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The Jump Number of Suborders of the Power Set Order

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Abstract. Let P be an ordered set induced by several levels of a power set. We give a formula for the jump number of P and show that reverse lexicographic orderings of P are optimal. The proof is based on an extremal set result of Frankl and Kalai.

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1. Introduction

For a linear extension L of an ordered set P a pair (x, y) which is adjacent in L but incomparable in P, is called a *jump* (or *setup*). The number of jumps of L is denoted by $s(P, L)$, and the *jump number* of P, denoted by $s(P)$, is defined by $s(P) = \min\{s(P, L) \mid L \text{ a linear extension of } P\}$. A linear extension L of P is called *optimal* if $s(L, P) = s(P)$. The *jump number problem*, a special scheduling task, is to determine $s(P)$ and to find optimal linear extensions of P.

This problem has gained a lot of attention in the last years as documented by many articles on this subject in this journal. For an introduction and references, see e.g., [1].

Let B_n denote the lattice of all subsets of an n-element set S. For a subset $\{l_1, l_2, \ldots, l_t\}$ of $\{0, \ldots, n\}$ with $l_1 < l_2 < \cdots < l_t$ we define $B_n(l_1, \ldots, l_t)$ to be the suborder of B_n which is induced by restricting B_n to the sets of cardinality $l_1, \ldots, l_t$. We shall give a formula for the jump number of this order by proving that reverse lexicographic orderings are optimal. The proof is based on this extremal set result:

THEOREM (Frankl [2], Kalai [5]) *Let $A_1, \ldots, A_m$ and $B_1, \ldots, B_m$ be subsets of a set with $|A_i| \leqslant a$, $|B_i| \leqslant b$ and $A_i \cap B_i = \emptyset$ for $i = 1, \ldots, m$. Then $A_i \cap B_j \neq \emptyset$ for all $i, j \in \{1, \ldots, m\}$ with $i > j$ implies $m \leqslant \binom{a+b}{a}$.*

Given a linear extension L, we call a pair (x, y) a *bump* if $x \prec_L y$ and $x \prec_P y$, i.e.,if (x, y) is a covering pair in L as well as in P. The number of bumps of L is denoted by $b(P, L)$ and $b(P)$ is defined to be $\max\{b(P, L) \mid L \text{ is a linear extension of } P\}$. Obviously, $s(P) + b(P) = |P| - 1$.

2. The Result

THEOREM.

$$s(B_n(l_1, \ldots, l_t)) = -1 + \sum_{k=1}^{t} \binom{n}{l_k} - \sum_{k=1}^{t-1} \binom{n - l_{k+1} + l_k}{l_k}.$$

Reverse lexicographic orderings of $B_n(l_1, \ldots, l_t)$ are optimal.

Proof. We have to show that

$$b(B_n(l_1, \ldots, l_t)) = \sum_{k=1}^{t-1} \binom{n - l_{k+1} + l_k}{l_k}.$$

In order to prove that the left side is less than or equal to the right side, it suffices to show that

$$b(B_n(l_k, l_{k+1})) \leqslant \binom{n - l_{k+1} + l_k}{l_k}.$$

Let L be a linear extension of $B_n(l_k, l_{k+1})$ with a maximal number of bumps $(A_1, C_1), \ldots, (A_m, C_m)$, where $A_i, C_i \subseteq S$ and $|A_i| = l_k$ and $|C_i| = l_{k+1}$. We assume also that the bumps are ordered as they occur in the linear extension L, i.e.,

$$A_1 \prec_L C_1 <_L A_2 \prec_L C_2 <_L \ldots <_L A_m \prec_L C_m.$$

Now $A_i \not\prec_L C_j$ and hence $A_i \not\subset C_j$ for $i > j$. Setting $B_j := S \backslash C_j$ we have $A_i \cap B_j \neq \emptyset$ for $i > j$ and can apply the forestanding Theorem. Thus

$$b(B_n(l_k, l_{k+1})) = m \leqslant \binom{n - l_{k+1} + l_k}{l_k}.$$

Now let S be ordered, say $S = [n] := \{1, \ldots, n\}$, and let L be a reverse lexicographic ordering of B_n, i.e., $A <_L B$ iff $\max((A \cup B) \backslash (A \cap B)) \in B$ for $A, B \subseteq S$. In order to prove that the left side of the equation above is greater or equal than the right side, it suffices to show that

$$b(B_n(l_k, l_{k+1}), L) \geqslant \binom{n - l_{k+1} + l_k}{l_k}.$$

We claim that all pairs (A, B) for which $A \subseteq [n] \backslash [l_{k+1} - l_k]$ and $B = A \cup [l_{k+1} - l_k]$, are bumps of L in $B_n(l_k, l_{k+1})$. But this is clear because $A \subseteq B$ and $A <_L B$ and, moreover, $A \prec_L B$, since $B_n(l_k, l_{k+1})$ is of height one. There are $\binom{n - l_{k+1} + l_k}{l_k}$ such pairs, which finishes the proof.

COROLLARY. $s(B_n) = 2^{n-1} - 1$.

This can also easily be argued directly as follows. A linear extension of an ordered set P induces a chain partition $C_1 \cup \cdots \cup C_r$. By $l(C_i)$ we denote the length of C_i, which is the number of elements of C_i minus 1. Now $b(P)$ equals $\Sigma\, l(C_i)$, where the C_i are induced by a linear extension L of P, which is chosen such that the sum is maximal. The chains have to be convex subsets of the order, which in the case of B_n implies that they are of length at most one. Now it is easy to see that $b(B_n) = 2^{n-1}$.

In [3], Gierz and Poguntke proved that $b(P) \leqslant \text{rank } M(P)$, where $M(P)$ denotes an incidence matrix indexed by elements of P, namely

$$(M(P))_{x,y} = \begin{cases} 1 & \text{if } x < y, \\ 0 & \text{else.} \end{cases}$$

In case of our Theorem, however, this bound does not help much, because $M(B_n(l_k, l_{k+1}))$ has rank $\binom{n}{l_k}$ if $l_k + l_{k+1} \leqslant n$ (cf. [4]).

It should be interesting to determine the jump number of other classical ordered sets, like the partition lattice, linear lattices, and so on. Nothing seems to be known on this.

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