JOURNAL OF COMBINATORIAL THEORY, Series A 56, 182–194 (1991)

Beyond the Erdős–Ko–Rado Theorem

PETER FRANKL AND ZOLTÁN FÜREDI

CNRS, Paris, France and
Mathematical Institute of the Hungarian Academy of Sciences,
*1364 Budapest, P.O.B. 127, Hungary**

Communicated by the Managing Editors

Received May 2, 1989

The exact bound in the Erdős–Ko–Rado theorem is known [F, W]. It states that if $n \geqslant (t+1)(k-t+1)$, and $\mathscr{F}$ is a t-intersecting family of k-sets of an n-set ($|F \cap F'| \geqslant t$ for all $F, F' \in \mathscr{F}$), then $|\mathscr{F}| \leqslant \binom{n-t}{k-t}$. Define $\mathscr{A}_r = \{F \subset \{1, 2, ..., n\} : |F| = k, |F \cap \{1, 2, ..., t+2r\}| \geqslant t+r\}$. Here it is proved that for $n > c\sqrt{t/\log(t+1)}\,(k-t+1)$ one has $|\mathscr{F}| \leqslant \max_r |\mathscr{A}_r|$. © 1991 Academic Press, Inc.

1. DEFINITIONS

2^S is the set of all subsets of S. $\binom{S}{k}$ denotes the set of all k-subsets of the set S ($k \geqslant 0$). If $|S| = n$, then $|\binom{S}{k}| = \binom{n}{k}$. For integers $a < b$ let us denote $\{a, a+1, ..., b\}$ by $[a, b]$, and $[1, n]$ is abbreviated to $[n]$. So $[n]$ denotes the set $\{1, 2, ..., n\}$. A set system $\mathscr{F} \subset 2^S$ is called *k-uniform* if its members are k-sets. $\mathscr{F}$ is *t-intersecting* if $|F \cap F'| \geqslant t$ holds for each $F, F' \in \mathscr{F}$. For a subset A let $\mathscr{F}[A]$ denote the members of $\mathscr{F}$ containing A; i.e., $\mathscr{F}[A] = \{F \in \mathscr{F} : A \subset F\}$. The cardinality of $\mathscr{F}[\{x\}]$ for an element $x \in S$ is called the *degree* of $\mathscr{F}$ (at x), and it is denoted by $\deg(\mathscr{F}, x)$.

To avoid double indices it is usually supposed that the underlying set of $\mathscr{F}$ is $[n]$. In this case for $F \in \mathscr{F}$ the ith entry of F is denoted by $(F)_i$; i.e., $F = \{(F)_1, (F)_2, ...\}$, where $1 \leqslant (F)_1 < (F)_2 < \cdots \leqslant n$. We also correspond to F its characteristic vector $\mathbf{v}(F) = (v_1(F), v_2(F), ..., v_n(F))$, where

$$v_i(F) = \begin{cases} 1 & \text{if } i \in F, \\ 0 & \text{otherwise.} \end{cases}$$

* Research supported in part by the Hungarian National Science Foundation Grant 1812. This paper was written while the authors visited A.T. & T. Bell Laboratories, Murray Hill, NJ 07974 and Bell Communications Research Inc., Morristown, NJ 07960.

0097-3165/91 $3.00

A third representation of the set F is given by *walks* on the grid. The walk starts at the point $(0, 0)$ on the plane. In the ith step the walk moves from the point (x, y) to the point $(x+1, y)$ or $(x, y+1)$ according to whether $v_i(F)$ is 0 or 1. Obviously, the walk ends at the point $(n-|F|, |F|)$. The total number of shortest paths in the grid of the integer points on the plane from $(0, 0)$ to $(n-k, k)$ is $\binom{n}{k}$.

PROPOSITION 1.1. *The number of walks from* (0.0) *to* $(n-k, k)$ *reaching the line* $y=x+h$ (*where* $0 \leqslant h \leqslant k$) *is* $\binom{n}{k-h}$.

Proof. This follows from the well-known reflection principle. Reflecting the rest of the path after it has been reached the line $y=x+h$, one can obtain a 1 to 1 correspondence to the paths from $(0, 0)$ to $(k-h, n-k+h)$. ∎

The most frequently used form of (1.1) is the following. The number of walks from (0.0) to (a, b) $(a>b)$ which intersect the line $y=x$ only in $(0, 0)$ is

$$\frac{a-b}{a+b}\binom{a+b}{b}=\frac{a-b}{a}\binom{a+b-1}{b}. \tag{1.2}$$

We will use some standard inequalities, e.g., for $b>a>0$

$$1-\frac{a}{b}>e^{-a/(b-a)}, \tag{1.3}$$

$$r^r e^{-r}\sqrt{2\pi r}<r!<r^r e^{-r}\sqrt{2\pi r}\left(1+\frac{1}{12r-1}\right) \tag{1.4}$$

for all $r \geqslant 1$. For positive integers A, B, C one has

$$\max_x \binom{A}{x}\binom{B}{C-x} \geqslant \frac{1}{4}\frac{A+B}{\sqrt{ABC}}\binom{A+B}{C}, \tag{1.5}$$

and the left hand side takes its maximum when

$$x=\left\lceil \frac{(A+1)(C+1)}{A+B+2} \right\rceil -1. \tag{1.6}$$

In the formulas c means a sufficiently large, but effectively computable constant (all of our formulas are true with the choice $c=50$, for example). But in each formula c might have a new value.

2. Results

Suppose that $\mathscr{F}$ is a family of k-sets of $[n]$ such that any two of them intersect in at least t elements. Erdős, Ko, and Rado [EKR] proved that this condition implies $|\mathscr{F}| \leqslant \binom{n-t}{k-t}$ whenever $n > n_0(k, t)$. Equality holds if and only if $\mathscr{F}$ consists of all the k-element subsets of $[n]$ containing a fixed t-element subset. In the case $t=1$ they established the best possible bound $n_0(k, 1) = 2k$. Hsieh [H] lowered the bound for $n_0(k, t)$. The exact form of the Erdős–Ko–Rado theorem was proved by Frankl [F], for $t \geqslant 15$, and by Wilson [W], for all t, proving that $n_0(k, t) = (k-t+1)(t+1)$. For smaller values of n there are larger t-intersecting families. For example, if $n \leqslant 2k-t$, then the whole $\binom{[n]}{k}$ is t-intersecting. Define

$$\mathscr{A}_r = \left\{ F \in \binom{[n]}{k} : |F \cap [t+2r]| \geqslant t+r \right\}.$$

(To avoid trivialities from now on it is supposed that $n > 2k-t$, $k > t \geqslant 2$, $k-t > r \geqslant 0$.)

Conjecture 2.1 (See [F]). If $\mathscr{F}$ is of maximal cardinality, then $\mathscr{F} = \mathscr{A}_r$ for some r.

The case $n=4p$, $k=2p$, $t=2$ (and then $r=p-1$) was conjectured in [EKR], too. It is not difficult to pick up the largest $\mathscr{A}_r$.

Proposition 2.2. *$|\mathscr{A}_r|$ is the largest among the $\mathscr{A}_i$'s if*

$$n_r(k, t) = (k-t+1)\left(2 + \frac{t-1}{r+1}\right) \leqslant n < (k-t+1)\left(2 + \frac{t-1}{r}\right). \tag{2.3}$$

(For the proof see Section 4.) Moreover, if equality holds above, then $|\mathscr{A}_r| = |\mathscr{A}_{r+1}|$. In this paper we prove Conjecture 2.1 for $n > (k-t+1)c\sqrt{t/\log t}$.

Theorem 2.4. *Suppose that $\mathscr{F}$ is a t-intersecting family over $[n]$ of maximal cardinality. Suppose further that n is in the range of (2.3), and $t \geqslant 1 + cr(r+1)/(1+\log r)$. Then $\mathscr{F}$ is isomorphic to $\mathscr{A}_r$ (or to $\mathscr{A}_{r+1}$ in the case $n = n_r(k, t)$).*

As the case $r=0$ was solved, we may suppose that $r \geqslant 1$ throughout in this paper. (Although, our method works for $r=0$ as well, at least if $t \geqslant t_0$.) The proof is elementary. It uses the so called shifting operation, introduced in [EKR]. We follow the line of [F], where several properties of the shifted families were proved.

Wilson [W] used the eigenvalue method of Delsarte [D], and actually obtained a stronger result for $n > n_0(k, t)$. He proved that the *Shannon capacity* of the graph $K(n, k, t)$ is $\binom{n-t}{k-t}$. ($K(n, k, t)$ is the generalized Kneser graph with vertex set $\binom{[n]}{k}$, and two vertices F and F' are connected by an edge if $|F \cap F'| < t$.) The case $t = 1$ was proved by Lovász [L] in his celebrated paper on Shannon capacity. The general case was proved by Schrijver [S] for very large n. They both used the Johnson scheme, but Wilson ingeniously utilized the properties of the Hamming scheme.

However, this method does not seem to be suitable to settle Conjecture 2.1 in general, because for $n < n_0(k, t)$ the Shannon capacity of $K(n, k, t)$ exceeds $\max |\mathcal{A}_r|$. (Private communication of R. M. Wilson.)

The Convex Hull of t-intersecting Families. For any family $\mathcal{F} \subset 2^{[n]}$ one can associate the following *profile vector*, $\mathbf{f}(\mathcal{F}) = (f_0, f_1, ..., f_n) \in \mathbf{R}^{n+1}$, $f_i = |\mathcal{F} \cap \binom{[n]}{i}|$. Let $P_{n,t}$ be the convex hull of all profile vectors of t-intersecting families over $[n]$. Let $r \in [n]$, $K \subset [n]$, $k_1 = \min K$. Suppose that $r + t \leqslant k_1$. Define the t-intersecting family $\mathcal{F}(K, r)$ as follows.

$$\mathcal{F}(K, r) = \left\{ F \in \binom{[n]}{k} : |F \cap [t+2r]| \geqslant t + r \text{ for } k \in K, k < n + t - k_1 \right\} \cup \left\{ \binom{[n]}{k} : k \in K, k \geqslant n + t - k_1 \right\}.$$

P. L. Erdős, Frankl, and Katona [EFK] proved that for $t = 1$ all the extremal points (i.e., vertices) of the polytope $P_{n,1}$ can be obtained as a profile vector of a family $\mathcal{F}(K, 0)$.

Conjecture 2.5. (Cooper [C]). $P_{n,t}$ is the convex hull of the $\mathbf{f}(\mathcal{F}(K, r))$'s.

Corollaries. As $(k - t + 1)(t + 1) = n_0(k, t) \leqslant \frac{1}{4}(k + 1)^2$, one can formulate the exact version of the Erdős–Ko–Rado theorem as follows. If $\mathcal{F} \subset \binom{[n]}{k}$ is a t-intersecting family, then $|\mathcal{F}| \leqslant |\mathcal{A}_0|$. Theorem 2.4 can be reformulated, too.

COROLLARY 2.6. *If $\mathcal{F} \subset \binom{[n]}{k}$ is a t-intersecting family and $n > ck^{3/2}(\log k)^{-1/2}$, then $|\mathcal{F}| \leqslant \max_i |\mathcal{A}_i|$.*

Considering the complements of the sets of the family, like in (5.3), we also have the following.

COROLLARY 2.7. *If $n < k + ck^{2/3}$, then Conjecture 2.1 is true.*

3. Shifting

The following exchange operation, or *shifting*, was defined in [EKR]. Let $1 \leqslant i < j \leqslant n$, and suppose that $\mathscr{F}$ is a t-intersecting family of k-sets over $[n]$. Define the operator $P_{ij}: \mathscr{F} \to \binom{[n]}{k}$ as follows.

$$P_{ij}(F) = \begin{cases} (F \setminus \{j\}) \cup \{i\}, & \text{if } i \notin F, j \in F, (F \setminus \{j\}) \cup \{i\} \notin \mathscr{F}, \\ F & \text{otherwise.} \end{cases}$$

Let us set $P_{ij}(\mathscr{F}) = \{P_{ij}(F) : F \in \mathscr{F}\}$. Obviously, $|P_{ij}(\mathscr{F})| = |\mathscr{F}|$, and it is easy to see that $P_{ij}(\mathscr{F})$ is t-intersecting, too. Iterating the shifting operation for all pairs $1 \leqslant i < j \leqslant n$, after finitely many steps one obtains a family $\mathscr{F}^*$ having the property $P_{ij}(\mathscr{F}^*) = \mathscr{F}^*$ for every pair (i, j), $i < j$. Then $\mathscr{F}^*$ is called *shifted*. This can be reformulated in the following way.

$$\text{If } F \in \mathscr{F}^*, i \notin F, j \in F, i < j \text{ then } (F \setminus \{j\}) \cup \{i\} \in \mathscr{F}^* \text{ as well.} \tag{3.1}$$

From now on it is always supposed that $\mathscr{F}$ is shifted. The following lemma essentially appeared first in [F].

Lemma 3.2. *For all $F, F' \in \mathscr{F}$ there exists an i such that $|F \cap [i]| + |F' \cap [i]| \geqslant t + i$.*

Proof. Suppose on the contrary that Lemma 3.2 does not hold for each pair. Then let $\{F, F'\} \subset \mathscr{F}$ be such that $|F \cap F'|$ is minimal among the pairs do not satisfy Lemma 3.2. Let us denote the tth element of $F \cap F'$ by j; i.e., $(F \cap F')_t = j$. Then $|F \cap [j]| + |F' \cap [j]| < j + t = |[j]| + |F \cap F' \cap [j]|$, so there exists a $1 \leqslant i < j$ such that $i \notin F \cup F'$. Then $\{F, (F' \setminus \{j\}) \cup \{i\}\}$ is another counterexample for Lemma 3.2 with $|F \cap (F' - \{j\} + \{i\})| < |F \cap F'|$, a contradiction. ∎

Note that in Lemma 3.2 $F = F'$ is allowed, hence we have the following.

Corollary 3.3. *For all $F \in \mathscr{F}$ there exists an s such that $|F \cap [t + 2s]| \geqslant t + s$.*

Corollary 3.4. (Hujter [Hu]). *Suppose that $\mathscr{F}$ is maximal, $F \in \mathscr{F}$. Let $F' \in \binom{[n]}{k}$ such that $F' \cap [t+1, n] = F \cap [t+1, n]$. Then $F' \in \mathscr{F}$.*

In other words, in the first t elements of $[n]$ one can shift any direction. As we are looking for an $\mathscr{F}$ of the maximum size, from now on we may suppose that $\mathscr{F}$ is maximal (i.e., $\mathscr{F} \cup \{F'\}$ is not t intersecting any more for $F' \in \binom{[n]}{k} \setminus \mathscr{F}$) and $|\mathscr{F}| \geqslant |\mathscr{A}_r|$. The following proposition is easy.

PROPOSITION 3.5. *Let $\mathscr{G} \subset \binom{[n]}{k}$ be a t-intersecting family, and suppose that for some $i<j$ one has $P_{ij}(\mathscr{G}) = \mathscr{A}_r$. Then $\mathscr{G}$ is isomorphic to $\mathscr{A}_r$.*

4. ON THE SIZES OF THE EXTREMAL FAMILIES

Proposition 2.2 is proved in the following form.

PROPOSITION 4.1. *For $n \geqslant n_r(k, t) = (k-t+1)(2+(t-1)/(r+1))$ one has $|\mathscr{A}_r| \geqslant |\mathscr{A}_{r+1}|$. Here equality holds only if $n = n_r(k, t)$.*

Proof. $\mathscr{A}_r \backslash \mathscr{A}_{r+1}$ consists of those sets $F \in \binom{[n]}{k}$ for which $|F \cap [t+2r]| \geqslant t+r$ but $|F \cap [t+2r+2]| < t+r+1$. Thus $|F \cap [t+2r]| = t+r$ and $F \cap \{t+2r+1, t+2r+2\} = \varnothing$. So

$$|\mathscr{A}_r \backslash \mathscr{A}_{r+1}| = \binom{t+2r}{t+r}\binom{n-t-2r-2}{k-t-r}. \tag{4.2}$$

Similarly, $\mathscr{A}_{r+1} \backslash \mathscr{A}_r = \{F \in \binom{[n]}{k} : |F \cap [t+2r]| = t+r-1$ and $\{t+2r+1, t+2r+2\} \subset F\}$. Hence

$$|\mathscr{A}_{r+1} \backslash \mathscr{A}_r| = \binom{t+2r}{t+r-1}\binom{n-t-2r-2}{k-t-r-1}. \tag{4.3}$$

The ratio of (4.2) and (4.3) is

$$\frac{r+1}{t+r}\,\frac{n-k-r-1}{k-t-r},$$

which is at least 1 if and only if $n \geqslant n_r(k, t)$. ∎

From now on we suppose that $n_r(k, t) \leqslant n < n_{r-1}(k, t)$; i.e.,

$$(r+1)n \geqslant (k-t+1)(t+2r+1) \tag{4.4}$$

$$rn \leqslant (k-t+1)(t+2r-1). \tag{4.5}$$

Let $\mathscr{A} = \cup \mathscr{A}_s$ $(0 \leqslant s \leqslant k-t)$. Then Corollary 3.3 is equivalent to the fact that $\mathscr{F} \subset \mathscr{A}$. The family $\mathscr{A}_s$ consists of those sets whose associated walks from $(0, 0)$ to $(n-k, k)$ reach the line $y = x+t$ during the first $t+2s$ steps. Hence Proposition 1.1 implies that

$$|\mathscr{A}| \leqslant \binom{n}{k-t}. \tag{4.6}$$

Define $\mathscr{A}_s^0 = \mathscr{A}_s \backslash (\mathscr{A}_0 \cup \mathscr{A}_1 \cup \cdots \mathscr{A}_{s-1} \cup \mathscr{A}_{s+1} \cup \cdots)$, i.e., those members of $\mathscr{A}_s$ which do not belong to another $\mathscr{A}_i$. This means that for $F \in \mathscr{A}_s^0$ one has

$(F)_{t+i} > t+2i$ except for $i=s$, when $(F)_{t+s}=t+2s$. In other words, the walk touches the line $y=x+t$ at the point $(s, t+s)$, otherwise it goes below it. Hence (1.2) gives

$$|\mathscr{A}_s^0| = \frac{t}{t+s}\binom{t+2s-1}{s}\frac{n-2k+t}{n-k-s}\binom{n-t-2s-1}{k-t-s}. \tag{4.7}$$

LEMMA 4.8. *For $n_r(k,t) \leqslant n < n_{r-1}(k,t)$ one has*

$$|\mathscr{A}_r^0| > \frac{t(t-1)}{4\sqrt{r+1}(t+2r)(t+2r-1)}\binom{n}{k-t}.$$

Proof. (4.7) implies that

$$\frac{|\mathscr{A}_r^0|}{\binom{n}{k-t}} = \frac{t}{t+2r}\frac{n-2k+t}{n-t-2r}\frac{\binom{t+2r}{r}\binom{n-t-2r}{k-t-r}}{\binom{n}{k-t}}.$$

Here the second term is at least $(t-1)/(t+2r-1)$ by (4.4). With the values $A=t+2r$, $B=n-t-2r$, and $C=k-t$, (1.5) implies that

$$\max_x \binom{t+2r}{x}\binom{n-t-2r}{k-t-x}$$

$$\geqslant \frac{n}{4\sqrt{(t+2r)(n-t-2r)(k-t-r)}}\binom{n}{k-t}. \tag{4.9}$$

The product of the first and the last factors under the square root is less than $(r+1)n$, by (4.4). So the right hand side of (4.9) is at least $(1/(4\sqrt{r+1}))\binom{n}{k-t}$. Moreover (4.5) and (4.4) imply that

$$r-1 < -1 + \frac{(A+1)(C+1)}{A+B+2} \leqslant r.$$

Then (1.6) gives that (4.9) takes its maximum at $x=r$. ∎

Let $\mathscr{A}^{(2)} = \mathscr{A} \setminus \cup \mathscr{A}_i^0$. One can define $\mathscr{A}^{(2)}$ as the family of k-sets from $\mathscr{A}$ whose walks have at least 2 common points with the line $y=x+t$.

LEMMA 4.10. $|\mathscr{A}^{(2)}| \leqslant 2\binom{n}{k-t-1}$.

Proof. The number of walks reaching the line $y=x+t+1$ is $\binom{n}{k-t-1}$, by Proposition 1.1. Consider a walk W from $\mathscr{A}^{(2)}$ which does not reach this line. Call these walks of type II. Reflecting the portion of W between the first and the last common point with $y=x+t$, one can obtain an injection from the walks of type II to the walks reaching $y=x+t+1$. This implies that the number of walks of type II is also at most $\binom{n}{k-t-1}$. ∎

PROPOSITION. *Suppose that* $t-1 \geqslant 10(r+1)$. *Then for* $s > 4(r+1)$ *one has*

$$|\mathscr{A}^0_{s+1}| \leqslant \tfrac{1}{2}\,|\mathscr{A}^0_s|. \tag{4.11}$$

Proof. (4.7) gives that

$$\frac{|\mathscr{A}^0_{s+1}|}{|\mathscr{A}^0_s|} = \frac{t+2s}{s+1}\,\frac{t+2s+1}{t+s+1}\,\frac{n-k-s-1}{n-t-2s-2}\,\frac{k-t-s}{n-t-2s-1}.$$

Here the third term is at most 1, because we may suppose that $s+1 \leqslant k-t$. (Otherwise the definition of $\mathscr{A}^0_{s+1}$ is empty.) The last term is at most $(r+1)/(t+2r+1)$, by (4.4), (for $s \geqslant r$). Hence the right hand side is at most

$$\frac{t+2s}{s+1}\,\frac{t+2s+1}{t+s+1}\,\frac{r+1}{t+2r+1}. \tag{4.12}$$

Introducing $\alpha = (t-1)/(r+1)$, and $\beta = (s+1)/(r+1)$ we have that the formula in (4.12) is bounded above by

$$\frac{\alpha+2\beta}{\beta}\,\frac{\alpha+2\beta}{\alpha+\beta}\,\frac{1}{\alpha+2},$$

which is less than $\frac{1}{2}$ for $\alpha \geqslant 10$, $\beta \geqslant 4$. ∎

5. Proof of the Theorem

From now on we suppose that

$$t \geqslant 1 + c_1\,\frac{r(r+1)}{1+\log r}, \tag{5.1}$$

where c_1 is an absolute constant. In the proof $c_1, c_2, \ldots$ are effectively computable constants.

Let $\mathscr{F}$ be a t-intersecting family, $\mathscr{F} \subset \binom{[n]}{k}$. As it was shown in Section 3, we may suppose that $\mathscr{F}$ is shifted. Calculate r from (4.4) and (4.5). Suppose that $|\mathscr{F}|$ is maximal; i.e.,

$$|\mathscr{F}| \geqslant |\mathscr{A}_r|. \tag{5.2}$$

Our first observation is that without loss of generality we may suppose that

$$n \geqslant 2k. \tag{5.3}$$

Indeed, for $n<2k$ consider the family of complementers, $\mathcal{F}^c=\{[n]-F: F\in\mathcal{F}\}$. Then $\mathcal{F}^c$ is a $t'=n-2k+t$-intersecting family, $\mathcal{F}^c\subset\binom{[n]}{k'}$, where $k'=n-k$. It is easy to check that the values n, k', and t' in the formulas (4.4) and (4.5) give $r'=k-t-r$. Then (4.4) implies

$$2k(r+1)>n(r+1)\geqslant(k-t+1)(t+2r+1)$$
$$=(k-t+1)(2r+2)+(k-t+1)(t-1),$$

which implies

$$r\geqslant k-t-r=r'. \tag{5.4}$$

Then (5.1) and (4.4) give

$$t'-1=n-2k+t-1\geqslant\frac{t-1}{r+1}(k-t-r)$$
$$\geqslant c_1\frac{r}{1+\log r}r'>c_2\frac{r'(r'+1)}{1+\log r'}$$

for some $c_2<c_1$. Now we are able to prove our key lemma.

PROPOSITION 5.5. *$\mathcal{A}_s^0\cap\mathcal{F}=\emptyset$ for all but one s.*

Proof. By (5.2) and Lemma 4.8 we have that

$$|\mathcal{F}|\bigg/\binom{n}{k-t}\geqslant|\mathcal{A}_r^0|\bigg/\binom{n}{k-t}>c_3\frac{1}{\sqrt{r+1}}. \tag{5.6}$$

(In the last step we used (5.1).) Moreover, Lemma 4.10 and (4.4) give

$$|\mathcal{A}^{(2)}|\bigg/\binom{n}{k-t}\leqslant\frac{2(k-t)}{n-k+t+1}<2\frac{r+1}{t+r}. \tag{5.7}$$

The right hand side of (5.6) is larger than the right hand side of (5.7), implying $\mathcal{F}\backslash\mathcal{A}^{(2)}\neq\emptyset$. We have obtained that there exists an $\mathcal{A}_s^0$ such that $\mathcal{A}_s^0\cap\mathcal{F}\neq\emptyset$. Next we show that only one of such s exists.

Suppose that $F\in\mathcal{F}\cap\mathcal{A}_s^0$, $F'\in\mathcal{F}\cap\mathcal{A}_{s'}^0$, $s<s'$. By the definition of $\mathcal{A}_s^0$ we have that

$$|F\cap[i]|\leqslant\frac{t+i}{2}$$

holds for all $i\geqslant t$, with equality holding only for $i=s$. Similarly, $|F'\cap[i]|$

$\leqslant (t+i)/2$ for all $i \geqslant t$, and here equality holds for $i = s'$ only. Then for the sum we have

$$|F \cap [i]| + |F' \cap [i]| \leqslant t + i,$$

but here equality never holds, since $s \neq s'$. This contradicts Lemma 3.2. ▮

From now on s denotes the index of $\mathscr{A}_s^0$ for which $\mathscr{A}_s^0 \cap \mathscr{F} \neq \varnothing$. Our next aim is to prove that $\mathscr{F} \subset \mathscr{A}_s$. (5.6) and (5.7) imply that

$$\frac{|\mathscr{A}_s^0 \cap \mathscr{F}|}{|\mathscr{A}_s^0|} = \frac{|\mathscr{F}| - |\mathscr{F} \cap \mathscr{A}^{(2)}|}{|\mathscr{A}_s^0|} \geqslant \frac{|\mathscr{A}_r^0| - |\mathscr{A}^{(2)}|}{|\mathscr{A}_s^0|} \geqslant 1 - \frac{|\mathscr{A}^{(2)}|}{|\mathscr{A}_r^0|}. \tag{5.8}$$

Here the right hand side is larger than $\frac{1}{2}$ (by (5.7) and Lemma 4.8). Then (4.11) gives that

$$s \leqslant 4(r+1). \tag{5.9}$$

Proof. Otherwise (4.11) gives that

$$\frac{|\mathscr{A}_s^0 \cap \mathscr{F}|}{|\mathscr{A}_s^0|} \leqslant \frac{|\mathscr{A}_s^0|}{\max_i |\mathscr{A}_i^0|} = \frac{|\mathscr{A}_s^0|}{|\mathscr{A}_r^0|} \leqslant \frac{1}{2}$$

contradicting (5.8). ▮

We claim that

$$n \geqslant 3k - 2t + 1. \tag{5.10}$$

Indeed, (5.3) and (4.5) imply that

$$k - t - r \geqslant r. \tag{5.11}$$

Moreover, the inequality

$$(k-t+1)(t+2r+1) \geqslant (3k-2t+1)(r+1) \tag{5.12}$$

is equivalent to

$$k - t - r \geqslant \frac{r^2 + r}{t - r - 2}.$$

Here the right hand side is less than r, by (5.1), hence it holds by (5.11). This implies (5.12), which gives with (4.4) that

$$(r+1)n \geqslant (k-t+1)(t+2r+1) \geqslant (r+1)(3k-2t+1),$$

yielding (5.10). ▮

Equation (5.10) was proved only to legitimate the following definition for all $1 \leqslant i \leqslant k-t-s+1$. Define $D_i \in \binom{[n]}{k}$ as follows, $D_i \cap [t+2s+i+1] = [s+1, t+2s]$, and $(D_i)_{a+t+s} = t+2s+i+2a$ for $1 \leqslant a \leqslant k-t-s$. That is,

$$D_i = \{s+1, s+2, ..., t+2s-1, t+2s, t+2s+i+2, t+2s+i+4, ..., t+2s+i+2(k-t-s)\}.$$

Here $(D_i)_k = 2k-t+i \leqslant 3k-2t+1$.

CLAIM 5.13. *If $D_i \in \mathscr{F}$, and $F \in \mathscr{F} \setminus \mathscr{A}_s$ then the walk associated to F meets the line $y = x+t+i$.*

Proof. This is an easy consequence of Lemma 3.2. Let j be the smallest integer satisfying

$$|D_i \cap [j]| + |F \cap [j]| \geqslant t+j.$$

Then, as $|F \cap [t+2s]| < t$, we have that $j = t+2s+i+2a$ for some $a \geqslant 1$. Then

$$|F \cap [t+2s+i+2a]| \geqslant t+j-|D_i \cap [j]| = t+s+i+a.$$

Hence $(F)_{t+i+s+a} \leqslant (t+i)+2(s+a)$. ▮

A corollary of Claim 5.13 is the following. If $D_{k-t-s+1} \in \mathscr{F}$ then $\mathscr{F} \setminus \mathscr{A}_s = \varnothing$; i.e., $\mathscr{F} \subset \mathscr{A}_s$ and the proof is ready. Let $i = \max\{a: D_a \in \mathscr{F}\}$. (If this set is empty, we take $i = 1$.) The remaining case is whenever

$$i \leqslant k-t-s. \tag{5.14}$$

Claim 5.13 and Proposition 1.1 imply that

$$|\mathscr{F} \setminus \mathscr{A}_s| \leqslant 2\binom{n}{k-t-i} =: P_i.$$

(The factor 2 is needed, in fact, only in the case when the set $\{D_a: 1 \leqslant a \leqslant k-t-s\} \cap \mathscr{F}$ is empty. But we wanted to have a universal upper bound.)

Now we are going to prove a lower bound for $|\mathscr{A}_s \setminus \mathscr{F}|$. As $\mathscr{F}$ is shifted and maximal, Corollary 3.4 implies that all the k-sets are missing from $\mathscr{F}$ which one can obtain from D_{i+1} by shifting the elements of $D_{i+1} \cap [t+2s+i+2, n]$, and shifting (rightward) the elements of $D_{i+1} \cap [t]$. By (1.2) the number of images of D_{i+1} is

$$\binom{t}{s} \frac{n-2k+t-i}{n-k-s-i} \binom{n-t-2s-i-1}{k-t-s} =: M_i.$$

CLAIM 5.15. *For $1 \leqslant a < i$ one has $M_{a+1}/P_{a+1} > M_a/P_a$.*

Proof. We have that

$$\frac{M_{a+1}}{M_a}\frac{P_a}{P_{a+1}} = \frac{n-2k+t-a-1}{n-2k+t-a}\,\frac{n-k-s-a}{n-t-2s-a-1}\,\frac{n-k+t+a+1}{k-t-a}.$$

Here (5.10) and (5.14) imply that the first nominator is at least 1, so the first fraction is at least $\frac{1}{2}$. Similarly, $n \geqslant 2k-t-1-a$, so the second fraction is at least $\frac{1}{2}$, too. Finally, for the last term we have

$$\frac{n-k+t+a+1}{k-t-a} \geqslant \frac{n-k+t+2}{k-t-1} > \frac{t+r}{r+1} > 4,$$

by (4.4) and (5.1). ∎

CLAIM 5.16. $M_1/P_1 > 1$.

This implies that $M_i > P_i$; i.e., $|\mathscr{F}| < |\mathscr{A}_s|$, a contradiction. So the opposite of (5.14) must be true; i.e., $\mathscr{F} \subset \mathscr{A}_s$.

Proof. We have that

$$\frac{M_1}{|\mathscr{A}_s^0|} = \frac{\binom{t}{s}\dfrac{n-2k+t-1}{n-k-s-1}\binom{n-t-2s-2}{k-t-s}}{\dfrac{t}{t+s}\binom{t+2s-1}{s}\dfrac{n-2k+t}{n-k-s}\binom{n-t-2s-1}{k-t-s}}$$

$$= \frac{(t-1)(t-2)\cdots(t-s+1)}{(t+2s-1)\cdots(t+s+1)}\,\frac{n-2k+t-1}{n-2k+t}\,\frac{n-k-s}{n-t-2s-1}.$$

Using (1.3) we obtain that the first term is at least

$$\exp\left(-\sum_{a=1}^{s-1}\frac{2s}{t-i}\right) > \exp -\frac{2s(s-1)}{t-s}.$$

Using (5.9) and (5.1) we obtain that the right hand side is at least $\exp(-32\log r/c_1) > r^{-32/c_1} > c_3 r^{-1/3}$, if c_1 is sufficiently large. Obviously, the second and third fractions are at least $\frac{1}{2}$, so we have

$$M_1 > |\mathscr{A}_s^0|\,\frac{c_4}{\sqrt[3]{r}} \geqslant |\mathscr{A}_s^0 \cap \mathscr{F}|\,\frac{c_4}{\sqrt[3]{r}} > c_5\binom{n}{k-t}r^{-5/6}. \qquad (5.17)$$

In the last step we used (5.8). On the other hand

$$P_1 = 2\binom{n}{k-t-1} = \frac{2(k-t)}{n-k+t+1}\binom{n}{k-t} < 2\frac{r+1}{t+r}\binom{n}{k-t}. \quad (5.18)$$

Finally, (5.17), (5.18), and (5.1) give Claim 5.16. ∎

References

[C] B. Cooper, unpublished manuscript.

[D] P. Delsarte, An algebraic approach to the association schemes in coding theory, *Philips Res. Rep. Suppl.* **10** (1973).

[EFK] P. L. Erdős, P. Frankl, and G. O. H. Katona, Extremal hypergraph problems and convex hulls, *Combinatorica* **5** (1985), 11-26.

[EKR] P. Erdős, C. Ko, and R. Rado, Intersection theorems for systems of finite sets, *Quart. J. Math. Oxford Ser. (2)* **12** (1961), 313–320.

[F] P. Frankl, The Erdős–Ko–Rado theorem is true for $n = ckt$, *in* "Combinatorics" (A. Hajnal *et al.*, Eds.), Proc. Fifth Hungarian Colloq. Combin., Keszthely, 1976; "Proceedings Colloq. Math. Soc. J. Bolyai," Vol. 18, pp. 365–375, North-Holland, Amsterdam, 1978.

[H] W. N. Hsieh, Intersection theorems for systems of finite vector spaces, *Discrete Math.* **12** (1975), 1-16.

[Hu] M. Hujter, unpublished manuscript.

[L] L. Lovász, On the Shannon capacity of a graph, *IEEE Trans. Inform. Theory* **25** (1979), 1-7.

[S] A. Schrijver, Association schemes and the Shannon capacity: Eberlein polynomials and the Erdős–Ko–Rado theorem, *in* "Proceedings Colloq. Math. Soc. J. Bolyai," Vol. 25, pp. 671–688, Szeged, Hungary, 1978 North-Holland, Amsterdam, 1981.

[W] R. M. Wilson, The exact bound in the Erdős–Ko–Rado theorem, *Combinatorica* **4** (1984), 247–257.