Competition Graphs and Clique Dimensions

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ABSTRACT

Here it is proved that for almost all simple graphs over n vertices one needs $\Omega(n^{4/3}(\log n)^{-4/3})$ extra vertices to obtain them as a double competition graph of a digraph. On the other hand $O(n^{5/3})$ extra vertices are always sufficient. Several problems remain open.

1. DEFINITIONS

The *competition graph* of a digraph $D=(V, A)$ is a graph $G=(V, E)$ where $\{x, y\} \in E$ if and only if $x \neq y$ and for some $v \in V$ both xv and $yv \in A$. It is easy to see that, by adding sufficiently many isolated vertices, every graph G can be made into the competition graph of an acyclic digraph, e.g., let $V(D) = V \cup E$ and let ve be an arc in D if $e \in E$ and $v \in e$. Thus the *competition number*, $k(G)$, is defined to be the smallest integer k so that G together with k isolated vertices, $G \cup I_k$, is a competition graph of a digraph.

The *double competition graph* of a digraph $D = (V, A)$ is the graph $G = (V, E)$ where $xy \in E$ if and only if $x \neq y$ and for some $u, v \in V$, the arcs ux, uy, xv, and $yv \in A$. The double competition number of a graph, $dk(G)$, is the smallest integer k so that $G \cup I_k$ is the double competition graph of some digraph. A

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simple construction shows that for all graphs G one has

$$dk(G) \leq 2|E| . \tag{1.1}$$

The neighborhood graph, $N(H)$, of an (undirected) graph $H = (V, E)$ is the graph $G = (V, E(G))$ where $\{x, y\} \in E(G)$ if and only if $x \neq y$ and for some $v \in V$ both $\{x, v\}$ and $\{y, v\} \in E$. One can define the *embedding number* $g(G)$ of any graph G to be the smallest integer for which there is a graph H on $|V(G)| + g(G)$ vertices such that G is isomorphic to an induced subgraph of $N(H)$.

A *wheel* W_n is a cycle of length $n-1$ with an additional node adjacent to every node on the cycle, $(n \geq 4)$.

The set of all (labelled) graphs over the elements $\{1, 2, \ldots, n\}$ is denoted by $\mathcal{G}^n$. Obviously, $|\mathcal{G}^n| = 2^{\binom{n}{2}}$. G^n denotes any member of $\mathcal{G}^n$. The statement "almost all graphs have property P" means that there exists a sequence $\epsilon_1, \epsilon_2, \ldots$ tending to 0 such that the number of graphs $G^n \in \mathcal{G}^n$ having property P is at least $(1 - \epsilon_n)2^{\binom{n}{2}}$.

The *degree* of a vertex x of the graph G is denoted by $\deg(G, x)$ or $\deg(x)$ for short. For digraphs $\deg^+(x)$ denotes the outdegree. The *neighborhood* of x is denoted by $N(G, x)$ or $N(x)$. As usual, $\omega(G)$ denotes the size of the largest complete subgraph in G. The induced subgraph $G|S$ is a graph with the vertex set $S \cap V(G)$ and with the edges of G contained in S.

A finite *affine plane* of order q is a pair $(P, \mathcal{A})$ where P, the point set, is a q^2-element set and $\mathcal{A}$, the line set, is a family of q-element subsets covering every pair in P exactly once. There are affine planes for each prime power order. The set of lines can be decomposed into $q + 1$ q-element subfamilies consisting of pairwise disjoint lines. These are called *parallel classes*.

2. RESULTS

The competition and double competition graphs of digraphs have been studied by numerous authors, mainly from a practical point of view. A survey may be found in Raychaudhuri and Roberts [10], or in other papers and books of Fred Roberts. A recent Ph.D. thesis on this topic was written by Kim [9]. Usually, it is required that the associated digraphs are acyclic, so one can define $dk_a(G) = \min\{k$: such that there exists an acyclic digraph over $|V(G)| + k$ vertices inducing G as its double competition graph$\}$. We have that

$$dk(G) \leq dk_a(G) .$$

As a theoretical approach, to obtain the possible most general result, we will omit this constraint. Diny [4] calculated the $dk_a(G)$ for some classes of graphs, all of them have double competition number $dk_a(G) \leq 2$. Jones et al. [7] showed that the complete 3-partite graph has large double competition number.

$$\frac{2}{\sqrt{3}} \sqrt{n} \leq dk_a\left(K\left(\frac{n}{3}, \frac{n}{3}, \frac{n}{3}\right)\right) \leq \left(\frac{2}{3} + o(1)\right)n .$$

Here we give asymptotic description of the behavior of $dk(G)$ (and $dk_a(G)$).

Theorem 2.1. *For almost all graphs $G \in \mathcal{G}^n$ one has $dk(G) > 0.1 n^{4/3} (\log n)^{-4/3}$.*

Theorem 2.2. *For each graph $G \in \mathcal{G}^n$ one has $dk_a(G) \le (2 + o(1)) n^{5/3}$.*

3. PROOF OF THEOREM 2.1

We are going to use two simple lemmas from the theory of random graphs (for the proofs, see Bollobás [2]).

Lemma 3.1. *For almost all graphs $G \in \mathcal{G}^n$ the following holds. Every set $S \subset V(G)$, $|S| > 10 \log n$ contains at least $|S|^2/5$ edges of G.* ∎

Lemma 3.2. *For almost all graphs $G \in \mathcal{G}^n$ the clique number $\omega(G) \le 2 \log n$.* ∎

In this section "log" means logarithm with base 2.

Denote by $\mathcal{H}(v, d)$ the set of all directed graphs on the vertex set $\{1, 2, \ldots, v\}$ such that all arcs start from or end at $\{1, 2, \ldots, n\} = [n]$ and every out- and indegree restricted to $[n]$ is at most d. This means that $|N^+(x) \cap [n]| \le d$ and $|N^-(x) \cap [n]| \le d$ holds for every $x \in [v]$. Obviously,

$$|\mathcal{H}(v, d)| \le \binom{n}{d}^{2v} < n^{2dv} = 2^{2dv \log n} .$$

Let $\mathcal{G}_0^n$ denote the set of all graphs $G \in \mathcal{G}^n$, such that

(1). $|E(G|S)| \ge \frac{1}{5}|S|^2$ for all $|S| > 10 \log n$,
(2). $\omega(G) \le 2 \log n$,
(3). G is not the competition graph of any digraph $D \in \mathcal{H}(v, d)$ with $n \le v \le \binom{n}{2}$, and $vd < n^2/(10 \log n)$.

Claim 3.3. *$\mathcal{G}_0^n$ contains almost all graphs.*

Proof. This is obvious from the Lemmas and from the fact that in (3)

$$\sum_{v,d} |\mathcal{H}(v, d)| < n^4 2^{0.2n^2} = o(2^{\binom{n}{2}}) . \qquad ∎$$

Claim 3.4. *For $G \in \mathcal{G}_0^n$ one has $dk(G) > n^{4/3} (\log n)^{-4/3}/10$.*

Proof. Suppose that G is the double competition graph of the digraph D. Without loss of generality we may suppose that $V(D) \backslash V(G)$ does not contain any arc of $E(D)$. Let $V(D) = \{1, 2, \ldots, v\}$, $V(G) = [n]$. Denote the maximum restricted outdegree of D by d^+ (i.e., $d^+ = \max_x |N^+(x) \cap [n]|$), the maximum restricted indegree d^-, and $d = \max\{d^+, d^-\}$.

If $d < n^{2/3} (\log n)^{1/3}$, then $D \in \mathcal{H}(v, d)$, so the property (3) implies that

$$v \ge n^2/(10 d \log n) > n^{4/3} (\log n)^{-4/3}/10 .$$

If $d \geq n^{2/3}(\log n)^{1/3}$, then suppose that $d = d^+$ and $d = |N^+(x) \cap [n]|$ for some $x \in V(D)$. Then $N^+(x) \cap [n]$ contains at least $d^2/5$ edges of G, by (1). All of these edges are contained in some $N^-(y)$ as well, so we have that

$$\frac{d^2}{5} \leq |E(G|N(x))| \leq \sum_y \binom{|N^+(x) \cap N^-(y)|}{2}.$$

By (2), the right-hand side is at most $v\binom{\omega}{2} < v\omega^2/2 \leq 2v(\log n)^2$. This implies again that $v \geq 0.1d^2/(\log n)^2$. ∎

4. THE CONSTRUCTION FOR THEOREM 2.2

Let q be the smallest prime power such that $q^2(q+2) \geq n$. By adding isolated vertices to G, we may suppose without loss of generality, that $|V(G)| = q^2(q+2)$. Divide $V(G)$ into $q+2$ equal parts, $V(G) = V_1 \cup \ldots \cup V_{q+2}$, $|V_i| = q^2$.

Let $\mathcal{A}_i$ be a copy of an affine plane of order q on the point set V_i, $(1 \leq i \leq q+2)$. $\mathcal{A}_i$ has $q+1$ parallel classes. Label these classes by the integers $\{1, 2, \ldots, q+2\} \setminus \{i\}$, and denote them by $\mathcal{L}_{i1}, \mathcal{L}_{i2}, \ldots, \mathcal{L}_{i,q+2}$. Each parallel class, $\mathcal{L}_{ij}$, consists of q pairwise disjoint q-sets $L_{ij1}, \ldots, L_{ijq}$. Now we are going to define the vertex set of the digraph D. Let $V(D) = V \cup A \cup B \cup U \cup W$ where these five sets are pairwise disjoint, $V = V(G)$, and $|A| = |B| = \binom{q+2}{2}q^3$, $|U| = |W| = (q+2)\binom{q^2}{2}$. Divide A (and B) into disjoint q-sets, i.e.,

$$A = \cup A_{ij\alpha\beta}$$

where $1 \leq i < j \leq q+2$, $1 \leq \alpha, \beta \leq q$. (Similarly, $B = \cup B_{ji\alpha\beta}$.) As $L_{ij\alpha}$ and $A_{ij\alpha\beta}$ are both q element sets one has a bijection $a_{ij\alpha\beta}$ between them, i.e.,

$$A_{ij\alpha\beta} = \{a_{ij\alpha\beta}(x) : x \in L_{ij\alpha}\}.$$

Similarly, $B_{ji\alpha\beta} = \{b_{ji\alpha\beta}(y) : y \in L_{ji\beta}\}$, where again $1 \leq i < j \leq q+2$. Divide U (and W) into $q+2$ equal parts, $U = U_1 \cup \ldots \cup U_{q+2}$ $(W = W_1 \cup \ldots \cup W_{q+2})$. As U_i has $\binom{|V_i|}{2}$ elements there exists a bijection u_i between U_i and the pairs of V_i, i.e.,

$$U_i = \{u_i(x, y) : x, y \in V_i, x \neq y\},$$

where $u_i(x, y) = u_i(y, x)$. Similarly, $W_i = \{w_i(x, y) : x, y \in V_i, x \neq y\}$.

Now we are going to define the set of arcs, D. There will be arcs from $A \cup U$ to V and from V to $B \cup W$. Put an arc from $a_{ij\alpha\beta}(x)$ to x, and to $N(x) \cap L_{ji\beta}$. Moreover, put an arc from $u_i(x, y)$ to x and y if $\{x, y\}$ is an edge of G contained in V_i. Similarly, we have all the arcs $(y, b_{ji\alpha\beta}(y))$ for $y \in L_{ji\beta}$, from $N(y) \cap L_{ij\alpha}$ to $b_{ji\alpha\beta}(y)$ and from x and y to $w_i(x, y)$ if $\{x, y\}$ is an edge of G contained in V_i.

We claim that the double competition graph induced by D is just G with the additional isolated vertices $A \cup B \cup U \cup W$. This implies that

$$dk_a(G) \leq |V(D)| = q^3(q+2)(2q+1) = (2 + o(1))n^{5/3}.$$

It is clear that all the induced edges are in V. It is also clear that the edges of G are induced. Indeed, if $\{x, y\} \in E(G)$, $x, y \in V_i$, then the four arcs $(u_i(x, y), x)$, $(u_i(x, y), y)$, $(x, w_i(x, y))$ and $(y, w_i(x, y))$ induce $\{x, y\}$. If $x \in V_i$, $y \in V_j$ for some $1 \le i < j \le q+2$, then consider the lines of $\mathcal{L}_{ij}$ in V_i and $\mathcal{L}_{ji}$ in V_j. For some α and β $(1 \le \alpha, \beta \le q)$ we have $x \in L_{ij\alpha}$ and $y \in L_{ji\beta}$. Then, by definition, there are arcs from $a_{ij\alpha\beta}(x)$ to x and to $y(\in N(x) \cap L_{ji\beta})$ and from $x(\in N(y) \cap L_{ij\alpha})$ and y to $b_{ji\alpha\beta}(y)$.

The only thing remained to check is that D does not induce more edges than $E(G)$. The arcs from U to V and V to W could not imply more than the edges inside V_i $(1 \le i \le q+2)$. Suppose that the arcs from a vertex of A, say from $a_{ij\alpha\beta}(x)$, and the arcs to a vertex of B, say to $b_{st\delta\gamma}(y)$ imply an edge e, $e \not\in E(G)$. By definition, $N^+(D, a_{ij\alpha\beta}(x)) = \{x\} \cup (N(G, x) \cap L_{ji\beta})$, and the pairs $\{x, y\}$ with $y \in N(G, x) \cap L_{ji\beta}$ belong to $E(G)$. So $e \subset N(G, x) \cap L_{ji\beta}$. Similarly, $e \subset L_{ts\delta}$. Hence these two lines of the affine plane have large intersection (=more than 1 element). They must coincide, $L_{ji\beta} = L_{ts\delta}$ and so $(j, i) = (t, s)$. On the other hand, recall that $j > i$ and $t < s$, so $(j, i) \ne (t, s)$. This contradiction completes the proof. ∎

In the above construction the following fact was utilized implicitly.

Claim 4.1. *Suppose that G^n is a bipartite graph. Then $dk_a(G^n) \le n$.*

Proof. Denote the parts of G^n by A and B, i.e., $A \cup B = V(G^n)$, and all the edges go between A and B. Duplicate A and B, i.e., let $A_1 = \{a(x) : x \in A\}$ and $B_1 = \{b(y) : y \in B\}$. Then the following digraph over $A \cup B \cup A_1 \cup B_1$ induces G^n.

$$E(D) = :\{(a(x), x) : x \in A\} \cup \{(a(x), y) \text{ if } (x, y) \in E(G^n)\}$$
$$\cup \{(y, b(y)) : y \in B\} \cup \{(x, b(y)) \text{ if } (x, y) \in E(G^n)\}\,. \quad ∎$$

5. PROBLEMS

Given a graph $G = (V, E)$. Define its clique dimension in t rounds, $cd_t(G)$, as follows.

$$cd_t(G) = \min \sum_{i=1}^{t} n_i\,,$$

such that there are families $\mathcal{A}_1, \ldots, \mathcal{A}_t$, $|\mathcal{A}_i| = n_i$ such that each edge $e \in E$ is covered by each family (i.e., for all $1 \le i \le t$ there exist an $A_i \in \mathcal{A}_i$ with $e \subset A_i$), but this does not hold for the nonedges. It is obvious that

$$cd_t(G) \le |E| + t - 1\,,$$

more generally

$$cd_t(G) \le cd_k(G) + cd_{t-k}(G)\,.$$

Moreover,

$$cd_1(G) - |V| \le g(G) \le cd_1(G)\,,$$
$$cd_1(G) - |V| \le k(G) \le cd_1(G)\,,$$

and

$$\frac{1}{2}\, cd_2(G) - |V| \le dk(G) \le cd_2(G)\,.$$

Define $cd_t(n) = \max\{cd_t(G) : G \in \mathcal{G}^n\}$. The problem of determination of $cd_1(G)$ was posed by several authors (see [11]). For example, Boland, Brigham, and Dutton [1] determined that $g(W_n) = \lfloor \frac{n-2}{2} \rfloor$. The true order of magnitude of $cd_1(n)$ was proved by Erdös, Goodman, and Pósa [5].

$$cd_1(n) = \left[\frac{n^2}{4}\right].$$

For a related problem about clique decomposition see Chung [3], Györi and Kostochka [6], and Kahn [8].

Problem 5.1. *Determine $cd_1(n)$ for almost all graphs.*

Lemma 3.2 implies that it is at least $\Omega(n^2/(\log n)^2)$. P. Erdös (private communication) showed that the upper bound $O(n^2/\log n)$ is immediate from standard results from the theory of random graphs.

Problem 5.2. *Give estimates to $cd_t(n)$. Especially, close the gap, if possible, between Theorem* 2.1 *and* 2.2.

Probably, using the arguments of this article it can be shown that there is an absolute constant $c > 0$ such that $cd_t(n) \ge \Omega(n^{1+c^t})$.

Problem 5.3. *Give estimates for some narrower classes of graphs, e.g., interval graphs.*

Define $cd(G)$ as the $\min_t cd_t(G)$, and $cd(n) = \max\{cd(G) : G \in \mathcal{G}^n\}$.

Problem 5.4. *Estimate $cd(n)$.*

Problem 5.5. *The definition of $cd_t(G)$ can be easily extended for r-uniform hypergraphs. Investigate $cd_t^r(n)$.* □

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