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Graphs of Diameter 3 with the Minimum Number of Edges

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Abstract. The graph **G** is called a porcupine, if $\mathbf{G}|A$ is a complete graph for some set A, every other vertex has degree one, and its only edge is joined to A. In this paper a conjecture of Bollobás is settled almost completely. Namely, it is proved that if **G** is a graph on n vertices of diameter 3 with maximum degree D, $D > 2.31\sqrt{n}$, $D \neq (n-1)/2$ and it has the mimimum number of edges, then it is a porcupine.

1. Results and a Conjecture

Let d, D and n be positive integers, $d, D < n$. Denote by $\mathscr{H}_d(n, D)$ the set of all (simple) graphs of n vertices with diameter at most d, and maximal degree at most D. Put

$$e_d(n, D) = \min\{|E(G)|: G \in \mathscr{H}_d(n, D)\},$$

i.e. the minimum number of edges. Also, denote by $\mathscr{E}_d(n, D)$ the set of extremal graphs,

$$\mathscr{E}_d(n, D) = \{\mathbf{G} \in \mathscr{H}_d(n, D): |E(\mathbf{G})| = e_d(n, D)\}.$$

The study of the function $e_d(n, D)$ was initiated by Erdös and Rényi, and an excellent survey can be found in the 4th chapter of Bollobás' book [2]. In this note we deal with the case $d = 3$.

Define the class of graphs $\mathscr{P}(n, D, a)$ as follows, for $D \geq a \geq 1$. A graph $\mathbf{G} \in \mathscr{P}(n, D, a)$ if its maximum degree is at most D, and there exists a set $A \subset V(\mathbf{G})$, $|A| = a$, $|V(\mathbf{G})| = n$ with the following properties. The induced subgraph $\mathbf{G}|A$ is complete, and every vertex in $V(\mathbf{G})\backslash A$ has degree exactly one, and each of them is joined to some vertex of A. Let $\mathscr{P}(n, D) = \bigcup_a \mathscr{P}(n, D, a)$, $\mathscr{P}(n) = \bigcup \mathscr{P}(n, D)$. Sometimes we call these graphs *porcupine*. Obviously, every member of $\mathscr{P}(n)$ has diameter (at most) three, and for all graphs $\mathbf{G} \in \mathscr{P}(n, D, a)$ one has

$$|E(\mathbf{G})| = n - 1 + \binom{a-1}{2}. \tag{1.1}$$

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Moreover, if $\mathscr{P}(n, D, a) \neq \emptyset$, then $D \geq a$, and considering the degrees at A we have

$$n + a^2 \leq a(D + 2). \tag{1.2}$$

Theorem 1.3. *Let n, D be positive integers, $n > D \geq (4/\sqrt{3})\sqrt{n} - 2$, $(4/\sqrt{3} = 2.30\ldots)$. Let a be the minimum integer satisfying* (1.2). *Suppose that* $\mathbf{G}$ *is a graph on n vertices of diameter at most* 3, *maximum degree at most D. Then $|E(\mathbf{G})| \geq n - 1 + \binom{a-1}{2}$. Moreover, for $D \neq n - 1$, $D \neq (n-1)/2$, here equality holds only if $\mathbf{G} \in \mathscr{P}(n, D, a)$.*

(If $D \in \{n - 1, (n - 1)/2\}$, then there is one more extremal graph, see later (2.1) and (2.2).) Theorem 1.3 was proved for $D > (2n)^{2/3}$ by Erdös, Rényi and T. Sós [3] (also see in [2], p. 181.). Bollobás ([2], Problem 5.10, page 213.) raised the question whether the statement of the Theorem 1.3 is true for all D whenever $\mathscr{P}(n, D, a) \neq \emptyset$, (i.e. for $D > 2\sqrt{n}$). The theorem is not true for (much) smaller D's, as Bollobás [2] proved for $D = \lfloor c\sqrt{n} \rfloor$

$$(2/c^2)n\left(1 - \frac{1}{n^{1/7}}\right) < e_3(n, c\sqrt{n}) < (7/c^2)n,$$

where $0 < c < 0.1$ is fixed, and $n > n_0(c)$.

A similar statement seems to be true for $e_d(n, D)$ if d is an odd integer. To state it define $\mathscr{P}^d(n, D, a)$ as the class of graphs $\mathbf{G}$ on n vertices with maximum degree D such that there exists a set $A \subset V(\mathbf{G})$, $|A| = a$, $\mathbf{G}|A$ is a complete subgraph, and removing the edges of A from $E(\mathbf{G})$ one obtains trees, every tree T has a unique common point with A, and the distance of each vertex of T from A is not more than $(d - 1)/2$. Let $\mathscr{P}^d(n, D) = \bigcup_a \mathscr{P}^d(n, D, a)$.

Conjecture 1.4. Suppose that $\mathscr{P}^d(n, D) \neq \emptyset$ and let a be minimal integer such that $\mathscr{P}^d(n, D, a) \neq \emptyset$ (i.e. $D \geq n_d = (1 + o(1))(4n)^{2/(d+1)}$). Then $e_d(n, D) = n - 1 + \binom{a-1}{2}$. Moreover, $\mathscr{E}_d(n, D) = \mathscr{P}^d(n, D, a)$.

This conjecture remains open even in the case $d = 3$ whenever $2\sqrt{n} < D < 2.30\ldots\sqrt{n}$.

2. Proof of Theorem 1.3

It is easy to prove the following two statements.

$$e_3(n, D) = n - 1 \tag{2.1}$$

if and only if $D \geq n/2$, and the only extremal graphs are from $\mathscr{P}(n, D, \leq 2)$.

$$e_3(n, D) = n \tag{2.2}$$

holds for $n/2 > D \geq (n + 2)/3$ if $n \geq 8$, and for $n/2 > D \geq 2$ if $n = 5, 6, 7$. The only extremal graphs are from $\mathscr{P}(n, D, 3) \cup \{P_{2D+1}\}$, and in the case $5 \leq n \leq 7$ we have $\mathscr{E}_3(n, D) = \mathscr{P}(n, D, 3) \cup \{C_n, P_{2D+1}\}$, where P_{2D+1} is a graph on $2D + 1$ vertices obtained from a pentagon having two neighbours joined to $D - 2$ new points each.

Suppose that $\mathbf{G} \in \mathscr{E}_3(n, D)$, where

$$D \geq \frac{4}{\sqrt{3}}\sqrt{n} - 2. \tag{2.3}$$

As (1.1) shows, we may suppose that

$$|E(\mathbf{G})| \leq n - 1 + \binom{a-1}{2}, \tag{2.4}$$

where a is defined by (1.2). Our aim is to prove that $\mathbf{G} \in \mathscr{P}(n, D, a)$ (whenever $D \neq n - 1, D \neq (n-1)/2$).

The case $D \geq n/2$ is covered by (2.1). So we may suppose that $D \leq (n-1)/2$. Then (2.3) implies that $n \geq 15$. Theorem 1.3 obviously holds for $(n-1)/2 \geq n \geq (n+2)/3$ for $n \geq 15$ by (2.2). So from now on we may suppose that

$$D \leq (n+2)/3. \tag{2.5}$$

This and (2.3) imply that

$$n \geq 30. \tag{2.6}$$

Since a is the smallest integer satisfying (1.2), one has that (2.3) implies

$$a = \left\lceil \frac{1}{2}\left(D + 2 - \sqrt{(D+2)^2 - 4n}\right) \right\rceil \leq \left\lceil \frac{\sqrt{n}}{\sqrt{3}} \right\rceil. \tag{2.7}$$

Claim 2.8. *There are vertices of degree* 1.

Proof. Let $m = \min\{\deg_{\mathbf{G}}(p)\colon p \in V(\mathbf{G})\}$, $\deg(p) = m$ for some $p \in V(\mathbf{G})$. Suppose on the contrary that $m \geq 2$. Then

$$|E(\mathbf{G})| \geq \frac{3}{2}(n-1) - D. \tag{2.9}$$

Indeed, in the case $m \geq 3$ we obtain immediatey that $|E(G)| \geq \frac{3}{2}n$. In the case $m = 2$, let N_i (or $N_i(p, \mathbf{G})$) denote the set of points of $\mathbf{G}$ whose distance from p is exactly i. Let T be a spanning subtree of $\mathbf{G}$ such that $N_i(p, \mathbf{G}) = N_i(p, T)$. As $|N_0 \cup N_1 \cup N_2| \leq 1 + 2 + 2(D-1)$ and $N_0 \cup N_1 \cup N_2 \cup N_3 = V(\mathbf{G})$ we obtain that T has at least $n - 1 - 2D$ leaves. All of them have degree at least two in $\mathbf{G}$, hence

$$|E(\mathbf{G})| \geq |E(T)| + \frac{1}{2}(n - 1 - 2D) = \frac{3}{2}(n-1) - D,$$

proving (2.9).

The right hand side of (2.9) is at least $(7n - 13)/6$ by (2.5). This contradicts (2.4) and (2.7) for $n \geq 17$. □

Define the following partition of $V(\mathbf{G}) = X \cup Y \cup Z$. Let X denote the set of vertices having a neighbour of degree 1, let Y be the set of neighbours of X $(Y = N(X)\backslash X)$, and let Z be the rest of the points, $Z = V(\mathbf{G})\backslash(X \cup Y)$. We use the notations $|X| = x$, $|Y| = y$, $|Z| = z$. Observe, that $\mathbf{G}|X$ is a complete subgraph.

$Z = \varnothing$ implies that $\mathbf{G}$ contains a porcupine $P \in \mathscr{P}(n, x)$ as a subgraph. The minimality of $|E(\mathbf{G})|$ implies that actually $P = \mathbf{G}$, and we are done. So from now on we may suppose that $Z \neq \varnothing$.

Claim 2.10. $|E(\mathbf{G})| \geq n - 1 + \lfloor z/2 \rfloor + \binom{x}{2}$.

Proof. Let $\tau = \min\{|N(p) \cap Y| : p \in Z\}$, and suppose that this minimum is taken at the vertex $p \in Z$. Every point of X can be reached in two steps from Z via Y, so $N(p) \cap Y$ has at least x edges to X. Hence the number of edges between X and Y is at least $x + y - \tau$. We have additional τz edges from Z to Y, and $\binom{x}{2}$ edges in X. Altogether

$$|E(\mathbf{G})| \geq x + y - \tau + \tau z + \binom{x}{2} = n - 1 + (\tau - 1)(z - 1) + \binom{x}{2}. \qquad (2.11)$$

Here the middle term is at least $\lfloor z/2 \rfloor$ for $\tau \geq 2$ (as $z \geq 1$).

In the case $\tau = 1$ we proceed as in the argument proving Claim 2.8. There are at least z edges from Z to Y, but as every degree in Z is at least 2, we have that the total number of edges adjacent to Z is at least $\frac{3}{2}z$. This gives a $\frac{3}{2}z$ term in (2.11) instead of τz proving the Claim 2.10. □

Finally, (2.4) and 2.10 give that $\binom{a-1}{2} \geq \lfloor z/2 \rfloor + \binom{x}{2}$, implying

$$x \leq a - 1, \qquad (2.12)$$

and

$$(a - x - 1)(a + x - 2) \geq 2\lfloor z/2 \rfloor. \qquad (2.13)$$

On the other hand, recall that by the minimal choice of a the inequality (1.2) does not hold if we replace a by a $- 1$. Hence

$$x + y + z + (a - 1)^2 = n + (a - 1)^2 > (a - 1)(D + 2). \qquad (2.14)$$

Considering the degrees at the points of X and the number of incoming edges from Y we have

$$Dx \geq \sum_{p \in X} \deg(p) \geq y + x^2 - x. \qquad (2.15)$$

Rearranging the sum of (2.14) and (2.15) we have that

$$z > (a - x - 1)(D + 3 - a - x). \qquad (2.16)$$

Then (2.13) and (2.16) imply that

$$2a + 2x \geq D + 5. \qquad (2.17)$$

However, (2.7) gives that $a \leq \lceil \sqrt{n}/\sqrt{3} \rceil \leq (D + 5)/4$, which together (2.12) imply $2a + 2x \leq D + 3$. This contradicts (2.17), completing the proof of Theorem 1.3.

References

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