Superimposed Codes are Almost Big Distance Ones

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Abstract — **Let $T(r,n)$ denote the maximum number of subsets of an n-set satisfying the condition that no set is covered by the union of r others, while let $T^\star(\epsilon,n)$ be the maximum size of a $<\epsilon$ part intersecting family, i.e. of a family where the size of the intersection of any two sets is $<$ then the ϵ^{th} part of the smaller one. In this this paper, partially answering to a question posed by Hwang and Sós, we prove, that superimposed codes (r-cover-free families) are big distance ones (almost $< 1/r$ part intersecting families). More precisely, our theorem says that the rate of an r-cover-free family is $\leq$ then the rate of a $\log\log r/r$ part intersecting family, i. e. for $n > n_o(r)$ $\log T(r,n)/n \leq \log T^\star(\log\log r/r, n)/n$.**

I. Summary

The notion of the r-cover-free families was introduced by Kautz and Singleton in 1964 [9]. They initiated investigating binary codes with the property that the disjunction of any $\leq r$ ($r \geq 2$) codewords are distinct (UD_r codes). This led them to studying the binary codes with the property that none of the codewords is covered by the disjunction of $\leq r$ others (*Superimposed codes, ZFD_r codes;* P. Erdős, P. Frankl and Z. Füredi called the correspondig set system r-cover-free in [3]).

Since that many results have been proved about the maximum size of these codes. Various authors studied these problems basically from three different points of view, and these three lines of investigations were almost independent of each other. This is why many results were found first in information theory ([1], [7], [8], [9]), were later rediscovered in combinatorics ([2], [3], [11]), or in group testing ([5], [6]), and vice versa.

Let S be an n-element set. 2^S is the set of all subsets of S, and $\log x$ is always of base 2. $\mathcal{F} \subset 2^S$ is r-cover-free, if

$$A_0 \not\subseteq A_1 \cup A_2 \cup \ldots \cup A_r$$

holds for all distinct $A_0, A_1, \ldots, A_r \in \mathcal{F}$. $\mathcal{F}^\star \subset 2^S$ is $<\epsilon$ part intersecting, if

$$|A_i \cap A_j| < \epsilon \min\{|A_i|, |A_j|\}$$

for any distinct $A_i, A_j \in \mathcal{F}^\star$ holds. We denote by $T(r,n)$, and $T^\star(\epsilon,n)$ the maximum cardinality of the corresponding set systems, resp. Obviously, if a family is $< 1/r$ part intersecting then it is r-cover-free. Hence $T^\star(1/r,n) \leq T(r,n)$. This fact implied that it is easy to generate superimposed codes from big distance ones, see e.g. the paper by A, Györfi and Massey [4]. On the other hand nothing was known about what can be the gap between those two functions. This was posed as an open problem by Hwang and Sós. If the gap is only a constant in the exponent, this would mean that superimposed codes are essentially big distance ones, which would explain why the magnitude of exponent (of $T(r,n)$) is still unsolved (more then 30 years open problem!), since in this case this problem is equivalent to the problem of the correct magnitude of the exponent of the binary big distance codes, which is one of the major open problems in coding theory. On the other hand it is hard to handle the r-cover-free families, since one have to consider $r+1$ relations instead of binary ones. J. Körner [10] pointed out that this is the main reason why his graph entropy method can not give strong enough bounds for the r-cover-free families. In this paper we significantly decrease this gap showing that r-cover-free families are almost (up to a factor of $\log\log r$) $< 1/r$ part intersecting ones. Our main theorem is the following.

Theorem 1 *For for $n > n_o(r)$*

$$\frac{\log T(r,n)}{n} \leq \frac{\log T^\star(\log\log r/r, n)}{n}.$$

The proof is based on a (modified) set compression algorithm described by the second author in [12].

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