THE MINIMUM SIZE OF A MAXIMAL PARTIAL PLANE

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Abstract. A partial plane of order n is a family $\mathcal{L}$ of $n+1$-element subsets of an n^2+n+1-element set, such that no two sets meet more than 1 element. Here it is proved, that if $\mathcal{L}$ is maximal, then $|\mathcal{L}| \geq \lfloor 3n/2 \rfloor + 2$, and this inequality is sharp.

1. Examples for maximal partial planes

Let n be a positive integer, P a set of n^2+n+1 elements. It will be convenient to set $P = \{1, 2, \ldots, n^2+n+1\}$. A family $\mathcal{L}$ of $(n+1)$-element subsets of P is called a *partial plane of order n* if

$$|L \cap L'| \leq 1$$

holds for every pair $L, L' \in \mathcal{L}$. (By another terminology, $(P, \mathcal{L})$ is a $(n^2+n+1, n+1, 2)$-packing, and $\mathcal{L}$ is a *nearly-disjoint* family.) $\mathcal{L}$ is *maximal* if there is no other partial plane containing it. Let $f(n)$ denote the minimum number of sets in a maximal partial plane.

Let the lines $A_0, A_1, \ldots, A_n$ form a spread with center $\{n^2+n+1\}$ (e.g., $A_i := \{in+1, in+2, \ldots, in+n\} \cup \{n^2+n+1\}$ for $0 \leq i \leq n$), and $B_1, \ldots, B_n$ an orthogonal equipartition of $P \setminus \{n^2+n+1\}$, (e.g., $B_i = \{i, i+n, \ldots, i+n^2\}$). Then $\{A_0, \ldots, A_n, B_1, \ldots, B_n\}$ is a maximal partial plane. Considering this example Mullin [M] conjectured that $f(n) = 2n+1$. It is easy to check that $f(1) = 3$ and $f(2) = 5$. Mullin had several more maximal partial planes of size $2n+1$ as well. However, the conjecture fails to be true for $n \geq 3$, we have

THEOREM 1.1. $f(n) = \lfloor 3n/2 \rfloor + 2$.

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*Supported in part by the Hungarian National Science Foundation grant No. 1812
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Example for n odd. Let $P = P_0 \cup P_1 \cup \cdots \cup P_{(n-1)/2}$ be a partition, where $|P_0| = \frac{1}{2}(n+1)(n+2)$ and $|P_1| = \cdots = |P_{(n-1)/2}| = n$. Let $L_1, \ldots, L_{n+2}$ be a system of $(n+1)$-element sets over P_0 such that every pairwise intersection is nonempty, and every element of P_0 is contained in exactly two of these sets. Moreover, let $L'_i = P_i \cup \{p_i\}$, where $p_i \in P_0$ is choosen arbitrarily, $1 \leq i \leq (n-1)/2$. Then, $\mathcal{L} := \{L_1, \ldots, L_{n+2}\} \cup \{L'_1, \ldots, L'_{(n-1)/2}\}$ is a maximal partial plane. Indeed, if $|C \cap L| \leq 1$ for all $L \in \mathcal{L}$ for some $(n+1)$-set C, then

$$n + 2 \geq \sum_{i=1}^{n+2} |C \cap L_i| = 2|C \cap P_0| \tag{1.1}$$

implies that $|C \cap P_0| \leq \lfloor (n+2)/2 \rfloor = (n+1)/2$. Hence $|C \cap P| = \sum_{i=0,\ldots,(n-1)/2} |C \cap P_i| \leq n$.

Example for n even. Let again $P = P_0 \cup P_1 \cup \cdots \cup P_{(n-2)/2}$, where $|P_0| = \frac{1}{2}(n+1)(n+3) - \frac{1}{2}$, $|P_1| = \cdots = |P_{(n-2)/2}| = n$. There exists a nearly-disjoint system of $(n+1)$-element sets $L_1, \ldots, L_{n+3} \subset P_0$, such that every element of P_0 is covered twice or 3 times. To see this, label the elements of P_0 by sets of size 2 and 3 as follows: $P_0 = \{p(B) : B \in \mathcal{B}\}$, where $\mathcal{B} = \{\{1,2,3\}\} \cup \{\{i,j\} : 1 \leq i < j \leq n+3,\ \{i,j\} \neq \{4,5\}, \{6,7\}, \ldots, \{n+2, n+3\}\}$. We get $L_i = \{p(B) : i \in B\}$ for $1 \leq i \leq n+3$.

Moreover, let $L'_i = P_i \cup \{p_i\}$, where $p_i \in P_0$, $1 \leq i \leq (n-2)/2$. Then $\{L_1, \ldots, L_{n+3}, L'_1, \ldots, L'_{(n-2)/2}\}$ is a maximal partial plane. To prove the maximality we use (1.1) but the left hand side is replaced by $n+3$, and the equality sign $=$ by a greater-or-equal sign $\geq$.

2. The lower bound is sharp

In the proof of Theorem 1.1 we will use the following result of Seymour [S]: If $\mathcal{D}$ is a nearly-disjoint family over the underlying set Y, then it contains at least $|\mathcal{D}|/|Y|$ pairwise disjoint members. (This theorem is a special case of the Erdős-Faber-Lovász conjecture [E].) As the *dual* of a nearly-disjoint family is again nearly-disjoint, Seymour's theorem gives that there is a set $I \subset Y$ such that $|I \cap D| \leq 1$ for all $D \in \mathcal{D}$ and

$$|I| \geq |Y|/|\mathcal{D}|. \tag{2.1}$$

Proof of 1.1. The upper bound on $f(n)$ was given in the previous section. Now suppose that $\mathcal{L}$ is a maximal family over P with $|\mathcal{L}| = f(n)$. First we show, that one can suppose that

$$\cup\mathcal{L} = P. \tag{2.2}$$

If the point $p \in P$ is uncovered, and $q \in P$ is contained in at least two lines $L, L' \in \mathcal{L}$, $q \in L \cap L'$, then $\mathcal{L}' := \mathcal{L} \setminus \{L\} \cup \{L \setminus \{q\} \cup \{p\}\}$ is also a maximal partial plane. Indeed, if $\mathcal{L}' \cup \{A\}$ is partial plane for some $A \subset P$, $|A| = n+1$, then $\mathcal{L}$ also can be extended by either A or by $A \setminus \{q\} \cup \{p\}$. Repeating this operation, we either obtain an $\mathcal{L}^*$ consisting of pairwise disjoint sets, a contradiction to its maximality, or an $\mathcal{L}^*$ covering the whole P, proving (2.2).

Denote by $L_1, \ldots, L_b \in \mathcal{L}$ the lines having a point of degree one, i.e. for $1 \leq i \leq b$ one has $p_i \in L_i$ such that $p_i \notin L$ for all $L \in \mathcal{L} \setminus \{L_i\}$. The set $\{p_1, \ldots, p_b\}$ intersects every $L \in \mathcal{L}$ in at most one element, hence $b \leq n$. Let $C := P \setminus \cup\{L_i : 1 \leq i \leq b\}$. We have that $|C| \geq |P| - (n+1)b > 0$.

Considering the valencies of the points of P we obtain that

$$(n+1)|\mathcal{L}| \geq |P| + |C| \geq 2(n^2+n+1) - (n+1)b.$$

This implies that

$$|\mathcal{L}| \geq 2n + 1 - b. \tag{2.3}$$

Apply (2.1) to the restriction of $\mathcal{L}$ into C. We get the points $q_1, \ldots, q_c \in C$ such that no pair $q_i q_j$ is contained in any $L \in \mathcal{L}$, and $c \geq |C|/(|\mathcal{L}| - b)$. Then $\{p_1, \ldots, p_b, q_1, \ldots, q_c\}$ is nearly-disjoint to $\mathcal{L}$, so

$$n \geq b + c \geq b + (n^2 + n + 1 - (n+1)b)/(|\mathcal{L}| - b).$$

Rearranging we have $(n-b)(|\mathcal{L}| - n - 1 - b) \geq 1$, implying

$$|\mathcal{L}| \geq n + 2 + b. \tag{2.4}$$

Finally, the sum of (2.3) and (2.4) gives $2|\mathcal{L}| \geq 3n + 3$, finishing the proof.

3. A remark on the lotto problem

The above discussed question is related to the following, so-called lotto problem (see, e.g., [BV]). For $v \geq k \geq t$, let $l(v, k, t)$ denote the smallest cardinality of a family $\mathcal{F}$ of k-subsets of the v-element underlying set V such that $K \subset V$, $|K| = k$ implies that $|F \cap K| \geq t$ for some $F \in \mathcal{F}$. It is easy to see, that $l(n^2+n+1, n+1, 2) = n+2$, in contrast with Theorem 1.1.

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