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MAXIMAL INDEPENDENT SUBSETS IN STEINER SYSTEMS AND IN PLANAR SETS*

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Abstract. A set of points is independent if there are no three on a line. It is proved that $\Omega(\sqrt{n \log n}) < \alpha(n) < o(n)$, where $\alpha(n)$ denotes the maximum α such that every planar set of n points with no four on a line contains an independent subset of size α.

Key words. planar subsets, Steiner systems, linear hypergraphs, polarized set mappings

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1. Independent planar sets. A set of points on the Euclidean plane is called *independent* if there are no three on a line among them. Denote the size of the largest independent subset of the set S by $\alpha(S)$, and let $\alpha(n) = \min\{\alpha(S) : |S| = n$ and S does not contain more than three points on a line$\}$. These sets S are briefly called 3-*independent*.

Erdös [E] proposed in several places the problem to determine or to give bounds for $\alpha(n)$. An old result of Erdös and Hajnal [EH] (or in other words, the greedy algorithm) implies that $\alpha(S) \geqq \lfloor \sqrt{2n} \rfloor$ for every n-element 3-independent set S. Obviously, the sequence $\alpha(n)/n$ is monotone decreasing, lim $\alpha(n)/n$ exists. Erdös remarked that every known construction S contains at least $|S|/3$ independent points. The aim of this note is to improve both estimates using deep combinatorial theorems.

THEOREM 1.1. *There is a positive constant c such that* $c\sqrt{n \log n} < \alpha(n)$ *holds for all n. On the other hand* $\lim \alpha(n)/n = 0$ *whenever n tends to infinity.*

1.1. A construction giving $\alpha(n) = o(n)$. Let $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_t$ be distinct unit vectors linearly independent over the rationals with the property that if the directions of two integer linear combinations are the same, then the two vectors are essentially the same, i.e.,

$$\text{(P)} \qquad \sum a_i\mathbf{v}_i = \gamma\left(\sum b_i\mathbf{v}_i\right) \quad \text{and} \quad a_i, b_i \in \mathbf{N} \text{ implies } \gamma \in \mathbf{Q}.$$

Note that the linear independence of the system $\{\mathbf{v}_i\}$ implies that $a_i = \gamma b_i$. Define S^t as the set of 3^t integer linear combinations of $\{\mathbf{v}_i\}_{i=1}^t$ with coefficients 0, 1, or 2. Property (P) implies that there are no four points of S^t on a line, and the density version of the Hales–Jewett theorem, recently proved by Fürstenberg and Katznelson [FK], implies that $\alpha(S^t) = o(3^t)$ whenever t tends to infinity.

For completeness we recall the theorem applied above. For every positive ε there exists a $t(\varepsilon)$ such that if $I \subset \{0, 1, 2\}^T$, $|T| = t > t(\varepsilon)$, and $|I| > \varepsilon 3^t$, then we can find three sequences $s_i \in I$ $(i = 0, 1, 2)$ and a partition of the coordinate set T, $T = C \cup V$ such that $(s_i)_v = i$ in all coordinates $v \in V$, and $(s_0)_c = (s_1)_c = (s_2)_c$ holds for $c \in C$.

2. Independent sets in Steiner triple systems. A *hypergraph*, $\mathbf{H}$, is a pair $(V, \mathscr{E})$, where $\mathscr{E}$ is a family of subsets of V. The elements of V are called *vertices*, the $E \in \mathscr{E}$ are called hyperedges. A hypergraph is called *linear* or (almost disjoint) if $|E \cap E'| \leqq 1$ holds for all distinct $E, E' \in \mathscr{E}$. A *cycle* of length k is a sequence of distinct vertices and edges $x_1, E_1, x_2, E_2, \cdots x_k, E_k$ $(x_i \in V, E_i \in \mathscr{E})$ such that $\{x_i, x_{i+1}\} \subset E_i$ $(x_{k+1} = x_1$, by definition$)$. $(V, \mathscr{E})$ has *girth* at least g if it has no cycles of length $2, 3, \cdots, g-1$.

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Namely, the girth is at least 3 if and only if the hypergraph is linear. For the $x \in V$ we set $\mathscr{E}[x] = \{E : x \in E \in \mathscr{E}\}$. The *degree*, deg $(\mathbf{H}, x)$ or briefly deg (x), is the cardinality of $\mathscr{E}[x]$. The average degree $\bar{d}$ is the mean value of the degrees, i.e.,

$$\bar{d} = \frac{1}{|V|} \sum_{x \in V} \deg(x).$$

A set $I \subset V$ is *independent* if I contains no hyperedges $E \in \mathscr{E}$. $\alpha(\mathbf{H})$ denotes the maximum cardinality of an independent set. The *restriction* of $(V, \mathscr{E})$ to a subset W, denoted by $\mathbf{H}|W$, is given by $(V \cap W, \mathscr{E}|W)$, where $\mathscr{E}|W = \{E \in \mathscr{E} : E \subset W\}$.

A linear hypergraph $\mathbf{S}$ having 3-element hyperedges is called a partial Steiner triple system. $\mathbf{S}$ is a Steiner system, $S(|V|, 3, 2)$, if every pair of its vertices is contained in a (unique) three-tuple. The following result is due to Komlós, Pintz, and Szemerédi [KPSz]. There exists a positive constant c_3 such that if $\mathbf{S}$ is a partial Steiner family of girth at least 5 over n vertices, with average degree $\bar{d} \leqq d < n^{0.2} (d > c_4)$, then

$$\alpha(S) > c_3 n \sqrt{\frac{\log d}{d}}. \tag{2.1}$$

As a corollary of (2.1) Phelps and Rödl [PR] obtained a solution of a problem of Erdös and Hajnal ([EH66], see also, e.g., [E69, Prob. 19]), the true order of magnitude of the size of an independent set in a partial Steiner triple system. Namely, they proved that for every partial Steiner triple system $S = (V, \mathscr{E})$ with $|V| = n$ there exists a subset $I \subset V$ of size

$$|I| \geqq c_5 \sqrt{n \log n} \tag{2.2}$$

containing no edges from $\mathscr{E}$. Here c_5 is an absolute constant not depending on n. The proof of (2.2) utilizes the probabilistic method.

2.1. Large independent subsets of a planar set. Here we prove the lower bound in (1.1). Suppose that S is 3-independent planar set. Define $\mathscr{S}$ as the family of triples of S whose 3 points lie on a line. Obviously, $|A \cap B| \leqq 1$ holds for all distinct $A, B \in \mathscr{S}$, i.e., $\mathscr{S}$ is a partial Stiener triple system. Theorem 2.2 says that every partial Steiner system $\mathscr{S}$ on S contains a set $I \subset S$, $|I| > c_5 \sqrt{n \log n}$ such that $A \not\subset I$ for all $A \in \mathscr{S}$. Then the points of I are independent.

2.2. A remark on polarized set mappings. We can think that the above application of the Komlós, Pintz, Szemerédi theorem also answers the following question of Erdös and Hajnal ([EH58], see also [E69 Prob. 20]). Let V be the set of the first n positive integers. Let f be a function from the pairs of V to V such that $f(P) \notin P$. A set $I \subset V$ is said to be independent if for any $i \in I, j \in I, f(i, j) \notin I$. Denote by $g(n)$ the minimum of the largest independent set where the minimum is taken over all functions $f(i, j)$. Erdös and Hajnal proved that

$$c_6 n^{1/3} < g(n) < c_7 \sqrt{n \log n}.$$

Here the lower bound is obtained by the greedy algorithm, and the upper bound is the mean value of $g(f)$ whenever $f(i, j)$ is chosen independently and uniformly from $V \setminus \{i, j\}$.

Although the system of triples $\{(i, j, f(i, j)) : i, j \in V\}$ looks very much alike a Steiner family, the next theorem shows that the true order of the magnitude of $g(n)$ is not $\sqrt{n \log n}$. Hence this yields another example for the necessity of the constraint of the large girth in (2.1).

THEOREM 2.3.

$$\frac{2\sqrt{3}}{9}\sqrt{n}<g(n)<2\sqrt{n}.$$

Proof. The lower bound is a special case of a theorem of Spencer [S], which says that there exists an absolute constant c_8 such that every 3-hypergraph **H** over n vertices with average degree d contains an independent set of size at least

$$\alpha(\mathbf{H})>c_8\frac{n}{\sqrt{d}}. \tag{2.4}$$

This inequality is a weaker but more general version of (2.1).

The upper bound is given by the following example f. Let $V_1 \cup \cdots \cup V_a$ be a partition of V, where $a = \lceil \sqrt{n} \rceil$, and $V_i = \{(i, 1), (i, 2), \cdots, (i, b_i)\}$, where $\lfloor \sqrt{n} \rfloor = b_a \geqq b_{a-1} \geqq \cdots \geqq b_1$. For $i<j$, $x \neq y$, $y \leqq b_j$ let

$$f((i,x),(j,y))=(j,x),$$

and otherwise define f arbitrarily. Let I be an f-independent set and let us denote the projection of $V_i \cap I$ by I_i, i.e., $I_i = \{x : (i, x) \in I\}$. Then $I_i \cap I_j \neq \emptyset$, $|I_j| > 1$ is impossible, implying $|I| \leqq \lceil \sqrt{n} \rceil + \lfloor \sqrt{n} \rfloor - 1$. □

3. Problems. More generally, for $2 \leqq i < k$ let $\alpha_k^{(i)}(n) = \min \{\alpha^{(i)}(S) : |S| = n, |S \cap l| \leqq k$ for all lines $l\}$, where $\alpha^{(i)}(S)$ denotes the size of the largest i-independent subset. Define $\beta_k^{(i)}(n)$ as the smallest integer s such that every partial Steiner k-family over n elements has an i-independent set of size s. Clearly $\beta \leqq \alpha$.

The above proof, with a simple generalization of Spencer's theorem [S] (i.e., for every k-hypergraph has an i-independent set of size at least $O(n/\sqrt[i]{d})$) and the general Fürstenberg's theorem, give that for all fixed k we have

$$\beta_k^{(2)}(n) \leqq \beta_k^{(3)}(n) \leqq \cdots \beta_k^{(k-1)}(n) \leqq a_k^{(k-1)}(n)=o(n),$$

$$\Omega(n^{(i-1)/i})<\beta_k^{(i)}(n).$$

$$\Omega(\sqrt{n\log n})<\beta_k^{(2)}(n).$$

A theorem of Ajtai, Komlós, Pintz, Spencer, and Szemerédi [AKPSSz] implies that the second inequality is not sharp, i.e., $\lim \beta_k^{(i)}(n)n^{-(i-1)/i} = \infty$, and probably their methods give that

$$\Omega(n^{(i-1)/i}(\log n)^{1/i}) \leqq \beta_k^{(i)}(n). \qquad (?)$$

We can conjecture that the true order of magnitude of $\alpha(n)$ is much closer to the upper bounds because it is very difficult to realize a Steiner triple system on the plane (although there are 3-independent n-sets with $(n^2/6) - O(n)$ collinear triples. See [BGS] or an elementary construction in [FP].).

It also seems interesting to investigate the higher dimensional versions of this problem.

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