 + M, \eta_1 + M, \ldots, \eta_r + M, \tag{5.2}$$

such that $\Phi \subset \bigcup_{0\leq i\leq r}(\eta_i + M)$, $|\Phi| + 1 = (r + 1)m$. Then there is a 0-tree.

Proof. Let α be an arbitrary left quasi-stabilizer of Φ. We claim that it is a left stabilizer of $\bigcup(\eta_i + M)$. Suppose, on the contrary, that the coset $\alpha + \eta_t + M$ does not appear among those in (5.2). Then $(\alpha + \eta_t + M) \cap (\bigcup\eta_i + M) = 0$. This and the facts $|(\alpha + \eta_t + M) \cap (\alpha + \Phi)| \geq 2$, $\Phi \subset (\bigcup\eta_i + M)$ imply $|(\alpha + \Phi)\backslash\Phi| \geq 2$, a contradiction. Hence

$$\mathrm{QStab}_l(\Phi) = \mathrm{Stab}_l(\eta_0 + M \cup \cdots \cup \eta_r + M_r). \tag{5.3}$$

Even more, we have got that $(\alpha + \Phi) \subset \bigcup(\eta_i + M)$. This (and 2.1) implies that

$$\bigcup(\eta_i + M) = \Phi \cup (\Phi + \beta).$$

As $\Phi(A \cup \{x_1, x_2\}) = \Phi \cup (\Phi + \beta) + \varphi_1 + \varphi(x_1x_2)$, we have that

$$\mathrm{Stab}_l(\cup_i(\eta_i + M)) = \mathrm{Stab}_l(\Phi(A \cup \{x_1, x_2\})). \tag{5.4}$$

Apply Lemma 3.1 with $D = A \cup \{x_1, x_2\}$, $\Delta = \Phi(D)$. Then (1) is obvious, and (3) follows from (2.4). To check (2) let $\gamma \in \Gamma$, $\gamma \neq 0$. The 2.3 and (5.3) imply that $\gamma \in \mathrm{Stab}_l(\cup_i(\eta_i + M))$. Hence (5.4) gives $\gamma \in \mathrm{Stab}_l(\Delta)$.

Finally, let $v \in C(A)$ and $\alpha \in \mathrm{Stab}_l(\Delta) \cap C(A)$. Using (5.4) we have that $\alpha \in \mathrm{Stab}_l(\Phi \cup (\Phi + \beta))$. By 2.1 we have

$$(v + \alpha - v) + (\Phi \cup (\Phi + \beta)) \supset (v + \alpha - v + \Phi) = v + \alpha + \Phi - v. \tag{5.5}$$

Here $\Phi = (\cup_i(\eta_i + M))\backslash x$ for some $x \in \Phi \cup (\Phi + \beta)$, so

$$\alpha + \Phi = \alpha + (\Phi \cup (\Phi + \beta)\backslash x) = \Phi \cup (\Phi + \beta)\backslash(\alpha + x) \supset \Phi\backslash(\alpha + x).$$

Continuing the right-hand side of (5.5), we have

$$\supset (v + \Phi - v)\backslash(v + \alpha + x - v) = \Phi\backslash(v + \alpha + x - v).$$

Hence $(v - \alpha + v) + (\Phi \cup (\Phi + \beta))$ contains at least $|\Phi| - 1$ elements of Φ, so it contains at least $|\Phi \cup (\Phi + \beta)| - 2$ elements of $\Phi \cup (\Phi + \beta)$. So it intersects all of the cosets of $\Phi \cup (\Phi + \beta)$. But it consists of the same number of cosets, so it is a left stabilizer of $\Phi \cup (\Phi + \beta)$. Then, by (5.4), it is a left stabilizer of $\Phi(A \cup \{x_1, x_2\})$ as well. ∎

6. THE CASE $u = 1$.

Here we deal with the case $u = 1, b \geq 3$ in a more general form.

Case 6.1. Suppose that there is a subgroup $M < Z$, $|M| = m \geq 3$, and a coset decomposition

$$\eta_1 + M, \ldots, \eta_r + M,$$

such that $\Phi \supset \cup_{1 \leq i \leq r}(\eta_i + M)$, $|\Phi| = rm + 1$. Then there is a 0-tree.

Proof. Let η_0 denote the element $\Phi\backslash(\cup_{i=1,\ldots,r}(\eta_i + M))$. For an arbitrary $\alpha \in \mathrm{QStab}_l(\Phi)$, we have that $(\alpha + \eta_0 + M) \cap (\cup_{i>0}(\eta_i + M)) = \varnothing$. Hence in this case α is a left stabilizer of $\cup_{i>0}(\eta_i + M) = \Phi\backslash\{\eta_0\}$,

$$\mathrm{QStab}_l(\Phi) = \mathrm{Stab}_l(\Phi\backslash\{\eta_0\}). \tag{6.2}$$

Apply Lemma 3.1 with $D = A$, $\Delta = \Phi(A)\backslash\{\eta_0\}$. Then (1) is obvious, and (2) is implied by 2.3 and (6.2).

To check (3) we use Claims 4.1 and 4.2. As $|A| \geq 2$ and $m \geq 3$, we have that $|A| \geq 1 + m \geq 4$. Suppose, on the contrary, that $\Phi\backslash\{\eta_0\}$ contains no element of $(|A| - 1)N(A)$. Then Claim 4.1 implies that $\varphi_i + \varphi_j = \varphi_j + \varphi_i$ and $2\varphi_i = 2\varphi_j$ hold for all $\varphi_i, \varphi_j \in N(A)$. Equation (6.2) implies that

$$|N(A)| = |\{\varphi_i - \varphi_1 : \varphi_i \in N(A)\}| \leq |\Gamma| \leq |\mathrm{QStab}_l(\Phi)| \leq |\Phi| - 1 < |A|.$$

Then Claim 4.2 yields a 0-tree.

The proof of (4) is similar to the proof given in the previous section. Let $v \in C(A)$ and $\alpha \in \mathrm{Stab}_l(\Phi\backslash\{\eta_0\}) \cap C(A)$. First observe, that $(v + \alpha - v) + (\Phi\backslash\{\eta_0\})$ consists of the same number of cosets of the form $x + Z(\beta)$ as $\Phi\backslash\{\eta_0\}$; hence

$$|[\Phi\backslash\{\eta_0\}]\backslash[(v + \alpha - v) + (\Phi\backslash\{\eta_0\})]| \text{ is divisible by } b. \tag{6.3}$$

By 2.1 we have

$$\begin{aligned}(v + \alpha - v) + (\Phi\backslash\{\eta_0\}) &= (v + \alpha - v + \Phi)\backslash\{v + \alpha - v + \eta_0\} \\ &= v + (\alpha + \Phi) - v\backslash\{v + \alpha - v + \eta_0\}.\end{aligned} \tag{6.4}$$

Here $\alpha + (\Phi\backslash\{\eta_0\}) = \Phi\backslash\{\eta_0\}$, so

$$\alpha + \Phi = \alpha + (\Phi\backslash\{\eta_0\} \cup \{\eta_0\}) = \Phi\backslash\{\eta_0\} \cup \{\alpha + \eta_0\}.$$

Continuing the right-hand side of (6.4) we have

$$\begin{aligned}&= v + (\Phi\backslash\{\eta_0\} \cup \{\alpha + \eta_0\}) - v\backslash\{v + \alpha - v + \eta_0\} \\ &\qquad = (v + \Phi - v)\backslash\{v + \eta_0 - v\} \\ &\qquad\quad \cup \{v + \alpha + \eta_0 - v\}\backslash\{v + \alpha - v + \eta_0\}.\end{aligned}$$

Here the first term is Φ, hence $(v - \alpha + v) + (\Phi\backslash\{\eta_0\}) \subset (\Phi\backslash\{\eta_0\}) \cup \{\eta_0, v + \alpha + \eta_0 - v\}$. So it intersects all the cosets of $\Phi\backslash\{\eta_0\}$. Then, (6.3) implies that $v - \alpha + v$ is a left stabilizer of $\Phi\backslash\{\eta_0\}$. ∎

7. THE CASE $1 < u < b - 1$.

In this section we finish the case $b \geq 3$.

Case 7.1. Suppose that $\beta \in \Gamma$ with order $b > 2$ such that $|\Phi| \equiv u (\mathrm{mod}\ b)$ for some $2 \leq u < b - 1$. Then there is a 0-tree.

Proof. Consider the decomposition

$$\Phi = C_0' \cup (\xi_1 + Z(\beta)) \cup \cdots \cup (\xi_s + Z(\beta)),$$

where $C_0' \subset (\xi_0 + Z(\beta))$, and $C_0' = \{\xi_0, \xi_0 + \beta, \ldots, \xi_0 + (u-1)\beta\}$. As $u > 1$ (and $b \geq 3$), for an arbitrary $\alpha \in \mathrm{QStab}_l(\Phi)$ we have that $(\alpha + \xi_i + Z(\beta)) \subset \bigcup_{i\geq 0}(\xi_i + Z(\beta))$. Hence in this case α is a left stabilizer of $\bigcup_{i\geq 0}(\xi_i + Z(\beta))$. Moreover, $u \leq b - 2$ implies that $\alpha + \xi_0 + Z(\beta) \neq \xi_i + Z(\beta)$ for some $i \geq 1$, hence $\alpha + \xi_0 + Z(\beta) = \xi_0 + Z(\beta)$. Equivalently, we have $-\xi_0 + \alpha + \xi_0 + Z(\beta) = Z(\beta)$, i.e., $-\xi_0 + \alpha + \xi_0 = g(\alpha)\beta$ for some integer $0 \leq g =: g(\alpha) < b$. Then $\alpha = \xi_0 + g(\alpha)\beta - \xi_0$. Consider $\alpha + C_0' = (\xi_0 + g(\alpha)\beta - \xi_0) + \{\xi_0 + i\beta : i = 0, \ldots, u-1\}$. We have

$$\alpha + C_0' = \xi_0 + \{i\beta : i = g, g+1, \ldots, g+u-1\}.$$

This implies that $g \in \{0, 1, -1\}$, i.e., $\mathrm{QStab}_l(\Phi) \subset \{0, \xi_0 + \beta - \xi_0, \xi_0 - \beta - \xi_0\}$. As $\beta \in \Gamma$, and $\xi_0 + \beta - \xi_0 = -(\xi_0 - \beta - \xi_0)$, we have that

$$\Gamma \subset \mathrm{QStab}_l(\Phi) \subset \{0, \beta, -\beta\}. \tag{7.2}$$

As $1 < u < b - 1$, we have

$$b \geq 4 \tag{7.3}$$

Proposition 7.4. $N(A) = \{\varphi_1, \varphi_2\}$.

Proof. Suppose that there are three distinct members φ_1, φ_2, and φ_3 of $N(A)$. Equation (7.2) implies that $\varphi_3 - \varphi_1 = -\beta$. Consider $0 \neq \varphi_2 - \varphi_3 \in \Gamma \subset \{0, \beta, -\beta\}$. The $\varphi_2 - \varphi_3 = \beta$ (with $\varphi_2 - \varphi_1 = \beta$) implies $\varphi_3 = \varphi_1$, a contradiction.

If $\varphi_2 - \varphi_3 = -\beta$, then (together with the previous two equations), we have

$$\beta = \varphi_2 - \varphi_1 = (\varphi_2 - \varphi_3) + (\varphi_3 - \varphi_1) = -2\beta.$$

This gives $b = 3$, contradicting (7.3). ∎

Finally, 7.4 and Case 4.3 finish the proof of 7.1. ∎

8. THE CASE OF ORDER 2

From now on we may suppose that $b = 2$, i.e., for every $\gamma \in \Gamma$ we have $2\gamma = 0$. This implies that

$$\varphi_i - \varphi_j = \varphi_j - \varphi_i \tag{8.1}$$

holds for all $\varphi_i, \varphi_j \in N(A)$.

Proposition 8.2. $\gamma_1 + \gamma_2 = \gamma_2 + \gamma_1$ for all $\gamma_1, \gamma_2 \in \Gamma$.

Proof. Let $\gamma_1 = \varphi_i - \varphi_j$ and $\gamma_2 = \varphi_k - \varphi_l$. Equation (8.1) implies that $-\varphi_j + \varphi_k = -\varphi_k + \varphi_j$. We have

$$\begin{aligned}\gamma_1 + \gamma_2 &= \varphi_i - \varphi_j + \varphi_k - \varphi_l = (\varphi_i - \varphi_k) + (\varphi_j - \varphi_l)\\ &= \varphi_k - \varphi_i + \varphi_l - \varphi_j = \varphi_k - \varphi_l + \varphi_i - \varphi_j = \gamma_2 + \gamma_1. \quad \blacksquare\end{aligned}$$

Recall that $\Phi = \{\xi_0\} \cup C_1 \cup \cdots \cup C_s, C_0 = \{\xi_0, \xi_0 + \beta\}$, and $\beta + \Phi = \Phi + \beta$. We claim that $\beta \in \mathrm{Stab}_l(C_0 \cup C_1 \cup \cdots \cup C_s)$. Indeed, $2\beta = 0$ implies that

$$\beta + (\Phi \cup (\Phi + \beta)) = (\beta + \Phi) \cup (\beta + \Phi + \beta) = (\Phi + \beta) \cup \Phi. \tag{8.3}$$

Similarly, $\beta + (\Phi\backslash\{\xi_0\})$ is a union of cosets of the form $\eta + Z(\beta)$, and $\beta + (\Phi\backslash\{\xi_0\}) \subset \beta + \Phi = \Phi + \beta = \{\xi_0 + \beta\} \cup C_1 \cup \cdots \cup C_s$. These imply that $\beta \in \mathrm{Stab}_l(\Phi\backslash\{\xi_0\})$,

$$\beta + (\Phi\backslash\{\xi_0\}) = \Phi\backslash\{\xi_0\}. \tag{8.4}$$

Proposition 8.5. $\gamma + (C_0 \cup C_1 \cup \ldots C_s) = (C_0 \cup C_1 \cup \ldots C_s) + \gamma$ for all $\gamma \in \Gamma$.

Proposition 8.6. $\gamma + (C_1 \cup \ldots, C_s) = (C_1 \cup \ldots C_s) + \gamma$ holds for all $\gamma \in \Gamma$.

Proofs. We have that

$$\gamma + \Phi = \{\gamma + \xi_0\} \cup (\gamma + \xi_1 + Z(\beta)) \cup \cdots \cup (\gamma + \xi_s + Z(\beta)).$$

Equation (8.2) implies that $Z(\beta) + \gamma = \gamma + Z(\beta)$; hence

$$\Phi + \gamma = \{\xi_0 + \gamma\} \cup (\xi_1 + \gamma + Z(\beta)) \cup \cdots \cup (\xi_s + \gamma + Z(\beta)).$$

As $\gamma + \Phi = \Phi + \gamma$, the above two equations yield $\gamma + \xi_0 = \xi_0 + \gamma$ for all $\gamma \in \Gamma$. Then

$$\begin{aligned} &\gamma + (C_0 \cup C_1 \cup \cdots \cup C_s) = \gamma + (\Phi \cup \{\xi_0 + \beta\}) \\ &= (\gamma + \Phi) \cup \{\gamma + \xi_0 + \beta\} = (\Phi + \gamma) \cup \{\xi_0 + \beta + \gamma\} \\ &= (C_0 \cup C_1 \cup \cdots \cup C_s) + \gamma. \end{aligned}$$

The proof of the second statement is similar. ∎

As 4.3 covers the case $|N(A)| \leq 2$, we may suppose the opposite. Hence $|\Gamma| > 2$,

$$\{0, \beta, \gamma\} \subset \Gamma$$

for some $0 \neq \gamma \neq \beta$. Let M be the subgroup generated by β and γ. By (8.1) and 8.2 we have that $M =: \{0, \beta, \gamma, \beta + \gamma\}$, $|M| = 4$.

Case 8.7. Suppose that $|\Gamma| > 2$. Then, there is a 0-tree.

Proof. Suppose, first, that $\gamma + C_0 \in \{C_0, C_1, \ldots, C_s\}$. Then, as γ a quasi-stabilizer of Φ, we obtain that $\gamma \in \mathrm{Stab}_l(C_0 \cup C_1 \cup \ldots \cup C_s)$. The same holds for β by (8.3). Then 8.5 implies that $\beta, \gamma \in \mathrm{Stab}_r(C_0 \cup C_1 \cup \cdots \cup C_s)$. Since $\Phi \cup (\Phi + \beta)$ is a union of cosets of the form $\eta + M$, one can use Case 5.1.

If $(\gamma + C_0) \cap (C_0 \cup C_1 \cup \cdots \cup C_s) = \varnothing$, then γ (and β, by (8.4)) are left stabilizers of $C_1 \cup \cdots \cup C_s$. Then 8.6 implies that $\beta, \gamma \in \mathrm{Stab}_r(C_1 \cup \cdots \cup C_s)$. Since $\Phi \setminus \{\xi_0\}$ is a union of cosets of the form $\eta + M$, one can use Case 6.1. ∎

This last case completes the proof of Theorem 1.1.

9. QUASI-STABILIZERS IN A GROUP

Actually, we proved the following statement, which holds for any finite Abelian group Z and subset Φ:

Proposition 9.1. Let QStab(Φ) be the set of quasistabilizers of Φ. Then one of the following cases holds:

(S) $|\mathrm{Stab}(\Phi)| > 1$, and then $\mathrm{QStab}(\Phi) = \mathrm{Stab}(\Phi)$.
(0) $\Phi = \{\alpha, \alpha + \beta, \ldots, \alpha + b'\beta\}$, where the order of $\beta \in Z$ is larger than $b' + 1 > 1$ and $\mathrm{QStab}(\Phi) = \{0, \beta, -\beta\}$.
(1) For some $\delta \in \Phi$ one has $\mathrm{QStab}(\Phi) = \mathrm{Stab}(\Phi \setminus \{\delta\})$.
(2) For some $\delta \in Z \setminus \Phi$ one has $\mathrm{QStab}(\Phi) = \mathrm{Stab}(\Phi \cup \{\delta\})$. ∎

Especially, $|\mathrm{QStab}(\Phi)| > 3$ implies that it is a subgroup.

It would be interesting to characterize the set of quasistabilizers for every group.

10. COMPLEXITY QUESTIONS

One can ask, How difficult is it to find a 0-tree given a weighting of $\mathbf{K}_{m+1}$?

The construction given makes use of a maximum cardinality set such that the number of weights of spanning trees in it has cardinality at least one less than that of the set. The same construction can be applied to a maximal set, or even to an "apparently maximal set," namely, one of cardinality $k + 1$ for which there is a set of k tree weight sums (not necessarily all) that cannot be extended by adding more vertices. This gives a polynomial algorithm for constructing such a tree, by extending such a set and weight set or using the argument of the theorem here to produce a 0-tree if the set is unextendable.

Theorem 10.1. There is an algorithm to find a 0-tree with running time of $O(|\mathscr{E}|)$.

Proof. Suppose we already know a $k + 1$ element set $A \subset V$ and spanning trees $\{\mathbf{T}_1, \ldots, \mathbf{T}_k\}$ over A with distinct weights, i.e., $\Phi_0 = \{\varphi(\mathbf{T}_i) : 1 \leq i \leq k\}$, $|\Phi_0| = k$. (In the beginning of the algorithm $k = 1$.)

If we can find an $x \in V \backslash A$ and $a, b \in A$ with $\varphi(ax) \neq \varphi(bx)$, then either there are at least $k + 1$ distinct values among $\{\varphi(\mathbf{T}_i \cup \{a, x\})\} \cup \{\varphi(\mathbf{T}_i \cup \{b, x\})\}$ and then we can have $k + 1$ tree values over $A \cup \{x\}$, or Fact 2.2 holds for Φ_0.

Similarly, either there exist $a \in A$ and $x, y \in V \backslash A$ such that $|\Phi_0 + \varphi(ax) + \varphi(xy)) \cup (\Phi_0 + \varphi(bx) + \varphi(xy))| \geq k + 2$, and then we can extend A into $A \cup \{x, y\}$, or Fact 2.3 holds, too, for Φ_0.

The rest of the proof is just to check the proof of Theorem 1.1, that all the constructions given by 4.1–4.3 and finally by Lemma 3.1 can be also constructed effectivelly using the $\mathbf{T}_i$'s on A.

To achieve linear running time for $Z = Z_m$ requires a slightly careful database handling. ∎

Whether finding such a tree is in NC an open question.

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